Self-improving properties of Poincaré inequalities

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ABSTRACT

The following is known as a weighted Poincaré inequality:

$$\left(\frac{1}{v(Q)}\int_Q |f-f_{av}|^r dv\right)^{1/r} \le Cl(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f|^p dw\right)^{1/p}$$

where f_{av} is either a constant depending on f and cube Q or could be taken as $\int_Q f dv/v(Q)$. The above Poincaré inequality has an interesting property such that the left hand side can be raplaced with other weight (that satisfies a balancing condition) that is independent of v. For example, the above Poincaré inequality will imply

$$\left(\frac{1}{\mu(Q)}\int_Q |f-f_{av}|^q d\mu\right)^{1/q} \le Cl(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f|^p dw\right)^{1/p}$$

if and only if

$$\frac{l(\tilde{Q})}{l(Q)}\left(\frac{\mu(\tilde{Q})}{\mu(Q)}\right)^{1/q} \le C\left(\frac{w(\tilde{Q})}{w(Q)}\right)^{1/p}$$

where $\tilde{Q} \subset Q$ are cubes and $l(Q)$ is the length of cube Q.

We will also present a number of results (on $\mathbb{R}^n$ or certain metric spaces) obtained by various people.