Q-curvature

The orig. construction of the Q-curv. seeks to imitate

Yamabe eqn. in dim. $n \geq 3$ $\longrightarrow$

Gauss curv. prescription eqn. (**GCP**) in dim. 2.

It's closely related to the GJMS operators P_m. To some extent, the construction can go either way (P's to Q's or Q's to P's). There are now constructions of Q that are genuinely different than the original one we'll discuss here (in which Q comes from the GJMS series).

Let everything as acting on functions (0-densities) on an n-dimensional manifold M.

The Graham-Jenne-Mason-Sparling (**GJMS**) operators [J. London Math. Soc. 1992] were built using the Fefferman-Graham ambient construction, and by careful analysis of the construction, have the properties in the following (redundant) list. Here n is not nec. even.

- P_m exists for m even and $m - n \notin 2\mathbf{Z}^+$.
- $P_m = \Delta^{m/2} + \text{LOT}$.
- P_m is formally self-adjoint.
- For $f \in C^\infty(M)$, under a conformal change of metric

  $$\widehat{g} = e^{2\omega} g, \qquad \omega \in C^\infty(M),$$

  we have the conformal covariance relation

  $$\widehat{P}_m f = e^{-\frac{n+m}{2}\omega} P_m(e^{\frac{n-m}{2}\omega} f).$$

- Alternatively, P_m gives rise to a conformally invariant operator $\mathbf{P}_m : \mathcal{E}[-(n-m)/2] \to \mathcal{E}[-(n+m)/2]$.

- P_m has a polynomial expression in ∇ and the Riemann tensor (actually the Ricci tensor, according to a recent result of Graham) in which all coefficients are rational in the dimension n.

- Gover and Peterson, CMP 2003 show that there's an expression in which the only poles are given by factors $(n-2)(n-4)\cdots(n-m+2)$ in the denominators of these rational functions.

- On flat $\mathbf{R}^n$, $P_m = \Delta^{n/2}$.

- P_m has the form

$$\delta S_m d + \frac{n-m}{2} Q_m,$$

where Q_m is a local scalar invariant, and S_m is an operator on 1-forms of the form

$$(d\delta)^{m/2-1} + \text{LOT} \quad \text{or} \quad \Delta^{m/2-1} + \text{LOT}.$$

All the formulas mentioned above are universal.

Note that P_m is unable to detect changes in the $(d\delta)^{m/2-1}$ term in the principal part of S_m.

Remark P_m gives rise to a Q_m in an elementary way (just take $P_m 1$) **when** $m \neq n$. But the really important Q is $Q = Q_n$.

Remark P_2 is the conformal Laplacian

$$Y = \underbrace{\delta d}_{\Delta} + \frac{n-2}{4(n-1)} K.$$

This makes

$$Q_2 = \frac{\mathsf{K}}{2(n-1)} =: \mathsf{J},$$

(the $\boxed{\text{Schouten scalar}}$).

Here's an intuitive approach (more formal approach later) to constructing the Q-curvature. The Yamabe eqn. is

$$\underbrace{\left(\Delta + \frac{n-2}{2}\mathsf{J}\right)}_{Y} u = \frac{n-2}{2} \widehat{\mathsf{J}} u^{(n+2)/(n-2)},$$

where

$$\widehat{g} = e^{2\omega} g, \quad \omega \in C^{\infty}(M), \quad u := e^{(n-2)\omega/2}.$$

The GCP eqn. is

$$\Delta\omega + \mathsf{J} = \widehat{\mathsf{J}} e^{2\omega} \qquad (n = 2).$$

We get GCP from the Yamabe eqn. by slipping in a gratuitous 1,

$$\Delta\left(e^{(n-2)\omega/2}-1\right)+\frac{n-2}{2}\mathsf{J}e^{(n-2)\omega/2}$$
$$=\frac{n-2}{2}\hat{\mathsf{J}}e^{(n+2)\omega/2},$$

dividing by $(n-2)/2$, and eval. at $n=2$.

Similarly, take the higher-order Yamabe equation based on the **GJMS operators**,

$$\underbrace{\left(\delta S_m d+\frac{n-m}{2}Q_m\right)}_{P_m}u=\frac{n-m}{2}\widehat{Q}_m u^{(n+m)/(n-m)},$$

where

$$u=e^{(n-m)\omega/2}\ (n\notin\{m,m-2,m-4,...,2,0\}),$$

$$S_m=(d\delta)^{m/2-1}+\mathsf{LOT}.$$

We slip in the gratuitous 1,

$$\delta S_m d(e^{(n-m)\omega/2} - 1) + \frac{n-m}{2} Q_m e^{(n-m)\omega/2}$$
$$= \frac{n-m}{2} \widehat{Q}_m e^{(n+m)\omega/2},$$

divide by $(n-m)/2$, and evaluate at $n = m$:

$$\boxed{P\omega + Q = \widehat{Q} e^{n\omega}.}$$

That is, we **define** Q from the GJMS op. series, as

$$\left[\frac{2P_m 1}{n-m}\right]_{n=m}.$$

This construction of Q immediately gives its unusual **linear** conformal change law. Switching to a density viewpoint (more later on this), we have a conformally invariant operator $\mathbf{P}$: $\mathcal{E}[0] \to \mathcal{E}[-n]$, and $\mathbf{Q} \in \mathcal{E}[-n]$ satisfying

$$\boxed{\boxed{\widehat{\mathbf{Q}} = \mathbf{Q} + \mathbf{P}\omega.}}$$

For example, GCP looks like

$$\widehat{\mathsf{J}} = \mathsf{J} + \boldsymbol{\Delta}\omega$$

for J viewed as a (-2)-density.

In hindsight, we have answered the

Question: Is there a higher (**even**) dimensional generalization of the exponential class Gauss curvature prescription problem $\widehat{\mathsf{J}} = \mathsf{J} + \Delta\omega$?

But the Q-curvature also plays other important roles in conformal geometry, in that:

- Its integral has total metric variation the Fefferman-Graham obstruction tensor;

- It provides the geometric expression of the exponential class Beckner-Moser-Trudinger inequality;

- It provides the main term in **Polyakov formulas** for the quotient of functional determinants or torsion quantities, at 2 conformally related metrics;

- It provides one of the important terms in volume renormalization asymptotics at conformal infinity [Fefferman-Graham, MRL 2002].

Background: The Einstein (divergence free Ricci) tensor E is the total metric variation of the scalar curvature. This means that if we take a smooth curve of metrics $g(\varepsilon)$, denote $(d/d\varepsilon)|_{\varepsilon=0}$ by a $\bullet$, and suppose

$$g(0) = g, \qquad g^{\bullet} = h,$$

then

$$\left(\int K\, dv_g\right)^{\bullet} = \int h^{ab} E_{ab} dv_g.$$

This is how the Einstein-Hilbert action leads to the Einstein equation.

Background: In dimension 4, the Bach tensor $\mathcal{B}$ is the total metric variation of $|C|^2$, where C is the Weyl tensor.

Question: In general even dimension n, the Fefferman-Graham tensor $\mathcal{O}_{ab}$ is the obstruction to the power series construction of the ambient metric assoc. to a conformal structure. Is $\mathcal{O}_{ab}$ the total metric variation of anything natural?

Answer: Yes, the Q-curvature, according to Graham-Hirachi, math.DG/0405068. In fact, for the $(-n)$-density version $\mathbf{Q}$ of the Q-curv.,

$$\left(\int \mathbf{Q}\right)^{\bullet} = \int h^{ab}\mathcal{O}_{ab}dv_g.$$

This is sensible at least when h has compact support.

Background: Beckner's [Ann. M. 1993] generalization, from S^2 to S^n, of the celebrated Moser-Trudinger inequality, says that with normalized measure on the sphere (and taking n even for simplicity),

$$\log \int_{S^n} e^{n(\omega-\bar{\omega})} \leq \frac{n}{2(n-1)!} \int_{S^n} \omega P \omega,$$

where

$$P = \Delta\{\Delta + n - 2\}\{\Delta + 2(n-3)\}\cdot$$

$$\cdot\{\Delta + 3(n-4)\} \cdots \left\{\Delta + \tfrac{n}{2}\left(\tfrac{n}{2} - 1\right)\right\}.$$

Equality holds **iff** there is a diffeomorphism h of S^n for which $h^* g_{\text{round}} = e^{2\omega} g_{\text{round}}$.

Remark: See [Branson, JFA 1987] for an early sighting of the operator P.

Remark: (2D) Moser-Trudinger is

$$\log \int_{S^2} e^{2(\omega-\bar{\omega})} \leq \int_{S^2} \omega \Delta \omega.$$

But in higher dim., note that P is more delicate than just $\Delta^{n/2}$. Closely related inequalities figure in de Branges' resolution of the Bieberbach conjecture (the Lebedev-Mihlin inequality), and Perelman's work on the Poincaré conjecture (Gross' logarithmic Sobolev inequality).

These are sharp endpoint derivatives of borderline Sobolev imbeddings, or duals of such.

Question: Is there an expression of Beckner's inequality that just involves some local invariant? Something like the soln. of the Yamabe problem, which realizes the Sobolev imbedding $L^2_1 \hookrightarrow L^{2n/(n-2)}$ as the problem of mimimizing $\int K$ over volume 1 metrics?

Answer: For vol. 1 metrics $\widehat{g} = e^{2\omega} g$, where $g = g_{\text{round}}$,

$$0 \le \int_{S^n} \omega(\widehat{\mathbf{Q}} + \mathbf{Q}).$$

Remark: The borderline Sob. imbeddings are $L^2_r \hookrightarrow L^{2n/(n-2r)}$, and the Beckner-MT edge of the borderline is $L^2_{n/2} \hookrightarrow e^L$.

Remark: This gives a glimpse of an interesting 2-metric functional on a conformal class,

$$\mathcal{Q}(\widehat{g}, g) = \frac{1}{2} \int \omega(\widehat{\mathbf{Q}} + \mathbf{Q}).$$

This is alternating, and satisfies the cocycle condition

$$\boxed{\mathcal{Q}(\widehat{\widehat{g}}, g) = \mathcal{Q}(\widehat{\widehat{g}}, \widehat{g}) + \mathcal{Q}(\widehat{g}, g)}$$

for iterated conformal changes. Here $\widehat{\widehat{g}} = e^{2\eta} \widehat{g}$, $\widehat{g} = e^{2\omega} g$, where ω and η are smooth functions.

Indeed,

$$\mathcal{Q}(\widehat{\widehat{g}}, \widehat{g}) + \mathcal{Q}(\widehat{g}, g) =$$

$$\tfrac{1}{2}\int \eta(\widehat{\widehat{\mathbf{Q}}} + \widehat{\mathbf{Q}}) + \tfrac{1}{2}\int \omega(\widehat{\mathbf{Q}} + \mathbf{Q}) =$$

$$\tfrac{1}{2}\int \eta \underbrace{(2\widehat{\mathbf{Q}} + \widehat{\mathbf{P}}\eta)}_{2(\mathbf{Q}+\mathbf{P}\omega)+\mathbf{P}\eta} + \tfrac{1}{2}\int \omega(2\mathbf{Q} + \mathbf{P}\omega) =$$

$$\tfrac{1}{2}\int (\omega + \eta)(2\mathbf{Q} + \mathbf{P}(\omega + \eta)) =$$

$$\mathcal{Q}(\widehat{\widehat{g}}, g).$$

The underbrace step used conformal invariance, $\widehat{\mathbf{P}} = \mathbf{P}$. The last step used the formal self-adjointness of $\mathbf{P}$ to equate

$$2\int \eta\mathbf{P}\omega \quad \text{and} \quad \int \eta\mathbf{P}\omega + \int \omega\mathbf{P}\eta.$$

$\mathbf{Q}(g_1, g_2)$ is a cocycle whose variation (in g_1, in the ω direction) is $\int \omega\mathbf{Q}_1$.

Background: It's known that there are Polyakov formulas expressing functional determinant quotients within a conformal class as differential polynomials in the conformal factor.

For example, let Y be the conformal Laplacian; then

$$-\log \frac{\det \widehat{Y}}{\det Y} = \int_M \omega \,\mathrm{polyn}(\underbrace{\nabla \cdots \nabla}_{\geq 1} \omega, \nabla \cdots \nabla R)$$

$$+(\text{global term})$$

in even dims., for $\widehat{g} = e^{2\omega} g$. The global term vanishes if $\mathcal{N}(Y) = 0$ (a conformally invt. property); otherwise it records the variation of the global inner product on the null space.

Similarly with Y replaced by anything with decent elliptic and conformal behavior. This includes $\boxed{\text{detour torsion}}$ quantities developed in recent joint work with Rod Gover, generalizing Cheeger's $\boxed{\text{half-torsion}}$.

Question: Can the RHS above be expressed more invariantly?

Answer: In low even dims. (2,4,6), and conjecturally in all even dims., for A a power of a conformally covariant operator with suitable positive ellipticity properties,

$$-\log \frac{\det \widehat{A}}{\det A} = c \int_M \omega(\widehat{\mathbf{Q}} + \mathbf{Q}) + \int_M (\widehat{\mathbf{F}} - \mathbf{F})$$
$$+(\text{global term})$$

for some (universal) constant c, where $\mathbf{F}$ is some density-valued local invt. (which vary depending on what A is).

Formulas like this (together with the link between these cocycles and sharp inequalities mentioned above) make possible an attack on the extremal problem for the determinant (or torsion) as in [Onofri, CMP 1982], [Osgood-Phillips-Sarnak, JFA 1988] in 2D; [Branson-Chang-Yang, CMP 1992] and [Chang-Yang, Ann. M. 1995] in 4D; [Branson, Seoul Natl. U. Lec. Notes no. 4, 1993], [Branson, TAMS 1995] in 6D; [Branson-Peterson, in prep.] in 8D.