Spectral invariants

Let's see how issues of Q-curvature, etc. turn up starting with the problem of tracking the conformal change of $\det(Y)$, where (recall)

$$Y = \Delta + \frac{n-2}{4(n-1)}\mathsf{K} = \Delta + \frac{n-2}{2}\mathsf{J}.$$

Y may really be replaced by lots of things – Dirac-squared, one of the GJMS ops., even the coboundaries of a natural complex for conformal structure – and the following will go through *mutatis mutandi*.

What is $\det(Y)$ anyway? Take a compact Riem. mfld. (M,g), and the eigenvalues λ_j of Y. Then

$$\zeta(s) = \zeta_Y(s) = \sum_{\lambda_j \neq 0} |\lambda_j|^{-s}.$$

Note that Y may have (finitely many) neg. and 0 eigenvalues.

Let's assume Y has positive spectrum for simplicity. (This is a conformally invt. condition.) For the genl. case, 0 eigs. are no real problem – they may be handled by a global term. Negative eigenvalues are even easier – they actually present no bother at all.

Also for simplicity, let us (again) say n is even.

Because of the Weyl asymptotics, $\zeta(s)$ conv. absolutely in $\mathbf{Re}(s) > n/2$. It analytically continues to a mero. fcn. on $\mathbf{C}$ with (at worst) simple poles at $n/2$, $n/2 - 1$, $\cdots$, 1. In particular, it's regular at $s = 0$ and

$$\det(Y) := \exp(-\zeta'(0)).$$

For example, on the round $2, 4, 6$-spheres, $\det(Y)$ is:

$$\exp(\tfrac{1}{2} - 4\zeta_{\mathsf{R}}'(-1)) = 3.19531\ldots,$$

$$\exp\left(\tfrac{1}{3}\left\{\tfrac{1}{48} + 2\zeta_{\mathsf{R}}'(-3) + \zeta_{\mathsf{R}}'(-1)\right\}\right),$$

$$\exp\left(\tfrac{1}{30}\left\{-\tfrac{1}{45} + \zeta_{\mathsf{R}}'(-5) - \zeta_{\mathsf{R}}'(-1)\right\}\right),$$

where $\zeta_{\mathsf{R}}(s)$ is the Riemann zeta function.

As we're getting into the nuts and bolts a little, let's carry along a generalization that will be useful. Note that

$$\zeta(s) = \mathrm{Tr}_{L^2} Y^{-s}$$

(henceforth just $\mathrm{Tr}\, Y^{-s}$). We may insert a multiplication operator just prior to tracing, and have something "just as traceable":

$$\zeta(s, \omega) := \mathrm{Tr}\, \omega Y^{-s}, \qquad \omega \in C^\infty(M).$$

This is strictly more than the old info as

$$\zeta(s) = \zeta(s, 1).$$

The $\boxed{\text{Mellin transform}}$

$$(\mathcal{M}f)(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} f(t)dt$$

performs the convenient trick of carrying

$$\exp(-t\lambda) \mapsto \lambda^{-s}.$$

So it carries

$$Z(t,\omega) := \mathrm{Tr}\,\omega \exp(-tY) \mapsto \zeta(s,\omega).$$

In more detail, the kernel fcns. for the ops. being traced are

$$\omega(x) \sum_j e^{-\lambda_j t} \varphi_j(x) \varphi_j(y),$$

$$\omega(x) \sum_j \lambda_j^{-s} \varphi_j(x) \varphi_j(y),$$

where (λ_j, φ_j) is the spectral resolution. (I.e. φ_j is the corresp. eigenfcn.) $\mathcal{M}$ acts only on the $\exp(-\lambda_j t)$ factors to produce the λ_j^{-s} factors.

However, the $\boxed{\text{heat trace}}$ $Z(t,\omega)$ has the asymptotic $\boxed{\text{heat expansion}}$

$$Z(t,\omega) \sim \sum_{\text{even } i \geq 0} t^{(i-n)/2} \int \omega U_i$$

as $t \downarrow 0$. The U_i are $\boxed{\text{natural scalars}}$ built from ∇, the Riemann tensor, the metric, and the metric inverse using tensor product and contraction.

For example,

$$(4\pi)^{n/2} U_0 = 1,$$

$$(4\pi)^{n/2} U_2 = \frac{4-n}{12(n-1)} \mathsf{K},$$

$$(4\pi)^{n/2} U_4 = \frac{1}{180} \left(90 \left(\frac{4-n}{12(n-1)} \right)^2 \mathsf{K}^2 \right.$$
$$\left. - |\mathsf{r}|^2 + |\mathsf{R}|^2 - \frac{3(6-n)}{2(n-1)} \Delta \mathsf{K} \right).$$

The heat expansion actually accomplishes the analytic continuation of the zeta function:

$$\Gamma(s)\zeta(s,\omega) =$$
$$\left(\int_0^1 + \int_1^\infty\right) t^{s-1}(\mathrm{Tr}\,\omega\exp(-tY))dt =$$
$$\sum_{\text{even } i\leq m} \left(s - \frac{n-i}{2}\right)^{-1} \int \omega U_i$$
$$+ \int_0^1 t^{s-1} O(t^{(m-n+2)/2})dt$$
$$+ \int_1^\infty t^{s-1}(\mathrm{Tr}\,\omega\exp(-tY))dt.$$

The last term is entire; the next to last is regular in a right half-plane whose boundary is moving to the left as $m\uparrow$. The first term shows us the poles and residues. Since $1/\Gamma(s)$ has zeros at the nonpositive integers, $\zeta(s,\omega)$ is regular at $0, -1, -2, \cdots$.

In particular, at $s = 0$,

$$\zeta(0,\omega) = \int \omega U_n.$$

In the following, let's pay (even) less attention to analytic questions (like interchanging limit processes). They can be handled.

We now think about the conformal variation of $\zeta(s)$, and esp. $\zeta(0)$, $\zeta'(0)$. Take a conformal curve of metrics $g_\varepsilon = e^{2\varepsilon\omega} g_0$, and let a $\bullet$ denote $(d/d\varepsilon)|_{\varepsilon=0}$. In part.,

$$\begin{aligned} g^\bullet &= 2\omega g, \\ Y^\bullet &= \frac{n-2}{2} Y\omega - \frac{n+2}{2}\omega Y \\ &= -2\omega Y + \frac{n-2}{2}[Y,\omega], \end{aligned}$$

since

$$Y_\varepsilon = e^{-(n+2)\varepsilon\omega/2} Y_0 e^{(n-2)\varepsilon\omega/2}.$$

We have:

$$\begin{aligned}
\zeta(s)^\bullet &= \Big(\mathrm{Tr}\, Y^{-s}\Big)^\bullet \\
&= -s\mathrm{Tr}(Y^\bullet Y^{-s-1}) \\
&= -s\mathrm{Tr}\Big(-2\omega Y^{-s} + \underbrace{\frac{n-2}{2}[Y,\omega]Y^{-s-1}}_{\text{contrib. } 0}\Big) \\
&= 2s\zeta(s,\omega).
\end{aligned}$$

From this we get

$$\zeta(0) \text{ is a conformal invariant,}$$

$$\zeta'(0)^\bullet = 2\zeta(0,\omega) = 2\int \omega U_n.$$

The first statement is the $\boxed{\text{conformal index}}$ property [Branson-Ørsted, Compositio M. 1986].

Another conclusion, which seems a little off-topic but isn't, is

$$\left(\int U_i\right)^\bullet = (n-i)\int \omega U_i.$$

This comes from looking at what's happening at $s=(n-i)/2$ (either a value or a residue). The conformal invce. of $\zeta(0)=\int U_n$ is of course a special case.

If we turn things around and look at conformal **anti**-variations, or $\boxed{\text{conformal primitives}}$, we're finding that

$$\begin{aligned} \text{Prim}\int \omega U_i &= \frac{1}{n-i}\int U_i \qquad (i\neq n), \\ \text{Prim}\int \omega U_n &= \tfrac{1}{2}\zeta'(0). \end{aligned}$$

Like all indef. integrals, these primitives are really only well-def. up to a constant summand.

2 ways of getting a handle on the difference between the values of $\zeta'(0)$ in conformally related metrics now present themselves – one straightforward, and one tricky.

First way: If we know U_n very well, we may write it in each metric g_ε and integrate in ε from 0 to 1. The result will be

$$\zeta'(0)_{g_1} - \zeta'(0)_{g_0}\,,$$

where $g_1 = e^{2\omega} g_0$. In fact, to do this we only need to integrate **polynomials** in ε. The downside is the $\boxed{\text{recognition problem}}$. The functional we get is (I claim) very geometric in reality, but the formula doesn't show it. We need to somehow reassemble the diffl. polyn. in ω that we get into curvatures, etc.

Second way: We note that the $1/(n-i)$ would not look so bad at $i=n$ if the dimension could be raised (as it was in the def. of the Q-curvature). In dim. N,

$$\mathrm{Prim}\int \omega U_n = \frac{1}{N-n}\int U_n.$$

So formally,

$$\zeta'(0) = 2\,\mathrm{Prim}\int \omega U_n = \left[\frac{2}{N-n}\int U_n\right]_{N=n}.$$

All these primitives were def. up to a constant summand, so really what we're getting is

$$\zeta'(0)_{\widehat{g}} - \zeta'(0)_g = \left[\frac{2}{N-n}\int(\widehat{\mathbf{U}}_n - \mathbf{U}_n)\right]_{N=n},$$

where $\mathbf{U}_n$ is the $(-n)$-density version of U_n.

If the continuation in dimension scares you (as perhaps it should), note that, if it produces an answer, this will at the very least solve the recognition problem from the first method!

It would be nice if $\widehat{\mathbf{U}}_n - \mathbf{U}_n$ in dim. N would just exhibit an obvious factor of $N-n$. But the story is more interesting than that – it doesn't, and you need Q-curvature to find the "hidden" factor.

It's interesting that the 2nd method doesn't require us to know $\mathbf{U}_n$ as well in dim. n. But we need to know $\int \mathbf{U}_n$ in many dims.

If we have $\underline{\underline{\text{a}}}$ Q-curvature for which

$$\int \mathbf{U}_n = c \int \mathbf{Q} \qquad \text{in dim. } n,$$

then

$$\int (\widehat{\mathbf{U}}_n - \mathbf{U}_n) = c \int (\widehat{\mathbf{Q}} - \mathbf{Q}) + (N-n) \int (\widehat{\mathbf{F}} - \mathbf{F}).$$

The first term on the right vanishes at $N = n$. The claim is that it's really

$$\frac{1}{2} c(N-n) \int \omega(\widehat{\mathbf{Q}} + \mathbf{Q}) + O((n-n)^2).$$

Take the Yamabe-like eqn.

$$\left\{P^0\left(e^{(N-n)\omega/2}-1\right)+\frac{N-n}{2}Qe^{(N-n)\omega/2}\right\}dv_g$$

$$=\frac{N-n}{2}\widehat{Q}e^{-(N-n)\omega/2}dv_{\widehat{g}},$$

divide by $(N-n)/2$ and multiply by $e^{(N-n)\omega/2}$:

$$\left\{e^{(N-n)\omega/2}P^0\left(\frac{e^{(N-n)\omega/2}-1}{(N-n)/2}\right)+Qe^{(N-n)\omega}\right\}dv_g$$

$$=(Qdv)_{\widehat{g}}.$$

Now rearrange:

$$(Q\,dv)_{\widehat{g}} - (Q\,dv)_g =$$

$$e^{(N-n)\omega/2} P^0 \left(\frac{e^{(N-n)\omega/2} - 1}{(N-n)/2} \right) dv_g$$

$$+(e^{(N-n)\omega} - 1)(Q\,dv)_g$$

$$= P^0 \left(\frac{e^{(N-n)\omega/2} - 1}{(N-n)/2} \right) dv_g$$

$$+(e^{(N-n)\omega/2} - 1) P^0 \left(\frac{e^{(N-n)\omega/2} - 1}{(N-n)/2} \right) dv_g$$

$$+(e^{(N-n)\omega} - 1)(Q\,dv)_g.$$

Now integrate and divide by $N-n$, noting that something in the range of P^0 integrates to 0:

$$\frac{1}{N-n}\int\Big((Q\,dv)_{\widehat{g}}-(Q\,dv)_g\Big)=$$
$$\int\left\{\frac{e^{(N-n)\omega/2}-1}{N-n}P^0\left(\frac{e^{(N-n)\omega/2}-1}{(N-n)/2}\right)\right.$$
$$\left.+\frac{e^{(N-n)\omega}-1}{N-n}Q\right\}dv_0$$

Eval. at $N=n$:

$$\left[\frac{1}{N-n}\int\Big((Q\,dv)_{\widehat{g}}-(Q\,dv)_g\Big)\right]_{N=n}=$$
$$\int\left\{\frac{1}{2}\omega P^0\omega+\omega Q\right\}dv_0=$$
$$\tfrac{1}{2}\int\omega\Big\{(Q\,dv)_{\widehat{g}}+(Q\,dv)_g\Big\}=$$
$$\mathcal{Q}(\widehat{g},g).$$

To restate,

$$\boxed{\boxed{\left[\frac{1}{N-n}\int\left(\widehat{\mathbf{Q}}-\mathbf{Q}\right)\right]_{N=n}=\tfrac{1}{2}\int\omega\left(\widehat{\mathbf{Q}}+\mathbf{Q}\right).}}$$

Now recall the point of all this:

$$\zeta'(0)_{\widehat{g}}-\zeta'(0)_g=\left[\frac{2}{N-n}\int(\widehat{\mathbf{U}}_n-\mathbf{U}_n)\right]_{N=n},$$

and

$$\int(\widehat{\mathbf{U}}_n-\mathbf{U}_n)=c\int(\widehat{\mathbf{Q}}-\mathbf{Q})+(N-n)\int(\widehat{\mathbf{F}}-\mathbf{F}).$$

As a result,

$$\boxed{\boxed{\zeta'(0)_{\widehat{g}}-\zeta'(0)_g=c\int\omega\left(\widehat{\mathbf{Q}}+\mathbf{Q}\right)+\int(\widehat{\mathbf{F}}-\mathbf{F})}}$$

where c is determined by

$$\int\mathbf{U}_n=c\int\mathbf{Q}\qquad\text{in dim. } n,$$

and $\mathbf{F}$ appears because of the "inflation" to higher dims. via

$$\int(\widehat{\mathbf{U}}_n-\mathbf{U}_n)=c\int(\widehat{\mathbf{Q}}-\mathbf{Q})+(N-n)\int(\widehat{\mathbf{F}}-\mathbf{F}).$$

Note that there is some ambiguity in $\mathbf{F}$ in the boxed formula above – in particular, we could change $\mathbf{F}$ by adding a multiple of $\mathbf{U}_n$.