Dimension 4

With our current setup, the 4D calculation [Branson-Ørsted, PAMS 1991] is now a leisurely task.

At the level of $\mathbf{U}_4$, there are 4 indep. natural scalars:

$$\mathsf{K}^2,\ |\mathsf{r}|^2,\ |\mathsf{R}|^2,\ \Delta\mathsf{K}.$$

Other bases are better for our purposes, like

$$\mathsf{J}^2,\ |\mathsf{P}|^2,\ |\mathsf{C}|^2,\ \Delta\mathsf{J}.$$

Here

$$\mathsf{P} := \frac{\mathsf{r} - \mathsf{J}g}{n-2}$$

is the Schouten tensor.

The fact that $\int \mathbf{U}_4$ is conformally invariant puts 1 condition on the coefficients, so that we are down to the 3 invariants

$$\mathbf{Q} = \Delta\mathsf{J} + 2(\mathsf{J}^2 - |\mathsf{P}|^2),\quad |\mathsf{C}|^2,\ \Delta\mathsf{J}.$$

Any linear combination of Q and $|\mathsf{C}|^2$ is usable as a Q-curvature. So (with this slightly soft meaning of Q),

$$\mathbf{U}_4 = c\mathbf{Q} + a\Delta\mathsf{J}, \quad \int \mathbf{U}_4 = c\int \mathbf{Q} \qquad \text{(in dim. 4).}$$

For the Yamabe operator,

$$(4\pi)^{n/2}\mathbf{U}_4 = \frac{1}{180}\Big(|\mathsf{C}|^2 - (n-6)(n-2)|\mathsf{P}|^2 + \tfrac{1}{2}(n-6)(5n-16)\mathsf{J}^2 + 3(n-6)\Delta\mathsf{J}\Big).$$

In 4D, this is

$$\frac{1}{180}\Big(|\mathsf{C}|^2 + 4|\mathsf{P}|^2 - 4\mathsf{J}^2 - 6\Delta\mathsf{J}\Big) = \frac{1}{180}\Big(-2\mathbf{Q} + |\mathsf{C}|^2 - 4\Delta\mathsf{J}\Big).$$

Recall that the amount of $\mathbf{Q}$ we want to "break off" here is determined by the integral in 4D. It may even be determined on a test manifold, like standard S^4. Here we're breaking off $-\mathbf{Q}/90$.

The 4π power is *a priori* an issue in analytic continuation, since it has an essential singularity at ∞ (which is being used as the accumulation point, if you just think of meromorphic continuation). This is easily handled though, if we first drain out the (predictable) 4π power to return to rational functions.

The GJMS operator being used is the Paneitz operator, [Paneitz, preprint 1983]

$$\mathsf{P} = \Delta^2 + \delta T d + \frac{n-4}{2}\mathbf{Q} \qquad (n \geq 3),$$

where

$$T = (n-2)\mathsf{J} - 4\mathsf{P}\cdot, \qquad \mathbf{Q} = \frac{n}{2}\mathsf{J}^2 - 2|\mathsf{P}|^2 + \Delta\mathsf{J}.$$

(We need to extend $\mathbf{Q}$ correctly to higher dims. to take advantage of the continuation argument we made.)

The (or a) correct extension of

$$\frac{1}{180}\int\left(-2\mathbf{Q}+|\mathsf{C}|^2)\right)$$

to higher dims. is thus

$$\frac{1}{180}\int\Big(-2\mathbf{Q}+|\mathsf{C}|^2-(n-4)^2|\mathsf{P}|^2 \\ +\tfrac{1}{2}(n-4)(5n-24)\mathsf{J}^2\Big).$$

We may just treat $\mathbf{Q}-\frac{1}{2}|\mathsf{C}|^2$ as our new alternative Q-curvature, say $\underline{\mathbf{Q}}$.

The conclusion is that

$$\zeta'(0)_{\widehat{g}}-\zeta'(0)_g = \\ -\frac{(4\pi)^{-2}}{90}\left\{\int\omega\left(\widehat{\underline{\mathbf{Q}}}+\underline{\mathbf{Q}}\right)-2\int\left(\widehat{\mathsf{J}}^2-\mathsf{J}^2\right)\right\}$$

Let's immediately treat the max/min problem within the standard conformal class on S^4. Let g be the round metric, and $\widehat{g} = e^{2\omega} g$ another metric in the conformal class.

Note that the functional determinant is not invariant under uniform scaling, so should be penalized to avoid trivialities. If $\tilde{g} = \alpha^2 g$ for $\alpha > 0$ constant, then

$$\begin{aligned} \zeta(s)_{\tilde{g}} &= \sum_j (\alpha^{-2}\lambda_j)^{-s} = \alpha^{2s}\zeta(s)_g, \\ \zeta'(0)_{\tilde{g}} &= \zeta'(0)_g + 2(\log\alpha)\zeta(0)_g. \end{aligned}$$

Recall the $\zeta(0)$ is a conformal invariant (the conformal index), so we didn'really need the subscript g we put on it.

To penalize the scaling effect, we need another quantity that scales – how about the volume? This is a good choice, as it actually appears in Beckner's inequality. Since

$$\begin{aligned} \log \mathrm{vol}(\tilde{g}) &= \log(\alpha^n \mathrm{vol}(g)) \\ &= n(\log \alpha) + \log \mathrm{vol}(g), \end{aligned}$$

the quantity

$$\mathcal{D}(g) := \zeta'(0) - \frac{2\zeta(0)}{n} \log \mathrm{vol}(g)$$

is insensitive to uniform scaling.

Recall that our c constant was chosen by

$$\int \mathbf{U}_n = c \int \mathbf{Q} \qquad \text{in dim. } n,$$

and that $\int \mathbf{U}_n = \zeta(0)$.

Thus

$$\mathcal{D}(g) = \zeta'(0) - \frac{2c\int \mathbf{Q}}{n} \log \mathrm{vol}(g),$$

$$\mathcal{D}(\widehat{g}) - \mathcal{D}(g) = \zeta'(0)_{\widehat{g}} - \zeta'(0)_g$$

$$-\frac{2c\int \mathbf{Q}}{n} \log \frac{\mathrm{vol}(\widehat{g})}{\mathrm{vol}(g)}$$

$$= c\int \omega(\widehat{\mathbf{Q}} + \mathbf{Q}) + \int (\widehat{\mathbf{F}} - \mathbf{F}) - \frac{2c\int \mathbf{Q}}{n} \log \frac{\mathrm{vol}(\widehat{g})}{\mathrm{vol}(g)}.$$

These last paragraphs were quite general.

Back in the case at hand,

$$c = -\frac{(4\pi)^{-2}}{90}, \qquad \mathrm{vol}(S^4) = \frac{(4\pi)^2}{2},$$
$$\mathbf{F} = -\frac{(4\pi)^{-2}}{90}\mathsf{J}^2. \qquad \mathbf{Q} = 6.$$

So (with $⨍$ as normalized $\int$ and $\mathbf{J}$ the (-2)-density J),

$$\mathcal{D}(\hat{g}) - \mathcal{D}(g) =$$
$$-\frac{1}{45}\left\{⨍\omega(\underbrace{\widehat{\mathbf{Q}} + \mathbf{Q}}_{\mathbf{P}\omega+2\mathbf{Q}}) + ⨍(\hat{\mathbf{J}}^2 - \mathbf{J}^2)\right.$$
$$\left.+\frac{1}{15}\log ⨍ e^{4\omega}\right\}$$
$$= -\frac{1}{15}\left\{\frac{1}{3}⨍\omega(\underbrace{\widehat{\mathbf{Q}} + \mathbf{Q}}_{\mathbf{P}\omega+12}) - \log ⨍ e^{4\omega}\right\}$$
$$-\frac{1}{45}⨍(\hat{\mathbf{J}}^2 - \mathbf{J}^2)$$
$$= -\frac{1}{15}\left\{\frac{1}{3}⨍\omega\mathbf{P}\omega - \log ⨍ e^{4(\omega-\bar{\omega})}\right\}$$
$$-\frac{1}{45}⨍(\hat{\mathbf{J}}^2 - \mathbf{J}^2).$$

This is really shaping up, as the $\{\cdot\}$ of the first term is exactly the thing estimates by Beckner-Moser-Trudinger – it's ≥ 0 with equality iff $\widehat{g}$ is a round metric (a conformal diffeomorph of the round one we started with).

There is that other term though. But this is also precisely estimated, with the same extremals, by an argument of Paul Yang. (This argument can be generalized quite a bit to handle some "trailing terms" in higher dimensional calculations.)

In the setting of compact conformal manifolds, take the $\boxed{\text{Yamabe quotient}}$

$$\mathcal{Y}(u) := \frac{\int uYu}{\|u\|_q^2}, \qquad q := \frac{2n}{n-2},$$

where

$$Y = \Delta + \frac{n-2}{2}\mathsf{J}.$$

The $\boxed{\text{Yamabe number}}$ is

$$\mu = \inf_{u>0} \mathcal{Y}(u).$$

If we think of u as a $(2-n)/2$-density, $\mathcal{Y}(u)$ is conformally invariant. If we think of u as a function, conformal covariance is expressed by

$$\mathcal{Y}_{\widehat{g}}(e^{(2-n)\omega/2}u) = \mathcal{Y}_g(u).$$

Take a positive function u, which we secretly think of as a metric $\widehat{g} = e^{2\omega}g$ via $u = e^{(n-2)\omega/2}$:

$$\begin{aligned}
&\mu \leq \frac{\int uYu}{\|u\|_q^2} \leq \frac{\int u^2|Yu/u|}{\|u\|_q^2} \\
&\leq \frac{\|u^2\|_{n/(n-2)}\|Yu/u\|_{n/2}}{\|u\|^2_{2n/(n-2)}} = \|Yu/u\|_{n/2} \\
&= \left\|\frac{n-2}{2}\widehat{\mathsf{J}}e^{2\omega}\right\|_{n/2} = \frac{n-2}{2}\left\{\int \left|\widehat{\mathsf{J}}\right|^{n/2} dv_{\widehat{g}}\right\}^{2/n} \\
&= \frac{n-2}{2}\left\{\int \left|\widehat{\mathbf{J}}\right|^{n/2}\right\}^{2/n}.
\end{aligned}$$

So

$$\left(\frac{2\mu}{n-2}\right)^{n/2} \leq \int \left|\widehat{\mathbf{J}}\right|^{n/2}.$$

If g minimizes the Yamabe quotient, then J is const. and

$$\mu = \frac{\int \frac{n-2}{2}\mathsf{J}\, dv_g}{\mathrm{vol}(g)^{(n-2)/n}} = \frac{n-2}{2}\mathsf{J}\,\mathrm{vol}(g)^{2/n},$$

so

$$\left(\frac{2\mu}{n-2}\right)^{n/2} = \mathsf{J}^{n/2}\mathrm{vol}(g) = \int \mathsf{J}^{n/2} dv_g,$$

and

$$0 \leq \int \left\{\left|\widehat{\mathbf{J}}\right|^{n/2} - \mathbf{J}^{n/2}\right\}.$$

This is nicer when $n/2$ is even (or the Yamabe number is ≥ 0). Recall that our current interest is dim. 4.

Just to stop and smell the roses, if n/m is an integer > 1, the same argument works with Q_m in place of $\mathsf{J} = Q_2$ – this is one virtue of having the higher-order Yamabe problems. The conclusion is

$$0 \leq \int \left\{ \left| \widehat{\mathbf{Q}}_m \right|^{n/m} - \mathbf{Q}_m^{n/m} \right\}.$$

The corresponding Yamabe-like fcnl. is

$$\frac{(P_m u, u)_{L^2}}{\|u\|_q^2}, \qquad q = \frac{2n}{n-m},$$

and (recall that) the corresp. prescription eqn. is

$$P_m u = \frac{n-m}{2} \widehat{Q}_m u^{(n+m)/(n-m)}, \;\; u = e^{(n-m)\omega/2}.$$

Implicitly, the inequality being used is the Sobolev imbedding $L^2_{m/2} \hookrightarrow L^q$, for which (on the sphere) Beckner has a sharp form that applies exactly. One difference is that these higher-order Yamabe problems are not solved in general. But on the sphere the Yamabe fcnls. are minimized at the round metrics, so it all works.

Back to S^4, we now know that

$$0 \leq \fint (\widehat{\mathbf{J}}^2 - \mathbf{J}^2).$$

In order to have equality, $\widehat{g}$ has to be a Yamabe quotient minimizer (we started the string of ineqs. by saying $\mu \leq \mathcal{Y}(u) \leq \cdots$). On the sphere, these are the round metrics.

Thm. [Branson-Chang-Yang, CMP 1992] In the standard conformal class on S^4, the penalized $-\log\det$ fcnl. $\mathcal{D} = \mathcal{D}_Y$ is maximized exactly at the round metrics.

Recall the assertion that there was nothing special about Y (as the operator whose det is being taken) in this. At least among ops. with decent conformal behavior this is true. For the square of the Dirac op. D on spinors, all is the same, except that the coefs. on the 2 sub-functionals, instead of being $-1/15$ and $-1/45$, are two (specific) positive numbers. Thus

Thm. • • • $\mathcal{D}_{D^2}$ is mimimized • • •.

One can get the coefficients to disagree. [Branson, CMP 1996] shows that this happens in 4D with the $\mathcal{D}_P$ of the Paneitz op. It's an open question whether one still gets the "correct" extremals.

The way in which $\mathbf{J}^2$ arrived in the above formulas is obscured somewhat by the dimensional continuation process. Thus, for the sake of better understanding, it's useful to contemplate how this goes completely within 4D (but still using some Q-technology).

What happened was that we needed a conformal primitive for

$$\int \omega\Big(\underbrace{-2\mathbf{Q} + |\mathsf{C}|^2}_{-2\underline{\mathbf{Q}}} - 4\Delta \mathsf{J}\Big).$$

For the $\underline{\mathbf{Q}}$ term, the answer was $\mathcal{Q}(\widehat{g}, g)$ as worked out before. For the $\Delta\mathsf{J}$ term,

$$\Big(\int \mathsf{J}^2\Big)^{\bullet} = 2\int \mathsf{J}\Delta\omega = 2\int \omega\Delta\mathsf{J}.$$

A really old-fashioned approach would have had us getting out formulas like

$$(\Delta \mathsf{J}\, dv)^{\frown} = \Big\{ \Delta \mathsf{J} + \Delta^2 \omega - 2\mathsf{J}\Delta\omega + 2\langle d\mathsf{J}, d\omega\rangle$$

$$-\Delta |d\omega|^2 - 2(\Delta\omega)^2 + 2\langle d\omega, d\Delta\omega\rangle$$

$$+2|d\omega|^2\Delta\omega - 2\langle d\omega, d|d\omega|^2\rangle \Big\} dv,$$

and for the conformal primitive, multiplying the s-homog. part (in ω) by $(s+1)^-1$, multiplying by ω, and integating. This gives a small taste of the **recognition problem**. After some integrations by parts (in which there is no normal form to work toward), one has to find the

$$(\mathsf{J}^2 dv)^{\frown} - \mathsf{J}^2 dv.$$