Continuation in dimension

Recall that a Q-curvature $\mathbf{Q}$ in even dimension n has

$$\widehat{\mathbf{Q}} = \mathbf{Q} + \mathrm{P}\omega,$$

where $\mathrm{P} : \mathcal{E} \to \mathcal{E}[-n]$ is an operator of the form

$$\mathrm{P} = \delta \left\{ (d\delta)^{n/2-1} + \mathsf{LOT} \right\} d.$$

P is automatically conformally invariant.

Let's do the original Q-curvature construction in the case $n = 2$, where $\mathbf{Q} = \mathsf{J}$. This is really the derivation of the $\boxed{\text{Gauss curv. presc. eqn.}}$ from the higher-dimensional $\boxed{\text{Yamabe eqn.}}$ The packaging in terms of $\boxed{\text{stabilization}}$ (inflating the dimension by taking the product with flat tori) was suggested by Robin Graham. The original treatment inflates the dimension in an invariant-theoretic sense.

Start with a Riemannian 2-manifold (M, g), and take the product with a standard flat m-torus (T, k) for $m = 0, 1, 2, \cdots$. Then change the metric

$$g + k = \begin{pmatrix} g_{ab} & 0 \\ 0 & k_{\alpha\beta} \end{pmatrix}$$

conformally, using a conformal factor ω that depends only on the M-parameter x.

The Yamabe equation

$$\left(\Delta + \frac{n-2}{2}\mathsf{J}\right) e^{(n-2)\omega/2} = \frac{n-2}{2}\widehat{\mathsf{J}} e^{(n+2)\omega/2}$$

in our situation says that

$$\left\{\Delta[g+k] + \frac{m}{2}\mathsf{J}[g+k]\right\} e^{m\omega/2} =$$
$$\frac{m}{2}\mathsf{J}[e^{2\omega}(g+k)]e^{(m+4)\omega/2}.$$

Dividing by $e^{m\omega/2}$, this may be written

$$\frac{m}{2}\Big(\mathsf{J}[e^{2\omega}(g+k)]e^{2\omega}-\mathsf{J}[g+k]\Big)=$$
$$e^{-m\omega/2}\underbrace{\Delta[g+k]e^{m\omega/2}}_{\Delta[g]e^{m\omega/2}}.$$

The RHS in this last eqn. is **polyn.** in m, and in fact (since $\Delta=\delta d$) one with no constant term. After evaluation at some $x\in M$, these are true polynomials (with numerical coefficients). In particular, everything is indep. of $y\in T$. This establishes the LHS (eval. at x) as a polyn. in m, indep. of the T-parameter y.

What's happening here is: there was an $(m+1)^{-1}$ involved in defining J from K. But the formula for the difference of J quantities at conformally related metrics is supplying an $m+1$ factor to keep things polynomial.

When going to higher order, it's important to be able to conclude the polynomial nature as above, rather than by explicit formulas.

We have either

- ∞ many eqns. (param. by m) on M; or
- a polynomial eqn. (in the vbl. m) at each $x \in M$.

Taking the 2nd viewpoint and harvesting the termwise eqns., the m^0 level gives

$$\Delta[g]1 = 0.$$

What the m^1 level provides can be seen by truncating power series:

$$\left(1-\frac{m\omega}{2}\right)\Delta[g]\left(1+\frac{m\omega}{2}\right)=$$
$$\frac{m}{2}\left(\mathsf{J}[e^{2\omega}(g+k)]e^{2\omega}-\mathsf{J}[g+k]\right)+O(m^2),$$

where $O(m^2)$ denotes a **polynomial** with a factor of m^2. (Recall that all this is happening at some $x \in M$.)

This is

$$-\frac{m\omega}{2}\underbrace{\Delta[g]1}_{0}+\frac{m}{2}\Delta[g]\omega=$$
$$\frac{m}{2}\mathsf{J}[e^{2\omega}g]e^{2\omega}-\frac{m}{2}\mathsf{J}[g]+O(m^2),$$

since to get the m^0 term in the polynomial $\mathsf{J}[e^{2\omega}(g+k)]e^{2\omega}$, we may evaluate at $m=0$.

This last equation **really** doesn't have anything (even typographical) to remind us of the torus T. The m^1 coef. in the last eqn. is Gauss curvature prescription,

$$\Delta[g]\omega + \mathsf{J}[g] = \mathsf{J}[e^{2\omega}g]e^{2\omega} \qquad \text{(dim. 2).}$$

Now generalize this to a way of getting the critical Q-curvature prescription

$$P_n[g]\omega + Q_n[g] = Q_n[e^{2\omega}g]e^{n\omega}$$

(in the non-density version) from the subcritical (Yamabe type) Q-prescription equations in higher dimensions:

$$\left(P_n^0[g] + \frac{N-n}{2}Q_n[g]\right)e^{(N-n)\omega/2} = \frac{N-n}{2}Q_n[e^{2\omega}g]e^{(N+n)\omega/2}.$$

Here n is the $\boxed{\text{target dim.}}$ and N is the $\boxed{\text{running dim.}}$

This is the conformal covariance law

$$\widehat{P}_n = \Omega^{-(N+n)\omega/2} P_n \Omega^{(N-n)\omega/2}$$

for the GJMS operator

$$P_n = P_n^0 + \frac{N-n}{2} Q_n$$

applied to the function 1.

We want to use this in the situation where the manifold is a product of the n-dim. M and a flat m-torus T (so $N = n + m$):

$$\left\{ P_n^0[g+k] + \frac{m}{2} Q_n[g+k] \right\} e^{m\omega/2} =$$
$$\frac{m}{2} Q_n[e^{2\omega}(g+k)] e^{(m+2n)\omega/2}.$$

We divide by $e^{m\omega/2}$ to get

$$\frac{m}{2} \Big(Q_n[e^{2\omega}(g+k)] e^{n\omega} - Q_n[g+k] \Big) =$$
$$e^{-m\omega/2} P_n^0[g+k] e^{m\omega/2}.$$

Since P_n^0 has the form $\delta(\bullet)d$, this establishes

$$Q_n[e^{2\omega}(g+k)]e^{n\omega} - Q_n[g+k]$$

as a **rational function** in m (at each $x \in M$, indep. of $y \in T$). (Recall that we can keep the poles under control; the rightmost one is at $n-2$.) This is just as good, for our purposes, as the polynomial behavior we had in the Yamabe $\to$ GCP case, since

rat'l. fcns. agree at ∞ many points $\iff$

they agree $\iff$

their power series expansions

(at some regular point) agree termwise.

And the engine behind these equivalences is (not complex variables but) polynomial continuation.

At the m^0 level we have $P_n^0 1 = 0$. At the m^1 level, it's

$$\frac{m}{2} P_n^0 \omega = \frac{m}{2} \left(Q_n[e^{2\omega} g] e^{n\omega} - Q_n[g] \right).$$

But this is the critical Q-curvature prescription equation

$$P_n[g]\omega + Q_n[g] = Q_n[e^{2\omega} g] e^{n\omega} \qquad \text{(dim. } n\text{)},$$

since $P_n = P_n^0$ in dim. n.