Some invariant theory

Setting: Smooth manifolds M of **even** dimension n.

Riemannian metric: A positive definite symmetric section g of $T^*M \otimes T^*M$.

g determines a unique torsion-free affine *connection* ∇,

$$\nabla g = 0, \qquad \nabla_X Y - \nabla_Y X = [X, Y],$$

which in turn determines the *Riemann curvature* R:

$$[\nabla_a, \nabla_b] X^c = R^c{}_{dab} X^d.$$

In the last formula, we're using *abstract index notation*. In particular,

$$\nabla_a X^b \text{ means } (\nabla X)_a{}^b.$$

The index d above, occuring once up and once down, denotes a *contraction* (or *trace*).

Raising and lowering of indices: Via the metric tensor $g = (g_{ab})$ and its inverse $g^\sharp = (g^{ab})$, defined by

$$g^{ab} g_{bc} = \delta^a{}_c.$$

For example,

$$\nabla_a X_b = g_{bc} \nabla_a X^c.$$

Natural tensors: Tensors built polynomially from R, ∇, g and $g^\sharp$ using tensor product and contraction. For example,

the Ricci tensor	$\mathsf{r}_{ab} := \mathsf{R}^c{}_{acb}$
the scalar curvature	$\mathsf{K} := g^{ab} \mathsf{r}_{ab} = \mathsf{r}^b{}_b$,
	$(\nabla^a \mathsf{r}^{bc}) \nabla_b \mathsf{r}_{ac}$.

The last 2 are *natural scalars*.

Conformal structure: A Riemannian metric, defined only up to positive function multiples:

$$[g] = \{e^{2\omega} g \mid \omega \in C^\infty(M)\}.$$

A choice of metric from within a conformal class is sometimes called a *conformal scale*.

Densities: Given a conformal structure, would like an object that's like a scalar function, but which responds to conformal change of metric

$$\widehat{g} = e^{2\omega} g, \qquad \omega \in C^\infty(M)$$

by acquiring a factor of $e^{w\omega}$ for some w.

Such an object is a section of a line bundle with trivializations parameterized by (U, g), where U is open and g is a scale. $f \in \mathcal{E}[w]$ has

$$(f)_{(U,\widehat{g})} = e^{w\omega}(f)_{(U,g)}.$$

For example, the metric determinant $\det(g)$ is a $2n$-density.

In fact, w-densities may be defined wo/ reference to conformal structure to be trivialized on (U, g), where U is a coordinate chart, so that

$$(f)_{(V,h)} = (\det(h)/\det(g))^{w/2n}(f)_{(U,g)},$$

g and h **any** metrics.

Once you have density bundles, you have tensor-density bundles like

$$\mathcal{E}^a[w] = \mathcal{E}[w] \otimes \mathcal{E}^a.$$

Here we denote a tensor bundle by its index structure, and blur the distinction between bundles and their section spaces. As another example, the 3-forms would be

$$\mathcal{E}_{[abc]},$$

the $[\cdot]$ denoting antisymmetrization.

Connection on density bundles:
Determined by

$$\nabla \det(g) = 0 \qquad (\text{so } \nabla \det(g)^r = 0).$$

This really insures that calculus with ∇ can be carried out without worrying about density weights.

Conformal metric: There's a tensor-density $\mathbf{g} \in \mathcal{E}_{(ab)}[2]$ which represents the conformal structure, just as a metric g represents a Riemannian structure. In the trivialization (U, g), it just gives g.

Natural scalars have a grading by *level*. If we multiply the metric by a positive constant α^2,

$$\widetilde{g} = \alpha^2 g,$$

then a polynomial A has *level* 2ℓ if

$$\widetilde{A} = \alpha^{-2\ell} A.$$

What's really happening with the level is:

$$\widetilde{\mathsf{R}} = \mathsf{R}, \quad \widetilde{\nabla} = \nabla, \quad \tilde{g} = \alpha^2 g, \quad \tilde{g}^\sharp = \alpha^{-2} g.$$

Thus the level measures the net number of indices that must be raised before contracting to a scalar:

$$2\ell = 2N_{g^\sharp} - 2N_g.$$

(This is how we know it's even.)

R has 2 excess down indices and ∇ has 1. So $g^\sharp$ will be used

$$2\ell = 2N_{\mathsf{R}} + N_{\nabla}$$

times more than g. I.e., the level is "secretly" a derivative count – each ∇ is 1 derivative, and R is 2 derivs. (of the metric).

It's convenient to view level -2ℓ natural scalars as (-2ℓ)-densities. This naturally happens if we replace each g in the formula by $\mathbf{g}$, and each $g^\sharp$ by

$$\mathbf{g}^\sharp = (\mathbf{g}^{ab}) \in \mathcal{E}^{(ab)}[-2], \qquad \mathbf{g}^{ab}\mathbf{g}_{bc} = \delta^a{}_c.$$

F.ex., the scalar curvature is

$$\mathbf{g}^{ab}\mathsf{r}_{ab} \in \mathcal{E}[-2]$$

in this viewpoint.

Conformal change for natural scalar densities: To track how natural densities change under the usual conformal change

$$\widehat{g} = e^{2\omega}g,$$

we just need to know how the ingredients g, $g^\sharp$, ∇, R change. The change of g is just above, and that of $g^\sharp$ is

$$\widehat{g}^\sharp = e^{-2\omega}g^\sharp.$$

For ∇,

$$
\begin{aligned}
(\nabla_a X^b)\widehat{\ } &= \nabla_a X^b + \underbrace{(\omega_{|a}\delta_c{}^b + \omega_{|c}\delta_a{}^b - \omega_|{}^b g_{ac})}_{\widehat{\Gamma}_{ac}{}^b - \Gamma_{ac}{}^b} X^c \\
&= \nabla_a X^b + \omega_{|a} X^b + \omega_{|c}\delta_a{}^b X^c - \omega_|{}^b X_a.
\end{aligned}
$$

To get to other tensor densities, we just need to note that ∇_a is a derivation, commutes with contractions, and agrees with the above ∇_a on (scalar) densities.

To see how R changes: first, the partial decomposition of R into Weyl and Schouten parts is

$$
\mathsf{R}^a{}_{bcd} = \mathsf{C}^a{}_{bcd} - 2\mathsf{P}_{b[c}\delta^a{}_{d]} + 2\mathsf{P}^a{}_{[c}\mathbf{g}_{d]b},
$$

where

$$
\widehat{\mathsf{C}}^a{}_{bcd} = \mathsf{C}^a{}_{bcd}
$$

is the $\boxed{\text{Weyl conformal curvature tensor}}$. P is the $\boxed{\text{Schouten tensor}}$,

$$
\mathsf{P} = \frac{r - \mathsf{J}g}{n-2}, \qquad \mathsf{J} = \frac{\mathsf{K}}{2(n-1)}.
$$

Note that

$$\mathsf{J} = \mathsf{P}^a{}_a, \qquad \mathsf{J}_{|a} = \mathsf{P}_{ab|}{}^b.$$

P changes by

$$\widehat{\mathsf{P}}_{ab} = \mathsf{P}_{ab} - \omega_{|ab} + \omega_{|a}\omega_{|b} - \frac{1}{2}\omega_{|c}\omega_{|}{}^c\mathbf{g}_{ab}.$$

As a result,

$$\widehat{\mathbf{J}} = \mathbf{J} - \omega_{|a}{}^a - \frac{n-2}{2}\omega_{|a}\omega_{|}{}^a.$$

In the last formula, using the (-2)-density version $\mathbf{J}$ of J as a (-2)-density made a difference – viewing it as a function, there would be a "weight factor" $e^{-2\omega}$ in front of the RHS.

By induction from the above, a level 2ℓ natural scalar $\mathbf{A}$, viewed as a (-2ℓ)-density, has a conformal change law

$$\widehat{\mathbf{A}} = \mathbf{A} + \mathbf{X}_1[\mathbf{A}](d\omega, \mathbf{g}, \mathbf{g}^\sharp, \nabla, R) + \cdots + \mathbf{X}_{2\ell}[\mathbf{A}](d\omega, \mathbf{g}, \mathbf{g}^\sharp, \nabla, R),$$

where

- ∇ and R are computed in g (not $\widehat{g}$);
- $\mathbf{X}_s$ is a universal polynomial formula for a (-2ℓ)-density, s-multilinear in ω;
- $2\ell = s + N_\nabla + 2N_R$ for a monomial term in $\mathbf{X}_s[A]$.

Let $\mathbf{I}_{2\ell}$ be the space of natural (-2ℓ)-densities A, and let $\mathbf{I}^t_{2\ell}$ be the subspace of those for which

$$\mathbf{X}_s[A] = 0, \qquad s > t.$$

In fact, this is the same as just requiring $\mathbf{X}_{t+1}[A] = 0$ – universality does the rest. Take a curve of metrics $g_\varepsilon = e^{2\varepsilon\omega} g_0$; then

$$\mathbf{X}_s[A]_{\varepsilon\omega} = \varepsilon^s \mathbf{X}_s[A]_\omega,$$
$$\left.\frac{d^s}{d\varepsilon^s}\right|_{\varepsilon=0} A_{\varepsilon\omega} = \mathbf{X}_s[A]_0.$$

If $\mathbf{X}_s[A]_0 = 0$ *universally*, then it's 0 replacing g_0 by $g_{\varepsilon_0\omega}$, so

$$\frac{d^s}{d\varepsilon^s} A_{\varepsilon\omega} = 0;$$

thus all higher derivatives also vanish.

$\mathbf{I}^0_{2\ell}$ is the space of $\boxed{\text{local conformal invariants}}$ ($\widehat{A} = A$) of level 2ℓ. F.ex.

$$|\mathsf{C}|^2 = \mathsf{C}^{abcd}\mathsf{C}_{abcd} \in \mathbf{I}^0_4.$$

To check whether some A in $\mathbf{I}_{2\ell}$ is in $\mathbf{I}^0_{2\ell}$, we just need to check whether its $\boxed{\text{conformal variation}}$

$$\mathbf{X}_1[A](d\omega) =: (\mathbf{b}A)\omega$$

vanishes. Note that $\mathbf{b}A$ is a **linear differential operator** on functions. Furthermore, it can be written with a right d factor:

$$\mathbf{b}A = Td,$$

for some linear differential operator T.

$\mathbf{b}A$ has a $\boxed{\text{formal adjoint}}$ $(\mathbf{b}A)^*$. If $\mathbf{b}A$ is $\boxed{\text{formally self-adjoint}}$, it can be written δT^*, and thus $\delta S d$. F.ex.

$$(\mathbf{b}\mathsf{J})\omega = -\omega_{|a}{}^{a} =: \Delta\omega = \delta d\omega.$$

Thus (to coin a notation) $\mathsf{J} \in \mathbf{I}^{\mathsf{FSA}}_2$.

Note that on the k-forms $\mathcal{E}^k = \mathcal{E}_{[a_1 \cdots a_k]}$,

$$\begin{aligned}(d\varphi)_{a_0 \cdots a_k} &= (k+1)\nabla_{[a_0}\varphi_{a_1 \cdots a_k]},\\ (\delta\varphi)_{a_2 \cdots a_k} &= -\nabla^b \varphi_{ba_2 \cdots a_k}.\end{aligned}$$

The $\boxed{\text{Q-curvature space}}$ is

$$\mathbf{I}^{\mathsf{Q}} = \mathbf{I}_n^1 \cap \mathbf{I}_n^{\mathsf{FSA}}.$$

A $\boxed{\text{Q-curvature}}$ is an element $\mathbf{Q}$ of $\mathbf{I}^{\mathsf{Q}}$ with

$$\mathbf{bQ} = \delta S d = \delta \left\{ (d\delta)^{n/2-1} + \mathsf{LOT} \right\} d$$

$$(= \Delta^{n/2} + \mathsf{LOT}).$$

In other words, a Q-curvature $\mathbf{Q}$ has

$$\widehat{\mathbf{Q}} = \mathbf{Q} + \mathbf{P}\omega,$$

where $\mathbf{P} : \mathcal{E} \to \mathcal{E}[-n]$ is an operator of the form

$$\mathbf{P} = \delta \left\{ (d\delta)^{n/2-1} + \mathsf{LOT} \right\} d.$$

Such an operator is **automatically** conformally invariant, and is called a $\boxed{\text{critical GJMS operator}}$.

The name refers to the landmark paper by Graham, Jenne, Mason, and Sparling, J. London Math. Soc. 1992. Q-curvatures were constructed in even dimensions using the whole series of GJMS operators in Branson, Seoul Natl. U. lecture notes #4, 1993. So the $\mathbf{P}$'s came first, but in hindsight the critical $\mathbf{P}$ is implicit in $\mathbf{Q}$.

Thm. There exists a Q-curvature.

The original proof involved higher (than n, the "target dimension") dim. manifolds.

One may view this as taking products with tori. It's been called "analytic continuation in dimension," but this terminology is overly frightening – it has nothing to do with non-integer dimensions. These days there are many proofs, including some taking place entirely within manifolds of the target dimension.

Example: In dimension 2, $\mathbf{I}_2$ is generated by J, which is a Q-curvature, recalling

$$\widehat{\mathbf{J}} = \mathbf{J} - \omega_{|a}{}^{a} - \frac{n-2}{2}\omega_{|a}\omega_{|}{}^{a}.$$

The corresponding $\mathbf{P}$ is Δ.

Related to $\mathbf{I}^{\mathsf{FSA}}$ is a potentially larger space, $\mathbf{I}^{\mathsf{ix}}$, the $\boxed{\text{conformal index densities}}$. If we take the conformal variation of $\int \mathbf{A}$ for $\mathbf{A} \in \mathbf{I}_n$, we get

$$\int (\mathbf{b}\mathbf{A})\omega = \int \omega \underbrace{(\mathbf{b}\mathbf{A})^* 1}_{:=\partial \mathbf{A}}.$$

This will be universally 0 iff $(\mathbf{b}\mathbf{A})^*$ has the form Td, iff $\mathbf{b}\mathbf{A}$ has the form δT^*. By universality again, infinitesimal invariance is the same as invariance, so

$$\mathbf{A} \in \mathbf{I}^{\mathsf{ix}} \iff \int \mathbf{A} \text{ is conformally invt.}$$

Note that $(-n)$-densities are just what can be integrated given a conformal structure, since their conformal scaling cancels that of the Riemannian measure dv_g.

Prop. $$\begin{cases} \partial \mathbf{I}_n \subset \mathbf{I}_n^{\mathsf{div}} \subset \mathbf{I}^{\mathsf{ix}}, \\ \mathbf{I}^{\mathsf{Q}} + \partial \mathbf{I}_n \subset \mathbf{I}_n^{\mathsf{FSA}} \subset \mathbf{I}^{\mathsf{ix}}, \\ \partial\partial = 0, \\ \text{and } \mathbf{I}^{\mathsf{ix}} \text{ is strictly larger than } \mathbf{I}_n^{\mathsf{div}}. \end{cases}$$

Proof: If $\mathbf{A} \in \mathbf{I}_n$, then $\mathbf{b}\mathbf{A} = Td$,

$$\partial \mathbf{A} = (\mathbf{b}\mathbf{A})^* 1 = \delta T^* 1 \in \mathbf{I}_n^{\mathsf{div}}.$$

$\partial\partial\mathbf{A} = 0$ since $(\int \delta F)^\bullet = 0^\bullet = 0$.

If $\mathbf{A} \in \mathbf{I}_n^{\mathsf{FSA}}$, then $\mathbf{b}\mathbf{A} = (\mathbf{b}\mathbf{A})^* = \delta S d$,
$\partial \mathbf{A} = \delta S d 1 = 0$.

$\mathsf{Pff} \in \mathbf{I}^{\mathsf{ix}} \setminus \mathbf{I}_n^{\mathsf{div}}$ since $\chi(S^n) = 2$.

The only nontrivial assertion is $\partial \mathbf{I}_n \subset \mathbf{I}_n^{\mathsf{FSA}}$. If $\mathbf{A} = \partial F$, then

$$\mathcal{H}(\widehat{g}, g) = \int (\widehat{F} - F)$$

is a $\boxed{\text{conformal primitive}}$ for $\mathbf{A}$. This means that $\mathcal{H}$ is a 2-metric functional on the conformal class which

- is $\boxed{\text{alternating}}$, $\mathcal{H}(g_0, g_1) = -\mathcal{H}(g_1, g_0)$;

- is a $\boxed{\text{cocycle}}$,
  $\mathcal{H}(g_2, g_0) = \mathcal{H}(g_2, g_1) + \mathcal{H}(g_1, g_0)$;

- has conformal variation $\mathbf{A}$, in the sense that

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{H}(e^{2\varepsilon\omega} g, g) = \int \omega \mathbf{A}.$$

Note that taking

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{H}(e^{2\varepsilon\eta}g, g_0)$$

for any other metric in the conformal class will give the same answer, by the cocycle condition.

The corresp. sec variation (in the direction η) is $\int \omega(\mathbf{b}\mathbf{A})\eta$; but this must be symmetric, meaning $\mathbf{b}\mathbf{A}$ is FSA.

An important open question is:

Conjecture. $\boxed{\boxed{\mathbf{I}^{\mathsf{Q}} + \partial \mathbf{I}_n = \mathbf{I}_n^{\mathsf{FSA}}.}}$

(We know $\subset$.)

Example: In dim. 4, $\mathbf{I}_4$ is generated by J^2, $|\mathsf{P}|^2$, $|\mathsf{C}|^2$, and $\Delta\mathsf{J}$. $|\mathsf{C}|^2$ is a local conformal invariant; $\Delta\mathsf{J}$ is an exact divergence, so both are conformal index densities. In fact

$$\mathbf{I}_n^{\mathsf{div}} = \mathsf{span}\{\Delta\mathsf{J}\}.$$

The $\boxed{\text{Pfaffian}}$ (Euler characteristic density) is (up to a constant factor)

$$\mathsf{Pff} = |\mathsf{C}|^2 + 8(\mathsf{J}^2 - |\mathsf{P}|^2).$$

The conformal index densities thus are at least 3-dim. in a 4-dim. space. Since $\mathbf{I}^{\mathsf{ix}} = \mathcal{N}(\partial)$ and

$$\left(\int \mathsf{J}^2\right)^{\bullet} = \underbrace{2\int \mathsf{J}\Delta\omega}_{\int(\mathbf{b}(\mathsf{J}^2))\omega} = 2\int \omega\Delta\mathsf{J},$$

we have $\partial(\mathsf{J}^2) = 2\Delta\mathsf{J}$. So $\mathbf{I}^{\mathsf{ix}}$ is exactly 3-dim.

This also shows

$$\partial\mathbf{I}_4 = \mathsf{span}\{\Delta\mathsf{J}\}.$$

Let

$$\mathbf{Q} = \Delta \mathsf{J} + 2(\mathsf{J}^2 - |\mathsf{P}|^2).$$

The conformal change of $\mathbf{Q}$ is

$$\widehat{\mathbf{Q}} = \mathbf{Q} + \mathbf{P}\omega, \qquad \begin{aligned} \mathbf{P} &= \Delta^2 + \delta T d, \\ T &= 2\mathsf{J} - 4\mathsf{P}\cdot, \end{aligned}$$

where $(\mathsf{P} \cdot \varphi)_a = \mathsf{P}^b{}_a \varphi_b$. This shows $\mathbf{Q}$ is a Q-curvature. In fact,

$$\mathbf{I}^{\mathsf{Q}} = \operatorname{span}\{\mathbf{Q}, |\mathsf{C}|^2\} \qquad \text{(see below).}$$

Now $\mathbf{I}^{\mathsf{Q}} + \partial \mathbf{I}_4$ and $\mathbf{I}^{\mathsf{ix}}$ are each $\operatorname{span}\{\mathbf{Q}, \Delta\mathsf{J}, |\mathsf{C}|^2\}$. So $\mathbf{I}_n^{\mathsf{FSA}}$, which is caught in between, is also.

If $\mathbf{I}_4^1$ is more than 2-dim., then some lin. comb. of J^2, $|\mathsf{P}|^2$ is in $\mathbf{I}_4^1$. But the sec order terms are

$$\mathsf{J}^2: \qquad -2\mathsf{J}\omega_{|a}\omega_{|}{}^{a} + \omega_{|a}{}^{a}\omega_{|b}{}^{b}$$

$$|\mathsf{P}|^2: \qquad 2\mathsf{P}^{ab}\omega_{|a}\omega_{|b} - \mathsf{J}\omega_{|a}\omega_{|}{}^{a} + \omega_{|}{}^{ab}\omega_{|ab}.$$

Conclusion: $\mathbf{I}_4^1 = \mathrm{span}\{\mathbf{Q}, |\mathsf{C}|^2\}$.

Example: Going up to dimension 6, we finally get $\mathbf{I}^{\mathsf{ix}}$ to be strictly larger than $\mathbf{I}_6^{\mathsf{FSA}}$. The dimensions of the spaces in the Proposition above (within the 17-dimensional $\mathbf{I}_6$) are

$$\overset{[6]}{\partial\mathbf{I}_6} \subset \overset{[7]}{\mathbf{I}_6^{\mathsf{div}}} \subset \overset{[11]}{\mathbf{I}^{\mathsf{ix}}},$$

$$\overset{[6]}{\mathbf{I}^{\mathsf{Q}}} + \overset{[6]}{\partial\mathbf{I}_6} \subset \overset{[10]}{\mathbf{I}_6^{\mathsf{FSA}}} \subset \overset{[11]}{\mathbf{I}^{\mathsf{ix}}}.$$

In addition, $\dim \mathbf{I}_6^0 = 3$ (the local conformal invariants).

A good choice of $\mathbf{Q}$
Gover-Peterson, CMP 2003 is

$$\Delta^2 \mathsf{J} + 8|\nabla \mathsf{P}|^2 + 16\mathsf{P}_{ab}\mathsf{P}^{ab}{}_{|c}{}^{c} - 8\mathsf{J}\mathsf{J}_{|c}{}^{c}$$

$$-32\mathsf{P}_{ab}\mathsf{P}^{a}{}_{c}\mathsf{P}^{bc} - 16\mathsf{J}|\mathsf{P}|^2 + 8\mathsf{J}^3 + 16\mathsf{P}_{ab}\mathsf{P}_{cd}\mathsf{C}^{acbd}.$$

Example: A smaller situation where there's a difference between $\mathbf{I}_n^{\mathsf{FSA}}$ and $\mathbf{I}^{\mathsf{ix}}$ is the conformally flat case in dim. 6. Just taking flat conformal classes reduces the number of invariants. This is more involved than just crossing out any invt. mentioning C (though in dims. > 3, vanishing of C is equivalent to conformal flatness). The fact that

$$\mathsf{C}_{abcd|}{}^{a} = 2(n-3)\mathsf{P}_{b[d|c]}$$

shows there are "hidden" identities. In fact, the reduced invariant space $\mathbf{I}_{6,\flat}$ has dim. 8.

The dimensions of the important subspaces are:

$$\overset{[3]}{\partial\mathbf{I}_{6,\flat}} \subset \overset{[4]}{\mathbf{I}^{\mathsf{div}}_{6,\flat}} \subset \overset{[5]}{\mathbf{I}^{\mathsf{ix}}_{\flat}},$$

$$\overset{[1]}{\mathbf{I}^{\mathsf{Q}}_{\flat}} + \overset{[3]}{\partial\mathbf{I}_{6,\flat}} \subset \overset{[4]}{\mathbf{I}^{\mathsf{FSA}}_{6,\flat}} \subset \overset{[5]}{\mathbf{I}^{\mathsf{ix}}_{\flat}} .$$

Another model in which explicit calculations are accessible is the 4-dim. case **with boundary** – see Branson-Gilkey 1994 and its use in Chang-Qing 1997.

The idea is that the natural scalar density $\mathbf{U}_n$ that arises in the variation of the functional determinant must be in $\mathbf{I}_n^{\mathsf{FSA}}$, since it is the variation of something, and so the corresp. $2^{\underline{\text{nd}}}$ variation must be symmetric.

If we can express any such thing as something in the Q-space plus something with a local conformal primitive, we've attained the form

$$c\int \omega(\widehat{\mathbf{Q}}+\mathbf{Q})+\int(\widehat{\mathbf{F}}-\mathbf{F})$$

for the log(determinant quotient) of the Yamabe op., or of anything else with good conformal and ellipticity props.

Another approach uses diml. continuation. S. Alexakis will tell us on Friday about his proof of:

Conjecture/Theorem

[Alexakis, Princeton PhD dissertation 2005]: A natural scalar density $\mathbf{A}\in\mathbf{I}_n$ with $\int\mathbf{A}$ conformally invariant may be universally expressed as

$$a\,\mathsf{Pff}+\delta\eta+\mathbf{L}$$

where a is a constant, $\delta\eta$ is the divergence of a natural 1-form $(2-n)$-density η, and $\mathbf{L}$ is a local conformal invariant, $\widehat{\mathbf{L}}=\mathbf{L}$.

Corollary: The same is true with $\mathbf{Q}$ replacing Pff.

In fact, if we write the decomposition of $\mathbf{Q}$,

$$\mathbf{Q} = a\,\mathrm{Pff} + \delta\eta + \mathbf{L},$$

then in the conformally flat case, $\mathbf{L} = 0$. On the sphere, $\mathbf{Q}$ and Pff, are positive constants we can give explicitly, so (integrating over the sphere) a is also.

This gets us (something like) our other conjecture, and enough to guarantee the form

$$c\int \omega(\widehat{\mathbf{Q}} + \mathbf{Q}) + \int (\widehat{\mathbf{F}} - \mathbf{F})$$

we want for the log(det. quotient).

The key is that $\mathbf{U}_n$ satisfies

$$\left(\int \mathbf{U}_n\right)^{\bullet} = (N-n)\int \omega \mathbf{U}_n$$

for each N. $\mathbf{Q} = \mathbf{Q}_n$ does a similar thing. Since P_n satisfies the conformal covariance relation

$$P_n^{\bullet} = -n\omega P_n + \frac{N-n}{2}[P_n, \omega]$$

in dim. $N \geq n$,

$$\frac{N-n}{2}Q_n^{\bullet} = P_n^{\bullet}1 = -n\omega\frac{N-n}{2}Q_n + \frac{N-n}{2}P_n\omega - \left(\frac{N-n}{2}\right)^2 \omega Q_n,$$

so

$$\begin{aligned} Q_n^{\bullet} &= -n\omega Q_n + P_n^0\omega, \\ (Q_n dv)^{\bullet} &= (N-n)\omega Q_n + P_n^0\omega. \end{aligned}$$

Integrating, the exact divergence $P_n^0\omega$ goes away, so

$$\left(\int Q_n dv\right)^{\bullet} = (N-n)\int \omega Q_n\,,$$

as desired.

So, both $\mathbf{U}_n$ and $\mathbf{Q}$ extend rationally (with controlled poles) to higher dimensions. We also make a rational (or even polynomial) extension of the local conformal invt. $\mathbf{L}$, and add it in as part of an alternative $\mathbf{Q}$, say $\underline{\mathbf{Q}}$. Now

$$\int \mathbf{U}_n = c\int \mathbf{Q} \qquad \text{in dim. } n.$$

Recall that this was what we had to assume earlier to get going on the form

$$c\int \omega(\widehat{\mathbf{Q}}+\mathbf{Q})+\int(\widehat{\mathbf{F}}-\mathbf{F}).$$