# Detour torsion

This is joint work with Rod Gover.

A $\boxed{\text{detour torsion}}$ is a spectral invariant of natural elliptic complexes. It generalizes the only previously known special case, $\boxed{\text{Cheeger's half-torsion}}$ for the de Rham complex.

All manifolds are compact, Riemannian (or conformal), and even-dim. The goal is to get a quantity that has a Polyakov-type formula as we move within a conformal class, much like the formulas for $Y$ and other individual conformal operators.

These detour torsions are products and quotients of functional determinants which individually behave badly under conformal change, but which behave well in the well-chosen aggregate.

Let $d_k$, $\delta_k$, and

$$\Delta_k = \delta_{k+1} d_k + d_{k-1}\delta_k$$

be the usual Hodge-de Rham operators. The Hodge decomposition is

$$\mathcal{E}^k = \underbrace{\mathcal{R}(\delta) \oplus \mathcal{R}(d)}_{\mathcal{R}(\Delta)} \oplus \underbrace{(\mathcal{N}(d) \cap \mathcal{N}(\delta))}_{=:\mathcal{H}^k},$$

where $\mathcal{N}$ is the null space and $\mathcal{R}$ the range.

The $\boxed{\text{zeta functions}}$ of the complex are

$$\zeta(s, \Delta_k) := \mathrm{Tr}_{L^2}(\Delta_k|_{\mathcal{R}(\Delta_k)})^{-s}.$$

These converge absolutely, and uniformly on compacts, in $\mathbf{Re}(s) > n/2$, and analytically continue via the heat expansion, much like the Yamabe op. for which we worked this out in detail earlier. (That is, only the analytic properties, and not the conformal props., are important so far.) As before, we have a $\boxed{\text{functional determinant}}$

$$\det \Delta_k = \exp\left(-\zeta'(0, \Delta_k)\right).$$

In terms of eigenvalues,

$$\zeta(s, \Delta_k) = \sum_{\lambda_j \neq 0} \lambda_j^{-s}$$

for sufficiently large $\mathbf{Re}(s)$.

The nonzero form eigenvalues split into a list of $\delta d$ eigenvalues, say $\mu_a$, and a list of $d\delta$ eigenvalues, say $\nu_b$. A key point in all discussions of index and torsion quantities is that much information is repeated in considering these lists for various $k$. Specifically, the nonzero $\delta d$ eigenvalue list for $k$-forms is repeated as the nonzero $d\delta$ eigenvalue list for $(k+1)$-forms, since $d$ and $\delta$ commute with $\Delta$. This offers some scope for achieving interaction among the spectral invariants of the various $\Delta_k$.

To enrich our supply of zeta fcns., note that since $d_0$ and $\delta_1$ are formal adjoints, the Hodge decomposition shows that

$$d_0 : \mathcal{R}(\delta_1) \leftrightarrow \mathcal{R}(d_0) : \delta_1 \quad \text{bijectively.}$$

Thus

$$\zeta(s, d_0\delta_1) := \mathrm{Tr}(d_0\delta_1|_{\mathcal{R}(d_0)})^{-s}$$

$$= \mathrm{Tr}(\delta_1 d_0|_{\mathcal{R}(\delta_1)})^{-s} = \zeta(s, \Delta_0).$$

With this in place, we may take

$$\begin{aligned}\zeta(s, \delta_2 d_1) &= \mathsf{Tr}\left(\delta_2 d_1|_{\mathcal{R}(\delta_2)}\right)^{-s} \\ &= \zeta(s, \Delta_1) - \zeta(s, \Delta_0).\end{aligned}$$

Continuing in this way, we may define

$$\zeta(s, \delta_{k+1} d_k) \quad \text{and} \quad \zeta(s, d_{k-1}\delta_k),$$

regular at $s = 0$, with

$$\zeta(s, \delta_{k+1} d_k) = \zeta(s, d_k \delta_{k+1}).$$

**Caution 1**: Differential operators without appropriate ellipticity or sub-ellipticity properties will generally not have zeta functions regular at $s = 0$. Here, only the status of $\delta d$ and $d\delta$ as partial Laplacians of an elliptic complex allows us to define good zeta functions for them.

As before, we can get the (very useful) $\boxed{\text{local zeta fcn.}}$ by inserting a multiplication operator just before tracing:

$$\zeta(s, \Delta_k, \omega) := \mathrm{Tr}\left(\omega(\Delta_k|_{\mathcal{R}(\Delta_k)})^{-s}\right).$$

**Caution 2**: Partial $\zeta$ fcns. and local $\zeta$ fcns. are both OK, but **local partial zetas** are not. Let us abbrevate restriction to the correct range by an underline, as in

$$(\underline{\delta d})^{-s} = (\delta d|_{\mathcal{R}(\delta)})^{-s}.$$

Because this op. is of trace class for large $\mathbf{Re}(s)$, the op. $\omega(\underline{\delta d})^{-s}$ will be too. But there is no reason to expect regularity of this function at $s = 0$. (In fact, we'll be able to see that it generally isn't regular.)

As an operator from $\mathcal{E}^k$ to $\mathcal{E}^{k+1}$, the exterior derivative $d_k$ is of course independent of the metric. The coderivative

$$\delta_k : \mathcal{E}^k[2k-n] \to \mathcal{E}^{k-1}[2k-2-n]$$

is conformally invariant. Thus viewed as an operator $\mathcal{E}^k \to \mathcal{E}^{k-1}$,

$$\widehat{\delta}_k \varphi = e^{(2k-2-n)\omega} \delta_k (e^{-(2k-n)\omega} \varphi)$$

for any $\varphi \in \mathcal{E}^k$. Choose a scale $g_0$ within our conformal class and take the conformal curve of metrics

$$g_\varepsilon := e^{2\varepsilon\omega} g_0.$$

Then

$$\delta_k^\bullet \varphi = -(n-2k+2)\omega\delta_k\varphi + (n-2k)\delta_k(\omega\varphi).$$

We'll compute the conformal variation of each term in

$$\mathrm{Tr}\underline{\Delta}_k^{-s} = \mathrm{Tr}(\underline{\delta d}_k^{-s}) + \mathrm{Tr}(\underline{d\delta}_k^{-s}).$$

For the 1$^{\underline{\text{st}}}$ term,

$$\mathrm{Tr}((\underline{\delta d})_k^{-s})^\bullet = -s\mathrm{Tr}\,\underbrace{(\delta_{k+1}d_k)^\bullet}_{\delta_{k+1}^\bullet d_k}(\underline{\delta d})_k^{-s-1}$$

$$= -s\mathrm{Tr}\Big(\{-(n-2k)\omega\delta_{k+1}$$

$$+(n-2k-2)\delta_{k+1}\omega\}d_k(\underline{\delta d})_k^{-s-1}\Big)$$

$$= (n-2k)s\mathrm{Tr}\left(\omega(\underline{\delta d})_k^{-s}\right)$$

$$-(n-2k-2)s\mathrm{Tr}\left(\omega(\underline{d\delta})_{k+1}^{-s}\right).$$

In rewriting the last term, we took advantage of the fact that $\delta_{k+1} : \mathcal{R}(d_k) \to \mathcal{R}(\delta_{k+1})$ is bijective.

This last step is a key point: the variation of $\delta d$ on $k$-forms leads to terms in $\delta d$ on $\boxed{k\text{-forms}}$, **and** in $d\delta$ on $\boxed{(k+1)\text{-forms}}$. This is how the different form orders interact.

Restating in terms of zetas,

$$\boxed{\begin{aligned}&\zeta(s,(\delta d)_k)^\bullet = (n-2k)s\zeta(s,(\delta d)_k,\omega)\\&-(n-2k-2)s\zeta(s,(d\delta)_{k+1},\omega).\end{aligned}}$$

Similarly,

$$\boxed{\begin{aligned}&\zeta(s,(d\delta)_k)^\bullet = (n-2k+2)s\zeta(s,(\delta d)_{k-1},\omega)\\&-(n-2k)s\zeta(s,(d\delta)_k,\omega).\end{aligned}}$$

**Caution**: At first glance, seems as though the right sides of the boxed equations vanish at $s=0$. But remember the perils of local partial zetas. In fact, these expressions make elementary sense only for large $\mathrm{Re}(s)$. We only try to analytically continue certain well-chosen combinations.

Speaking of these, let

$$\kappa(s) := c_0\zeta(s,\Delta_0)+c_1\zeta(s,\Delta_1)+\cdots+c_n\zeta(s,\Delta_n).$$

We'd like make a good choice of the $c_k$. Modelling our expectations on what we had for a single good operator, like Yamabe, we'd like $\kappa(0)$ conformally invariant and $\kappa'(0)$ with computable differences for 2 metrics in a conformal class.

The coef. in $\kappa(s)^\bullet$ of $s\zeta(s,(\delta d)_k,\omega)$ is

$$(n-2k)(c_k+c_{k+1}),$$

while the coef. of $s\zeta(s,(d\delta)_k,\omega)$ is

$$-(n-2k)(c_k+c_{k-1}).$$

One distinguished choice for the coefficient list will thus be $1,-1,1,-1,\cdots$. This just detects the conformal invariance of the index of the de Rham complex (which has much more than just conformal invariance of course).

We could ask just that the variation only produce full Laplacians – that is, that the coefficients of

$$s\zeta(s,(\delta d)_k,\omega) \quad \text{and} \quad s\zeta(s,(d\delta)_k,\omega)$$

agree. This leads to

$$c_{k+1} = -c_{k-1} - 2c_k, \qquad k \geq 1,$$

which doesn't produce a unique coupling. In fact we'd like more – that the

$$(s\zeta(s,(\delta d)_k,\omega), s\zeta(s,(d\delta)_k,\omega))$$

coefficient pair in the variation be proportional to the

$$(\zeta(s,(\delta d)_k), \zeta(s,(d\delta)_k))$$

coefficient pair in the original quantity:

$$(n-2k)(c_k + c_{k+1}, -c_k - c_{k-1}) = \lambda(c_k, c_k)$$

for some $\lambda$.

Shifting $k$ in the equality of second components, we get the system

$$\begin{aligned}(n-2k-\lambda)c_k + (n-2k)c_{k+1} &= 0,\\ (n-2k-2)c_k + (n-2k-2+\lambda)c_{k+1} &= 0,\end{aligned}$$

the determinant of which is $\lambda(2-\lambda)$. The choice $\lambda = 0$ gives us the coefficient list $1, -1, 1, -1, \cdots$ associated with the index calculation. The choice $\lambda = 2$ gives the recursion

$$(n-2(k-1))c_k = -(n-2k)c_{k-1}.$$

A key point is that

> for this choice, $c_k$ may be taken to vanish for $k \geq n/2$.

If we set $c_{n/2} = \cdots = c_n = 0$, only the first half of the complex will be noticed by the calculation. This is the origin of the term **half-torsion**.

One normalization of the coefficient list is then

$$n, -(n-2), n-4, \cdots, \mp 4, \pm 2, 0;$$

that is,

$$c_k = (-1)^k \left((n-2k)_+\right).$$

Make this choice, and define the local kappa function by

$$\kappa(s,\omega) := c_0\zeta(s,\Delta_0,\omega) + c_1\zeta(s,\Delta_1,\omega)$$

$$+ \cdots + c_n\zeta(s,\Delta_n,\omega).$$

Then

$$\kappa(s)^\bullet = 2s\kappa(s,\omega).$$

Only full Laplacians appear above, so $\kappa(s,\omega)$ is regular at $s = 0$, and

$$\kappa(0) \quad \text{is a conformal invariant.}$$

The **Cheeger half-torsion** is (exp of minus) $\kappa'(0)$. We get Polyakov-type formulas for the local part of the infinitesimal conformal variation of this. We have to face up to the global part too, as it comes from null spaces; that is, cohomology.

Since the functional determinants of the Laplacians are defined as their $e^{-\zeta'(0)}$ quantities, the half-torsion is

$$\kappa'(0) = -\log \frac{(\det \Delta_0)^n (\det \Delta_2)^{n-4} \cdots}{(\det \Delta_1)^{n-2} (\det \Delta_3)^{n-6} \cdots},$$

where the terms abbeviated by $\cdots$ involve only the $\Delta_k$ for $k < n/2$.

To see where the global term comes from, recall that the analytic continuation is accomplished via the heat trace asymptotics on the other side of the **Mellin transform**

$$(\mathcal{M}f)(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} f(t) dt.$$

which takes

$$\exp(-t\lambda) \mapsto \lambda^{-s}.$$

for positive real $\lambda$. Thus it carries

$$\mathrm{Tr}(\omega \exp(-t\underline{\Delta}_k)) \mapsto \zeta(s, \Delta_k, \omega).$$

The $L^2$ trace on the left is closely related to the localized heat operator trace

$$Z(t, \Delta_k, \omega) := \mathrm{Tr}(\omega \exp(-t\Delta_k))$$
$$\sim \sum_{\text{even } i \geq 0} t^{(i-n)/2} \int \omega U_i \qquad \text{as } t \downarrow 0,$$

in which the $\Delta$ is not underlined.

The kernel functions involved are

$$\underline{\underline{\Delta}}: \qquad \omega(x) \sum_{\lambda_j \neq 0} e^{-\lambda_j t} \varphi_j(x) \otimes \varphi_j^*(y),$$

$$\Delta: \qquad \omega(x) \sum_{\lambda_j} e^{-\lambda_j t} \varphi_j(x) \otimes \varphi_j^*(y)$$

for $\{(\lambda_j, \varphi_j)\}$ an orthonormal spectral resolution. So the difference in $L^2$-traces is

$$\sum_{\lambda_j = 0} \int \omega |\varphi_j|^2 = \mathrm{Tr}\, \omega \mathcal{P}_k,$$

where $\mathcal{P}_k$ is the Hodge projection onto the harmonics. (The kernel fcn. of $\mathcal{P}_k$ is the difference of the kernels above.)

Thus

$$\mathrm{Tr}\, \omega \exp(-t \underline{\underline{\Delta}}_k) \sim \sum_{\text{even } i \geq 0} t^{(i-n)/2} A_i(\Delta_k, \omega)$$

as $t \downarrow 0$,

where

$$A_i(\Delta_k, \omega) := \begin{cases} \int \omega U_n - \mathsf{Tr}\, \omega \mathcal{P}_k & \text{if } i = n, \\ \int \omega U_i & \text{otherwise.} \end{cases}$$

The calculation using the heat expansion now goes:

$$\begin{aligned} \Gamma(s)\zeta(s, \Delta_k, \omega) = \sum_{i=0}^{m} \left( s - \frac{n-i}{2} \right)^{-1} A_i(\Delta_k, \omega) \\ + \int_0^1 t^{s-1} O(t^{(m-n+1)/2}) dt \\ + \int_1^\infty t^{s-1} (\mathsf{Tr} \omega \exp(-t\underline{\Delta}_k)) dt, \end{aligned}$$

so at $s = 0$,

$$\zeta(0, \Delta_k, \omega) = A_n(\Delta_k, \omega),$$

since $\Gamma(s)$ has a simple pole at $s = 0$ with residue 1.

Let

$$\tau(g) := \kappa'(0).$$

We now have

$$
\begin{aligned}
\tau(g)^\bullet &= 2\kappa(0,\omega) \\
&= 2\sum_k A_n(\Delta_k,\omega) \\
&= \underbrace{\int \omega \sum_k c_k U_n[\Delta_k]}_{=:2\tau_{\mathrm{loc}}(g,\omega)} \underbrace{-2\sum_k c_k \operatorname{Tr} \omega \mathcal{P}_k}_{=:2\tau_{\mathrm{glob}}(g,\omega)} \\
&=: 2\tau(g,\omega).
\end{aligned}
$$

In naming the $\tau$ quantities, we make explicit the dependence on the metric $g$; this will be useful in thinking of them as functionals on the conformal class. Let

$$\mathbf{U} := \sum_k c_k \mathbf{U}_n[\Delta_k],$$

so

$$\tau_{\mathrm{loc}}(g,\omega) = \int \omega \mathbf{U}.$$

Recall that $\mathrm{Tr}\,\omega\mathcal{P}_k$ is expressible in terms of any $L^2$-orthonormal bases

$$\{\psi_m^k\}_{m=1}^{b_k}$$

of the harmonic spaces $\mathcal{H}^k$, so

$$\tau_{\mathrm{glob}}(g,\omega) = -\sum_k c_k \sum_{m=1}^{b_k} \int \omega |\psi_m^k|^2 dv_g.$$

We now want to find a $\boxed{\text{conformal primitive}}$ for $2\tau(g,\omega)$. Recall that this is a 2-metric fcnl. $\mathcal{H}(\widehat{g},g)$ which is alternating and cocyclic,

$$\mathcal{H}(g,\widehat{g}) = -\mathcal{H}(\widehat{g},g),$$

$$\mathcal{H}(\widehat{\widehat{g}},g) = \mathcal{H}(\widehat{\widehat{g}},\widehat{g}) + \mathcal{H}(\widehat{g},g),$$

and whose variation in the $\widehat{g}$ argument is $2\tau(\widehat{g},\omega)$. When we get this, we'll have the desired formula for

$$\tau(\widehat{g}) - \tau(g).$$

We do this by finding conformal prims. for both the local and global parts. For the local part, it's covered by the previous lectures:

$$\mathbf{c}\int \omega(\widehat{\mathbf{Q}} + \mathbf{Q}) + \int (\hat{\mathsf{F}} - \mathsf{F}),$$

where $\mathbf{Q}$ is $\underline{\underline{\text{a}}}$ Q-curv., and $\mathsf{F}$ is a natural $(-n)$-density.

For the global part, we can actually find conformal primitives for the $\operatorname{Tr}\omega\mathcal{P}_k$ individually. It goes like this. (The idea is applicable to individual ops. like Yamabe too.)

Let $\mathbf{h} = \mathbf{h}^k$ be some basis of the real cohomology $H^k$, and let $\Psi$ be an ONB of the harmonics $\mathcal{H}_g^k$. The $\boxed{\text{de Rham map}}$

$$\mathcal{D}_g : \mathcal{H}_g^k \to H^k$$

is a canonical isomorphism (whose inverse is a Hodge projection).

Let

$$[\Psi/\mathbf{h}]$$

be the determinant of the basis change, $\mathcal{D}\Psi$ to $\mathbf{h}$, and

$$[g:\mathbf{h}] = |[\Psi/\mathbf{h}]|.$$

The $2^{\underline{\text{nd}}}$ notation for this reflects the fact that changes between different ONB $\Psi$ are orthogonal matrices, so have det. $\pm 1$.

A calculation shows

$$-\left(\log[\Psi/\mathbf{h}]^2\right)^{\bullet} = (n-2k)\operatorname{Tr}\omega\mathcal{P}_k.$$

But recall that

$$\tau_{\text{glob}}(\omega) = -\sum_{k=0}^{n/2-1}(-1)^k(n-2k)\operatorname{Tr}\omega\mathcal{P}_k.$$

So

$$\left(\sum_{k=0}^{n/2-1}(-1)^k\log\left[g:\mathbf{h}^k\right]^2\right)^{\bullet} = \tau_{\text{glob}}(\omega).$$

To get a 2-metric cocycle with the same conformal variation, subtract the $g$ version from the $\widehat{g}$ version.

$$\mathcal{H}_{\mathsf{glob}}(\widehat{g}, g) = \sum_{k=0}^{n/2-1} (-1)^k \log \frac{[\widehat{g} : \mathbf{h}^k]^2}{[g : \mathbf{h}^k]^2}.$$

Since this is uniquely determined, the apparent dependence on the coho. basis must wash out. It does, as the quotient above is the squared det. of the basis change between $\widehat{g}$ and $g$ ONB of the harmonic space; to coin a notation,

$$[\widehat{g} : g]_k^2.$$

Summing up,

$$\boxed{\begin{aligned}&\tau(\widehat{g})-\tau(g)=\mathcal{H}_{\mathsf{loc}}(\widehat{g},g)+\mathcal{H}_{\mathsf{glob}}(\widehat{g},g)\\ &=c\int\omega(\widehat{\mathbf{Q}}+\mathbf{Q})+\int(\widehat{\mathbf{F}}-\mathbf{F})\\ &+\sum_{k=0}^{n/2-1}(-1)^k\log[\widehat{g}:g]_k^2\end{aligned}}$$

This was the Cheeger half-torsion. In [Branson-Gover math.DG.0309085] it's shown that there are universal $\boxed{\text{detour complexes}}$

$$\mathcal{E}^0\xrightarrow{d}\mathcal{E}^1\xrightarrow{d}\cdots\xrightarrow{d}\mathcal{E}^{k-1}\xrightarrow{d}\mathcal{E}^k\xrightarrow{\quad L_k\quad}\mathcal{E}_k$$

$$\xrightarrow{\delta}\mathcal{E}_{k-1}\xrightarrow{\delta}\cdots\xrightarrow{\delta}\mathcal{E}_1\xrightarrow{\delta}\mathcal{E}_0,$$

where $\mathcal{E}_p=\mathcal{E}^p[2p-n]$, in which

$$\boxed{L_k=\delta\left\{(d\delta)^{n/2-k-1}+\mathsf{LOT}\right\}d.}$$

At the end are $\delta$ operators from the $\boxed{\text{de Rham co-complex}}$.

No orientation is needed to do this, and no metric – only a conformal structure.

If we do have an orientation, we can make all the bundles $\boxed{\text{true forms}}$:

$$\mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^{k-1} \xrightarrow{d} \mathcal{E}^k \xrightarrow{\quad \star L_k \quad} \mathcal{E}^{n-k}$$

$$\xrightarrow{d} \mathcal{E}^{n-k+1} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^{n-1} \xrightarrow{d} \mathcal{E}^n.$$

Fix $k$, and denote the coho. at $\mathcal{E}^p$ by $H^p_L$. Note that $H^k_L$ might have a different dimension than $H^k$. Having the same dimension is a $\boxed{\text{regularity}}$ property (of the (manifold, conformal structure) pair) in the sense of [Eastwood-Singer, JDG 1993].

The coboundaries have different orders, $n-2k$ for $L$, and 1 for everything else. We compensate in the standard way by pumping up the orders of the partial Laplacians – $(\delta d)^{n-2k}$ for example. All zetas respond via

$$\zeta^{\text{new}}(s) = \zeta^{\text{old}}((n-2k)s).$$

The regularity (and in fact the value) at $s=0$ is unchanged by this device, while the $\zeta'(0)$ quantity gets multiplied by $n-2k$.

We could actually use $(n-2k)/2$ powers. This would make the Laplacian at $\mathcal{E}^k$

$$(d\delta)_k^{(n-2k)/2} + L$$

instead of

$$(d\delta)_k^{n-2k} + L^2.$$

For the kappa quantity, it turns out we need the same relative coefficient list, truncated earlier of course:

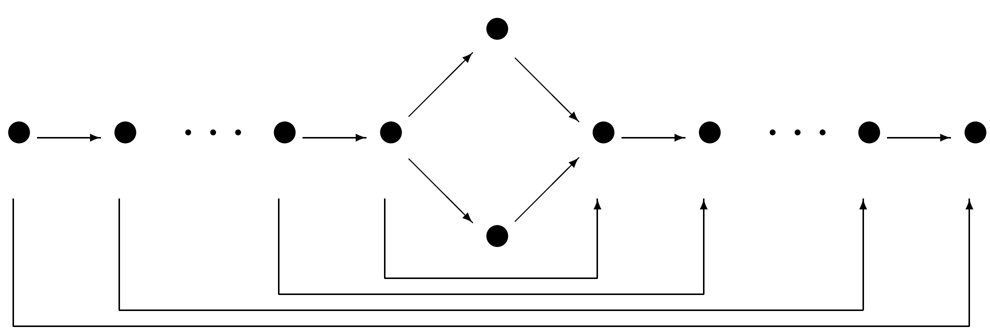
$$\kappa_k(s,\omega) := (-1)^k(n-2k)\zeta(s,(d\delta)_k^{n-2k}+L^2,\omega)$$
$$+\sum_{p=0}^{k-1}(-1)^p(n-2p)\zeta(s,\Delta_p^{n-2k},\omega)$$

What's important is the conformal covariance, and the precise conformal weights. We arrive at:

$$\boxed{\begin{aligned}&\tau_k(\widehat{g})-\tau_k(g)=\mathcal{H}^k_{\mathsf{loc}}(\widehat{g},g)+\mathcal{H}^k_{\mathsf{glob}}(\widehat{g},g)\\&=\mathbf{c}_k\int\omega(\widehat{\mathbf{Q}}+\mathbf{Q})+\int(\widehat{\mathbf{F}}_k-\mathbf{F}_k)\\&+\sum_{p=0}^{k}(-1)^p\log[\widehat{g}:g]^2_{p,k}.\end{aligned}}$$

For $k = 0$, the detour torsion is just the det. of the critical GJMS operator. Otherwise it may be viewed as a kind of det. of the non-elliptic $L_k$ (which is the Maxwell $\delta d$ when $k = n/2 - 1$).

There are many natural elliptic complexes in the conformally flat (CF) case – the gBGG diagrams. For conformal geometry, these have the shape



There are $n + 2$ dots in all in such a $\boxed{\text{regular diagram}}$. In the de Rham complex the zenith and nadir of the diamond hold the self- and anti-self-dual middle forms.

The longest arrow in the de Rham complex is the critical GJMS operator. All compositions can be made (density weights match up), but all are 0 in the CF case except one (linear combination of the two) around the diamond – this is also the shortest long arrow.

Sometimes the behavior of a complex can be concluded by an assumption weaker than conformal flatness.

In the $\boxed{\text{flat}}$ ($S^n$) case, the arrows are differential intertwining operators for representations of the conformal group $\mathrm{SO}_0(n+1,1)$, or its cover $\mathrm{Spin}_0(n+1,1)$. The representations involved are induced from representations of the maximal parabolic subgroup $MAN$ for which the nilpotent part $N$ acts trivially.

The representations are parameterized by an $M$ weight and an $A$ weight. Since $\mathfrak{m} = \mathfrak{so}(n)$, the $M$ parameter takes the form $[\lambda_1, \cdots, \lambda_{n/2}]$, where all $\lambda_a$ are integral, or all are properly half-integral, and

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n/2-1} \geq |\lambda_{n/2}|.$$

Such a the tuple $\lambda$ is $\boxed{\text{dominant}}$; it gives the coefficients when the highest weight of the $\mathfrak{m}$ module is expanded in the positive weights of the defining representation of $\mathfrak{so}(n)$. The $\mathfrak{a}$-weight can be any complex number; but according to the classification of invariant differential operators [Boe-Collingwood, J. Algebra and Math. Z. 1985], only values in $\frac{1}{2}\mathbf{Z}$ can occur in the source or target for invariant differential operators. Furthermore, the $(\mathfrak{a}, \mathfrak{m})$ weight $[\lambda_0 | \lambda_1, \cdots, \lambda_{n/2}]$ cannot occur for a differential operator unless $\lambda_0 - \lambda_1 \in \mathbf{Z}$.

The $(\mathfrak{a}, \mathfrak{m})$ weights are partitioned into orbits under the $\boxed{\text{affine Weyl group}}$. The $\boxed{\text{rho-shift}}$ of $[\lambda_0|\lambda_1, \cdots, \lambda_{n/2}]$ is

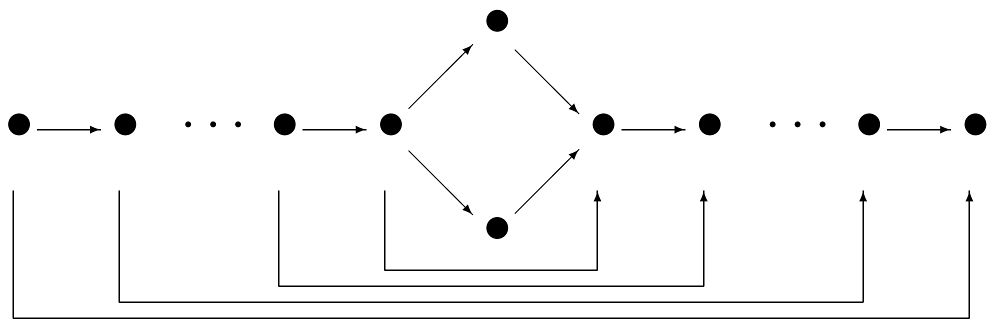
$$\left(\lambda_0 + \frac{n}{2} \,|\, \lambda_1 + \frac{n-2}{2}, \lambda_2 + \frac{n-4}{2}, \cdots, \lambda_{n/2} + 1, \lambda_{n/2}\right).$$

We shall use round vs. square parentheses to distinguish rho-shifted vs. not.

$$(\mu_0|\mu) \sim (\nu_0|\nu)$$

iff the $(n/2+1)$-tuples involved differ by a permutation and an even number of sign changes. An affine Weyl orbit is $\boxed{\text{regular}}$ iff the absolute values of the $n/2+1$ entries are distinct.

All regular affine Weyl orbits have $n+2$ elements which may be arranged as in the diagram,



in a unique way so that the weights $(\mu_0|\mu)$, are lexicographically decreasing as we move to the right or down, and all tuples to the right of the bar are $\boxed{\text{strictly dominant}}$ (dominant with $>$ signs).

By a theorem of Harish-Chandra, all intertwining operators (for principal series representations of $\mathrm{Spin}_0(n+1,1)$) must pass between bundles in the same affine Weyl orbit.

The Boe-Collingwood classification says that all **differential** intertwinors in a regular orbit pass between the bundles in the positions indicated in the picture above, and furthermore, there is a unique (up to constant multiples) nonzero differential intertwinor on $S^n$ corresponding to each arrow.

In addition, still in the flat case, any composition of two arrows (with the exception of one linear combination of the arrows around the diamond, corresponding to the shortest $\boxed{\text{long operator}}$) vanishes, and the leading symbol complex at any such composition (including a the composition of a long and short operator) is exact.

For the complex of short arrows or for any detour complex, in any conformal class where they are complexes (in particular flat conformal classes), one gets a detour torsion with the coefficient list

$$c_p = \begin{cases} (-1)^p 2\mu_0^{(p)}, & p < k, \\ 0, & p \geq k. \end{cases}$$

Again the detour torsion quotients (differences) are of the form

$$\boxed{\begin{aligned} &\tau_k(\hat{g}) - \tau_k(g) = \mathcal{H}^k_{\mathrm{loc}}(\hat{g}, g) + \mathcal{H}^k_{\mathrm{glob}}(\hat{g}, g) \\ &= \mathbf{c}_k \int \omega(\widehat{\mathbf{Q}} + \mathbf{Q}) + \int (\widehat{\mathbf{F}}_k - \mathbf{F}_k) \\ &+ \sum_{p=0}^{k} (-1)^p \log[\hat{g} : g]^2_{p,k}. \end{aligned}}$$

where now the **U** giving the variation of the local term is built from the kappa with these coefs. The correct coefs. (conformal weights) emerge from the global calc. so the coefs. above on the global term again become $\pm 1$.

Some things that have not been noted in this quick summary are:

- The problem with orders of operators appears again, even if we don't take a long detour – the coboundaries may have all kinds of different orders. To get elliptic Laplacians, we level the orders by taking a common multiple.

- Some of these gBGG are not amenable to this because the duals of the bundles in the beginning of the complex are not the bundles at the end – the duals live in a different gBGG. The "bad" gBGG in this sense are those in dim. $4k$ with no 0 entry in the weight. This other gBGG has the same $|$weights$|$, but with the other parity of negative ones – so the bad ones come in pairs. We get these into the detour torsion game anyway by $\oplus$ing the pair of gBGGs involved.