The energy of graphs and matrices

Given a complex $m \times n$ matrix A, we index its singular values as $\sigma_1(A) \geq \sigma_2(A) \geq \ldots$ and call the value $\mathcal{E}(A) = \sigma_1(A) + \sigma_2(A) + \ldots$ the *energy* of A, thereby extending the concept of graph energy, introduced by Gutman.

Let $2 \leq m \leq n$, A be an $m \times n$ nonnegative matrix with maximum entry α, and $\|A\|_1 \geq n\alpha$. Extending previous results of Koolen and Moulton for graphs, we prove that

$$\mathcal{E}(A) \leq \frac{\|A\|_1}{\sqrt{mn}} + \sqrt{(m-1)\left(\|A\|_2^2 - \frac{\|A\|_1^2}{mn}\right)} \leq \alpha \frac{\sqrt{n}\,(m+\sqrt{m})}{2}.$$

Furthermore, if A is any nonconstant matrix, then

$$\mathcal{E}(A) \geq \sigma_1(A) + \frac{\|A\|_2^2 - \sigma_1^2(A)}{\sigma_2(A)}.$$

Koolen and Moulton exhibited an infinite but sparse family of graphs with $\mathcal{E}(G) = (v(G)/2)\left(1 + \sqrt{v(G)}\right)$. We prove that for all sufficiently large n, there exists a graph $G = G(n)$ with $\mathcal{E}(G) \geq n^{3/2}/2 - n^{11/10}$, implying a conjecture of Koolen and Moulton.

We also characterize all square nonnegative matrices and all graphs with energy close to the maximal one. In particular, such graphs are quasi-random.

Finally we note that Wigner's semicircle law implies that

$$\mathcal{E}(G) = \left(\frac{4}{3\pi} + o(1)\right) n^{3/2}$$

for almost all graphs G.