Martingales, Optional Stopping, and the Critical Random Graph

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Talk based on work joint with:
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Martingales

Definition: An L_1 sequence of random variables $(X_1, X_2, \ldots)$ is called a martingale with respect to a σ-algebra filtration $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ if X_n is $\mathcal{F}_n$-measurable for all $n > 0$ and

$$\mathbf{E}[X_n \mid \mathcal{F}_{n-1}] = X_{n-1}\,. \tag{1}$$

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Stopping Times

Definition: A random variable $\tau : \Omega \to \{0, 1, \ldots\}$ is called a Stopping Time with respect to σ-algebra filtration $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ if for all n

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is a stopping time.

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Easy: If τ_1 and τ_2 are stopping times with respect to the same filtration, then $\tau_1 \wedge \tau_2$ is also a stopping time.

Optional Stopping Theorem

Theorem

Let X_n be a martingale, and τ a stopping time with $\tau \leq k$ a.s. for some integer $k > 0$ then

$$\mathbf{E}X_\tau = \mathbf{E}X_0 \, .$$

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Proof.

Observe that $\{\tau > i-1\} \in \mathcal{F}_{i-1}$, and so by conditioning on $\mathcal{F}_{i-1}$ we have $\mathbf{E}[X_i \mathbf{1}_{\{\tau > i-1\}}] = \mathbf{E}[X_{i-1} \mathbf{1}_{\{\tau > i-1\}}]$.

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$$\begin{aligned}
\mathbf{E}X_\tau &= \sum_{i=0}^{k-1} \mathbf{E}[X_i \mathbf{1}_{\{\tau = i\}}] + \mathbf{E}[X_k \mathbf{1}_{\{\tau > k-1\}}] \\
&= \sum_{i=0}^{k-2} \mathbf{E}[X_i \mathbf{1}_{\{\tau = i\}}] + \mathbf{E}[X_{k-1} \mathbf{1}_{\{\tau > k-2\}}] = \ldots = \mathbf{E}X_0\,.
\end{aligned}$$

Another Optional Stopping Theorem

Theorem

Let X_n be a martingale and τ a stopping time such that $\{X_{\tau \wedge n}\}$ is bounded a.s., then

$$\mathbf{E}X_{\tau} = \mathbf{E}X_0 \,.$$

The proof goes by truncating τ, using previous theorem and then finishing with the Dominated Convergence Theorem.

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Remark: If instead of a martingale X_n is a submartingale, then under the same assumptions as in the previous theorems we will have

$$\mathbf{E}X_\tau \geq \mathbf{E}X_0 \, .$$

The Critical Random Graph

The random graph $G(n, p)$ is obtained from the complete graph on n vertices, by independently retaining each edge with probability p and deleting it with probability $1 - p$. Consider $p = \frac{c}{n}$ where $c > 0$ is fixed., and let $\mathcal{C}_1$ denote the largest connected component of $G(n, p)$.

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Theorem (Erdos, Renyi '60)

- *If $c < 1$ we have $|\mathcal{C}_1| = \Theta(\log n)$ a.a.s.*
- *If $c > 1$ we have $|\mathcal{C}_1| = \Theta(n)$ a.a.s.*

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Question: What of $c = 1$?
Answer: [Bollobas; Luczak, Pittel, Wierman; Aldous] $n^{-2/3}|C_1|$ converges in distribution to some non-trivial random variable, i.e., $|C_1|$ is about $n^{2/3}$. Complicated proofs.

A BFS-Exploration Process

The process starts from a vertex v and explores the component containing v, denoted $C(v)$. Vertices in the process are either explored, active or neutral.

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- At time $t > 0$, take first active vertex and mark it explored; then make its neutral neighbors active.
- Process ends once there are no more active vertices.

Let Y_t=#(active vertices), N_t=#(neutral vertices). Number of explored vertices at time t is t. Observe

$$Y_t + N_t + t = n\,.$$

A BFS-Exploration Process (continued)

Formally, $Y_0 = 1$ and given $Y_1, \ldots, Y_{t-1}$, let η_t be random variable distributed as $\mathrm{Bin}(N_{t-1}, 1/n)$, and we have the recursion

$$Y_t = Y_{t-1} + \eta_t - 1\,,$$

defined only when $Y_{t-1} > 0$.

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defined only when $Y_{t-1} > 0$.

Observe,

$$|C(v)| = \min\{t : Y_t = 0\}\,.$$

An upper bound for $|\mathcal{C}_1|$ with a simple proof

Theorem (Nachmias, P.)

In $G(n, \frac{1}{n})$, *we have that for any* $n > 1000$ *and any* $A > 0$

$$\mathbf{P}(|\mathcal{C}_1| \geq An^{2/3}) \leq \frac{3}{A^2}\,.$$

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Proof: First observe that it suffices to bound

$$\mathbf{E}|C(v)| \leq 3n^{1/3}\,.$$

Since by symmetry,

$$\mathbf{E}|C(v)| = \frac{1}{n}\mathbf{E}\sum_{i=1}^{n}|C(v_i)| = \frac{1}{n}\sum_{j}\mathbf{E}|\mathcal{C}_j|^2\,,$$

and hence $\mathbf{E}|\mathcal{C}_1|^2 \leq 3n^{4/3}$, which finishes the proof.

Consider the martingale S_t defined by

$$S_0 = 1, \qquad S_t = S_{t-1} + \mathrm{B}in(n, \frac{1}{n}) - 1\,.$$

We can couple such that $S_t \geq Y_t$ for all $t > 0$.

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Fix an integer H and define a stopping time

$$\gamma = \min\{t : S_t = 0 \text{ or } S_t \geq H\}\,,$$

Then since S_t is a martingale, Optional Stopping Theorem gives $1 = \mathbf{E}S_\gamma \geq H\mathbf{P}(S_\gamma \geq H)$, hence

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A simple lemma: Let X be distributed $\mathrm{Bin}(n, 1/n)$ and let f be an increasing real function. Then,

$$\mathbf{E}[f(S_\gamma - H) \mid S_\gamma \geq H] \leq \mathbf{E} f(X)\,.$$

Write $S_\gamma^2 = H^2 + 2H(S_\gamma - H) + (S_\gamma - H)^2$ and apply the lemma with $f(x) = 2Hx + x^2$ to get that for $H > 3$

$$\mathbf{E}\left[S_\gamma^2 \mid S_\gamma \geq H\right] \leq H^2 + 2H + 2 \leq H^2 + 3H\,.$$

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Now $S_t^2 - (1 - \frac{1}{n})t$ is also a martingale. By optional stopping and our previous inequalities,

$$1 + (1 - \frac{1}{n})\mathbf{E}\gamma = \mathbf{E}(S_\gamma^2) = \mathbf{P}(S_\gamma \geq H)\mathbf{E}\left[S_\gamma^2 \mid S_\gamma \geq H\right] \leq H + 3\,,$$

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We conclude that for $H < n - 3$

$$\mathbf{E}\gamma \leq H + 3\,.$$

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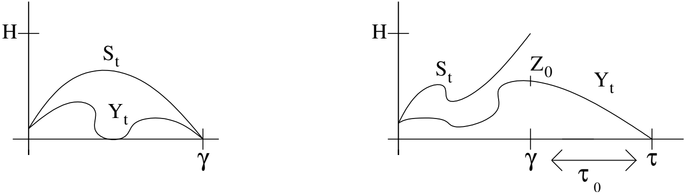
If $S_\gamma = 0$, then $\tau \leq \gamma$. Therefore: $\tau \leq \gamma + \tau_0 \mathbf{1}_{(S_\gamma \geq H)}\,.$



On the left is the case where $S_\gamma = 0$ and hence by the coupling $Y_\gamma = 0$. On the right, $S_\gamma \geq H$.

Let

$$Z_t = Y_{\gamma+t} + \sum_{j=1}^{t} \frac{j}{n} .$$

Recall that number of neutral vertices is $N_t = n - t - Y_t$, hence $\mathbf{E}[Y_t - Y_{t-1}|Y_{j-1}] \leq -\frac{t}{n}$.
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We conclude that Z_t is a supermartingale.

By Optional Stopping Theorem and the simple lemma, we have

$$H + 1 \geq \mathbf{E}\Big[S_\gamma \mid S_\gamma \geq H\Big] \geq \mathbf{E}\Big[Z_0 \mid S_\gamma \geq H\Big] \geq \mathbf{E}\Big[Z_{\tau_0} \mid S_\gamma \geq H\Big] .$$

Invoking the obvious inequality $Z_t \geq \frac{t^2}{2n}$, this yields

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By Cauchy-Schwarz,

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$$\mathbf{E}[\tau_0 \mid S_\gamma \geq H] \leq (2n(H+1))^{1/2} .$$

We are almost done. Recall:

$$\tau \leq \gamma + \tau_0 \mathbf{1}_{(S_\gamma \geq H)} .$$

Hence,

$$\mathbf{E}\tau \leq \mathbf{E}\gamma + \mathbf{E}[\tau_0 \mid S_\gamma \geq H]\,\mathbf{P}(S_\gamma \geq H)\,.$$

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Taking expectation and putting all the estimates together gives,

$$\mathbf{E}\tau \leq H + 3 + (2n(H+1))^{1/2} H^{-1} \leq H + 2(n/H)^{1/2} - 1\,,$$

where the second inequality holds if n/H is large.

Hence,

$$\mathbf{E}\tau \le \mathbf{E}\gamma + \mathbf{E}[\tau_0 \mid S_\gamma \ge H]\,\mathbf{P}(S_\gamma \ge H)\,.$$

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Optimize by taking $H = \lceil n^{1/3} \rceil$, this yields

$$\mathbf{E}|C(v)| \le 3n^{1/3}\,.$$

$\square$