On the CLT for spectral empirical measure of Wigner matrices

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Consider a $n \times n$ Wigner random matrix M_n with (real) eigenvalues $\lambda_1, \ldots \lambda_n$. Assume M_n is normalised such that when $n \to \infty$, its spectral empirical measure

$$\sigma_n(dx) = \frac{1}{n}\sum_{i=1}^{n} \delta_{\{\lambda_i\}}(dx) \ ,$$

converges weakly to the semi-circle law $\sigma(dx)$ on $[-1,1]$. This result is known since E.P. Wigner's discovery for the Gaussian ensembles. However, the problem of the Gaussian fluctuations of these limits, in other words correctly normalised CLT's, is not solved in full generality yet.

This talk is based on the works [2] and [1] which provide partial solutions to the problem.

In the first part, we assume that M_n belongs to the Gaussian unitary ensemble, so that the distribution function of the eigenvalues λ_k's is explicitly known relying on the family of Hermite polynomials. Let $F_n(u) = \sigma_n((-\infty, u])$ be the *empirical spectral distribution function.* We will restrict our attention to the empirical process indexed by the set of close intervals contained in the base interval $(-1,1)$, i.e. in the so-called "bulk" of the spectrum. More precisely, for $\overline{uv} = [u,v] \subset (-1,1)$, we define

$$F_n(\overline{uv}) = \sigma_n(\overline{uv}) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{u \le \lambda_i \le v} = F_n(v) - F_n(u) \ , \tag{1}$$

the last identity being valid with probability 1. We prove that $\mathbb{E}\, F_n(\overline{uv})$ converges at rate $\mathrm{O}\left(n^{-1}\right)$ to $F(\overline{uv}) = F(v) - F(u)$, uniformly for all closed intervals in $(-1+\eta, 1-\eta)$ for some $\eta > 0$ (here $F(u) = \sigma(-\infty, u]$). Our approach relies on the celebrated Plancherel-Rotach's formula for asymptotic expansions of the Hermite polynomials.

As for the random fluctuations around the mean, we first prove that

$$\mathrm{var}(n(F_n(\overline{uv}) - F(\overline{uv}))) = \pi^{-2}\log n + O(1).$$

A surprising fact is that the dominating term $\pi^{-2}\log n$ is constant for *all* closed intervals $\overline{uv} \subset (-1,1)$. Furthermore, we identify the covariance function of the empirical process $\{F_n(\overline{uv})\}$. The covariance between $\{F_n(\overline{ab})\}$ and $\{F_n(\overline{cd})\}$ for two subintervals $\overline{ab}$ and $\overline{cd}$ of $(-1,1)$ has a simple asymptotic form. Only three values are possibles, namely $\pm\frac{1}{2}(\pi n)^{-2}\log n + \mathrm{O}\left(n^{-2}\right)$ or $\mathrm{O}\left(n^{-2}\right)$, depending on whether one of the intervals is included in the other. In particular, the details of these intervals such as their lengths or their location in $(-1,1)$ have no influence on this form. This asymptotic covariance function is very similar to the one found by Wieand [3] for random unitary matrices where this author also established a central limit theorem. Actually we conjecture that such a CLT should also take place here.

In the second part of the talk, I will describe the main results obtained in [1] on functional CLT for spectral statistics of Wigner matrices with *general entries*. More precisely, we consider a fixed open set $\mathcal{U}$ of the complex plane including the interval $[-1, 1]$. Next define $\mathcal{A}$ to be the set of analytic functions $f : \mathcal{U} \to \mathbb{C}$. We then consider the empirical process $\mathbb{G}_n := \{G_n(f)\}$ indexed by $\mathcal{A}$ and defined as

$$G_n(f) := n \int_{-\infty}^{\infty} f(x)[\sigma_n - \sigma](dx) , \quad f \in \mathcal{A}. \tag{2}$$

The main result is that, under general moment conditions on the matrix entries, the empirical process $\mathbb{G}_n$ converges to a Gaussian process with identified mean and covariance functions. As a consequence, for any p elements $f_1, \ldots, f_p$ of $\mathcal{A}$, the finite-dimensional CLT holds, *i.e.* the vector $[G_n(f_1), \ldots, G_n(f_p)]$ converges weakly to a p-dimensional Gaussian distribution.

References

[1] Bai Z.D. and Yao J.-F., (2005). On the convergence of the spectral empirical process of Wigner matrices. *Bernoulli* **11**(6), 1059-1092

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[3] Wieand K., (2002). Eigenvalue distributions of random unitary matrices. *Probab. Theory Related Fields*, **123**, 202–224