On the convergence of the spectral empirical process of Wigner matrices

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Joint works with Z.D. Bai *and* B. Delyon

IMS Workshop on **Random matrix theory and its applications**, March 2006

1. **Introduction**
2. **Asymptotic covariances for the ESD (Gaussian U.E. case)**
3. **Functional CLT for a spectral empirical process**

The real orthogonal ensemble (O.E.)

(i) $W_n = (x_{ij})$, size $n \times n$ symmetric matrix with independent, zero-mean entries

(ii) Moments of second order

- $\mathbb{E}|x_{ii}|^2 = \sigma^2 > 0$
- for $i < j$, $\mathbb{E}|x_{ij}|^2 = 1$

1. Introduction: Wigner matrices

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2. Moments of second order
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The complex unitary ensemble (U.E.)

1. $W_n = (x_{ij})$, size $n \times n$ Hermitian matrix with independent, zero-mean entries
2. Second moments:
   - $\mathbb{E}|x_{ii}|^2 = \sigma^2 > 0$
   - for $i < j$, $\mathbb{E}|x_{ij}|^2 = 1$ and $\mathbb{E}x_{ij}^2 = 0$

Empirical spectral distribution F_n

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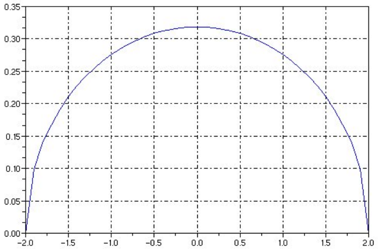
Theorem

EF_n converges to the semi-circle law, $F(dx) = \frac{1}{2\pi}\sqrt{4-x^2}\,dx, \quad x \in [-2, 2]$



1. Introduction: *Convergence rate, CLT*

Other types of LLN

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How about a CLT ?

- First $\sqrt{n}[F_n(x) - F(x)]$ does not converge in distribution
- $\sqrt{n}$ is an uncorrect norming in the "buddle of the spectrum"

$$Var[F_n(x) - F_n(x')] = \frac{2}{\pi^2}\frac{\log n}{n^2} + \mathrm{O}(n^{-2}), \quad -2 < x < x' < 2 \qquad (0.1)$$

Delyon and Yao, 2004, Gaussien U.E. case

2. Asymptotic covariances for thes ESD: Gaussian U.E.

More details:

- joint distribution of the eigenvalues $x_1, ...x_n$ of $W_n/\sqrt{2}$:

$$P(x_1, ...x_n) = \exp\left\{-\sum_i x_i^2 + 2\sum_{i<j} \log|x_j - x_i| + C\right\}$$

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- marginal distribution for two eigenvalues x, y of $W_n/\sqrt{2}$

$$s_n(x,y) = \frac{1}{n(n-1)}\left(K_n(x,x)K_n(y,y) - K_n(x,y)^2\right),$$

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where

$$K_n(x,y) = \sum_{j=0}^{n-1} \varphi_j(x)\varphi_j(y) = \frac{\sqrt{n}}{\sqrt{2}(x-y)}(\varphi_n(x)\varphi_{n-1}(y) - \varphi_n(y)\varphi_{n-1}(x)).$$

$$\varphi_n(x) = (2^n n!\sqrt{\pi})^{-1/2} e^{-x^2/2} H_n(x), \quad n \geq 0$$

$$H_n(x) = n - \text{th Hermite polynomial.}$$

Recall

$$\varphi_n(x) = (2^n n!\sqrt{\pi})^{-1/2} e^{-x^2/2} H_n(x), \quad n \geq 0$$

The Plancherel-Rotach formula

Let $\varepsilon > 0$ be fixed. For $\varepsilon \leq \phi \leq \pi - \varepsilon$, let $x = \sqrt{2n+1}\cos\phi$, then

$$(2n)^{1/4}\varphi_n(x) = \left(\frac{2}{\pi \sin\phi}\right)^{1/2} \sin\left[\frac{2n+1}{4}(\sin 2\phi - 2\phi) + \frac{3\pi}{4}\right] + \mathrm{O}\left(n^{-1}\right) ,$$

where the error term $\mathrm{O}\left(n^{-1}\right)$ is uniform on the interval $[\varepsilon, \pi - \varepsilon]$.

Convergence rate the expected E.S.D. for $\mathbb{E}F_n$

Proposition

Let $\eta > 0$ be fixed. We have

(i) *The density of a single eigenvalue, say $\lambda_{n,1}$ of $W_n/\sqrt{2n}$ has the expansion*

$$p_n(u) = \frac{1}{2\pi}\sqrt{4-u^2} + \mathrm{O}\left(n^{-1}\right) \ ,$$

where the term $\mathrm{O}(n^{-1})$ is uniform for all $|u| \leq 2-\eta$.

(ii)

$$\sup_{\overline{uv}\subset[-2+\eta,2-\eta]} |\,\mathbb{E}F_n(\overline{uv}) - F(\overline{uv})| = \mathrm{O}\left(n^{-1}\right) \ .$$

Delyon & Yao (2003)

Variance, covariances asymptotiques

- In the so-called "bulk" of the spectrum: $\overline{uv} = [u, v] \subset (-2, 2)$

$$F_n(\overline{uv}) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{u \le \lambda_i \le v} \ ,$$

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$$F_n(\overline{uv}) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{u \le \lambda_i \le v} \ ,$$

- Let $\overline{ab} \subset (-2, 2)$, $\overline{cd} \subset (-2, 2)$

$$\lim_{n \to \infty} \frac{n^2 \pi^2}{\log n} \operatorname{cov}\left(\frac{1}{2} F_n(\overline{ab}), \frac{1}{2} F_n(\overline{cd})\right) = \begin{cases} 1, & \text{if } \ a = c, \quad b = d, \\ \frac{1}{2}, & \text{if } \ a = c < b < d, \\ \frac{1}{2}, & \text{if } \ c < a < b = d, \\ -\frac{1}{2}, & \text{if } \ a < c < d \\ 0, & \text{otherwise} \end{cases}$$

Still an open conjecture:

$$\frac{n}{\sqrt{\log n}}[F_n(\overline{ab}) - F(\overline{ab})] \xrightarrow{w} \mathcal{N}(0,*), \quad (a,b) \subset (-2,2) \quad ??$$

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A related result: Wieand (2001), Diaconis & Evans (2001)

- M_n Haar-distributed form the unitary group
- for all i, $\lambda_{n,i} \in \mathscr{S}^1$

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- M_n Haar-distributed form the unitary group
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Theorem. As $n \to \infty$, the finite-dimensional distributions of the processes

$$\frac{n\pi}{\sqrt{\log n}}[F_n(\alpha,\beta) - \mathbb{E}F(\alpha,\beta)]$$

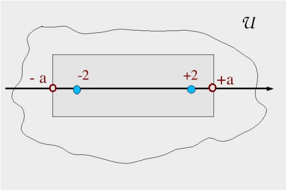
converge to those of a centered Gaussian process $\{Z_{\alpha\beta} \ : \ 0 \le \alpha < \beta < 2\pi\}$ with the convariance structure

$$\mathrm{cov}(Z_{\alpha\beta},\ Z_{\alpha'\beta'}) = \begin{cases} 1, & \text{if } \ \alpha=\alpha' \text{ and } \ \beta=\beta', \\ \frac{1}{2}, & \text{if } \ \alpha=\alpha' \text{ and } \ \beta\neq\beta', \\ \frac{1}{2}, & \text{if } \ \alpha\neq\alpha' \text{ and } \ \beta=\beta', \\ -\frac{1}{2}, & \text{if } \ \beta=\alpha', \\ 0, & \text{otherwise} \end{cases}$$

3. Functional CLT for a spectral empirical process

Bai & Yao (2005)

- Fix an open set $\mathcal{U} \supset [-2, 2]$



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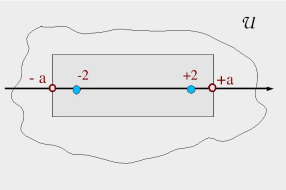

$\mathscr{A}$ = analytic functions defined on $\mathcal{U}$

- Consider integrals

$$G_n(f) := n \int_{-\infty}^{\infty} f(x)[F_n - F](dx) , \quad f \in \mathscr{A} .$$

3.a) Main theorem

Main result

- Under suitable conditions, the empirical process

$$G_n := \{G_n(f), \quad f \in \mathscr{A}\}$$

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- In particular, for any $f_1, \ldots, f_p$ of $\mathscr{A}$,

$$[G_n(f_1), \ldots, G_n(f_p)] \xrightarrow{w} \mathscr{N}_p(0, *).$$

Moment conditions

- **[M1]** (homogeneity of 4-th moments): $M \equiv \mathbb{E}|x_{ij}|^4$ for $i \neq j$;
- **[M2]** (uniform tails): for any $\eta > 0$, as $n \to \infty$,

$$\frac{1}{\eta^4 n^2} \sum_{i,j} \mathbb{E}\left[|x_{ij}|^4 \mathbf{1}_{\{|x_{ij}| \geq \eta\sqrt{n}\}}\right] = o(1).$$

Expansion on Tchebychev polynomials

- $\{T_k\}$ be the family of Tchebychev polynomials
- Orthogonal family:

$$\frac{1}{\pi}\int_{-1}^{1} T_i(t)T_j(t)\frac{1}{\sqrt{1-t^2}}dt = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j = 0 \\ \frac{1}{2}, & \text{if } i = j > 0 \end{cases}$$

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- Expansion on this basis: define

$$\tau_\ell(f) = \frac{1}{\pi}\int_{-1}^{1} f(2t)T_\ell(t)\frac{1}{\sqrt{1-t^2}}dt\ .$$

Theorem

Under the conditions **[M1]-[M2]**,

$$G_n = \{G_n(f), f \in \mathcal{A}\} \quad \overset{weakly}{\longrightarrow} \quad G := \{G(f): \ f \in \mathcal{A}\}$$

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$$\mathbb{E}[G(f)] = \frac{\kappa - 1}{4}\{f(2) + f(-2)\} - \frac{\kappa - 1}{2}\tau_0(f) + (\sigma^2 - \kappa)\tau_2(f) + \beta\tau_4(f) \ ,$$

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$$\begin{aligned}\operatorname{cov}(G(f), G(g)) &= (\sigma^2 - \kappa)\tau_1(f)\tau_1(g) + 2\beta\tau_2(f)\tau_2(g) + \kappa\sum_{\ell=1}^{\infty}\ell\tau_\ell(f)\tau_\ell(g) \\ &= \frac{1}{4\pi^2}\int_{-2}^{2}\int_{-2}^{2} f'(t)g'(s)V(t,s)dtds,\end{aligned}$$

where $\kappa = 1$ *or 2,* $\quad \beta = \operatorname{var}(|x_{12}|^2) - \kappa,$ $\quad$ *and*

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where $\kappa = 1$ *or 2,* $\quad \beta = \operatorname{var}(|x_{12}|^2) - \kappa,$ $\quad$ *and*

$$\begin{aligned} V(t,s) &= (\sigma^2 - \kappa + \frac{1}{2}\beta ts)(4 - t^2)^{1/2}(4 - s^2)^{1/2} \\ &\quad + \kappa \log\left(\frac{4 - ts + (4 - t^2)^{1/2}(4 - s^2)^{1/2}}{4 - ts - (4 - t^2)^{1/2}(4 - s^2)^{1/2}}\right). \end{aligned}$$

3.b). Application: CLT for Linear spectral statistics

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A linear spectral statistic

- f : a polynomial

$$\frac{1}{n}[f(\lambda_{n,1}) + \cdots + f(\lambda_{n,n})] = F_n(f) = \frac{1}{n}\operatorname{tr} f(M_n) \ .$$

CLT for Tchebychev polynomials

1. Assume the conditions **[M1]-[M2]** hold.
2. $T_1, \ldots, T_p$: be p first Tchebychev polynomials.
3. Denote $e(k) = \{k \text{ is even }\}$.

Then

$$[G_n(T_1), \ldots, G_n(T_p)] \xrightarrow{w} \mathscr{N}_p(w, \Sigma)$$

with

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with

$$\begin{aligned} w &= (m_\ell), \quad m_\ell = \frac{1}{2}\left[(\kappa - 1)e(\ell) + (\sigma^2 - \kappa)\delta_{\ell 2} + \beta\delta_{\ell 4}\right], \\ \Sigma &= \mathrm{diag}(a_1, \cdots, a_p), \quad a_\ell = \left(\frac{1}{2}\right)^2 \left[(\sigma^2 - \kappa)\delta_{\ell 1} + 2\beta\delta_{\ell 2} + \kappa\ell\right]. \end{aligned}$$

In particular, these statistics are asymptotically independent.

CLT for linear spectral statistics of Gaussian ensembles

The Gaussian U.E.

$$m = \begin{pmatrix} 0 & \vdots & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} (\frac{1}{2})^2 & & & & \\ & 2(\frac{1}{2})^2 & \cdots & & \\ & & \ddots & & \\ & & k(\frac{1}{2})^2 & & \\ & & \ddots & & \\ & & & & p(\frac{1}{2})^2 \end{pmatrix}$$

The Gaussian O.E.

$$m_k = \frac{1}{2} e(k) = \frac{1}{2}\{k \text{ is even }\}, \quad \Sigma = \begin{pmatrix} 2(\frac{1}{2})^2 & & & & \\ & 4(\frac{1}{2})^2 & \cdots & & \\ & & \ddots & & \\ & & 2k(\frac{1}{2})^2 & & \\ & & \ddots & & \\ & & & & 2p(\frac{1}{2})^2 \end{pmatrix}$$

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- **Results in this case already provided by (Johansson 1998)**
- Striking fact: the limit distribution is non central.

3.c) Related works on CLT, Gaussian fluctuations

Related CLT's

- Starting from [Costin and Lebowitz,1995]

[Johansson, 1998]

- Extended random ensemble

$$\text{Joint density of the entries of } W \propto \exp[-n\,\mathrm{tr}\{V(W)\}]$$

$V =$ polynomial of even degree

- Covers G.O.E.
- CLT for linear spectral statistics (Tchebychev polynomials)

Related CLT's

[Khorunzhy et al.,1996]

- orthogonal ensemble (with general entries)
- CLT for

$$n\{s_n(z_1) - \mathbb{E}s_n(z_1), \ldots, s_n(z_q) - \mathbb{E}s_n(z_q)\}$$

where s_n is the Stieltjes transform

$$s_n(z) = \frac{1}{n}\mathrm{tr}\left(\frac{1}{\sqrt{n}}W_n - zI\right)^{-1}.$$

and

$$z_j \in \mathbb{C}, \quad |\Im(z_j)| \geq 2$$

- CLT very close to fidi convergence part of our theorem

Related CLT's

[Sinai and Soshnikov,1998]

- assume $p = p(n) \to \infty, \quad p/\sqrt{n} \to 0.$
- the entries are symmetric, having moments of any order with a appropriate growth condition
- CLT for

$$tr \left(\frac{1}{\sqrt{n}} W_n \right)^p - \text{"its expectation"}$$

- Also a CLT for $G_n(f) - \mathbb{E}[G_n(f)]$ with f analytic on the disk of radius 2 without providing the asymtotic mean or variance.

3.d) Some idea of the proofs

Stieltjes transform

- the Stieltjes transform $s_H(z)$ of $H : \mathbb{R} \to \mathbb{R}$ with bounded variation

$$s_H(z) = \int_{-\infty}^{\infty} \frac{dH(x)}{x - z} , \qquad z \in \mathbb{C}^+ := \{u + iv, \ v > 0\} .$$

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- Stieltjes transform of F_n

$$s_n(z) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\lambda_{k,n} - z}.$$

- Stieltjes transform of the semi-circle law F

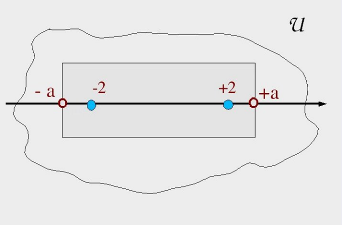
$$s(z) = \frac{1}{2}(z - \sqrt{z^2 - 4}).$$

Integral representation

- Take a contour $\gamma \subset \mathscr{U}$
- for $x \in (-a, a)$,

$$f(x) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - x} dz \ .$$

- Then



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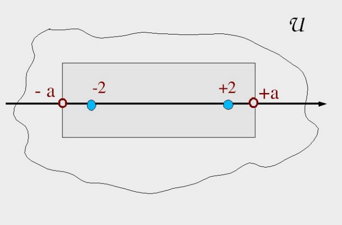

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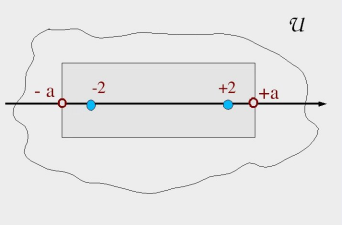

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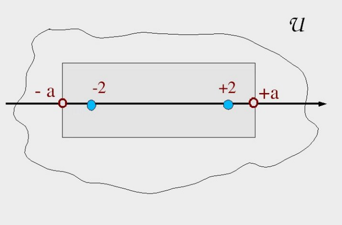

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$$\begin{aligned} G_n(f) &= \int f(x) n[F_n - F](dx) = \frac{1}{2\pi i} \int \oint_\gamma \frac{f(z)}{z-x} n[F_n - F](dx) dz \\ &= \frac{1}{2\pi i} \oint_\gamma f(z) dz \int \frac{1}{z-x} n[F_n - F](dx) \end{aligned}$$

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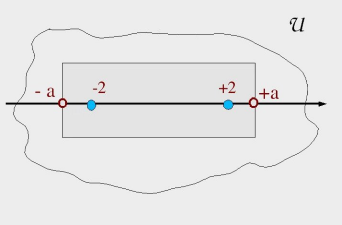

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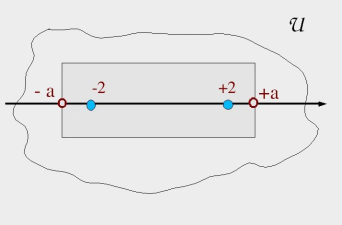

- Then

$$\begin{aligned} G_n(f) &= \int f(x) n[F_n - F](dx) = \frac{1}{2\pi i} \int \oint_\gamma \frac{f(z)}{z - x} n[F_n - F](dx) dz \\ &= \frac{1}{2\pi i} \oint_\gamma f(z) dz \int \frac{1}{z - x} n[F_n - F](dx) \\ &= -\frac{1}{2\pi i} \oint_\gamma f(z) n[s_n(z) - s(z)] dz \,. \end{aligned}$$

First reduction: prove the weak convergence of the process

$$\xi_n(z) = n[s_n(z) - s(z)], \quad z \notin [-2, 2]$$

to some Gaussian process.

The mean function

Proposition

Uniformly on $\mathscr{U}$, the mean function $\mathbb{E}\xi_n(z)$ tends to

$$\to [1 + s'(z)]s(z)^3 \left[\sigma^2 - 1 + (\kappa - 1)s'(z)\beta s^2(z)\right]$$

Set

$$\begin{aligned} \mathbb{C}_0 &= \{u+iv, \quad |v| \geq v_0\}, \quad v_0 \quad \text{small}, \\ \zeta_n(z) &= \xi_n(z) - \mathbb{E}\xi_n(z), \quad z \in \mathbb{C}_0 \end{aligned}$$

Proposition

Assume Conditions **[M1]-[M2]** *satisfied. For any set of p points* $\{z_s, s = 1, \ldots, p\}$ *of* $\mathbb{C}_0$*, then*

$$(\zeta(z_1), \ldots, \zeta(z_p)) \Rightarrow \mathcal{N}_p(0, \Gamma) \ ,$$

where

$$\begin{aligned} \Gamma(z_j, z_s) &= \frac{\partial^2}{\partial z_j \partial z_s} \left[(\sigma^2 - \kappa) s_j s_s + \frac{1}{2}\beta(s_j s_s)^2 - \kappa\, \ell n(1 - s_j s_s) \right] \\ &= s_j' s_s' \left[\sigma^2 - \kappa + 2\beta s_j s_s + \frac{\kappa}{(1 - s_j s_s)^2} \right] \ , \end{aligned}$$

with $s_j = s(z_j)$.

Tightness of the sequence of processes $(\zeta_n(z))$

It is enough to establish the following Hölder condition: for some positive constant K

$$\mathbb{E}|\zeta_n(z_1) - \zeta_n(z_2)|^2 \leq K|z_1 - z_2|^2 , \quad z_1, z_2 \in \mathbb{C}_0 .$$

Conclusions

1. Provided general weak convegence theorem for the spetral process, indexed by a general class of analytic functions

$$G_n(f) := n \int_{-\infty}^{\infty} f(x)[F_n - F](dx) , \quad f \in \mathscr{A} .$$

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(v) Still, I do not know any proof of a result assumed as a folklore by many random matrix people:

$$\frac{n}{\sqrt{\log n}}[F_n(\overline{ab}) - F(\overline{ab})] \xrightarrow{w} \mathscr{N}(0, *), \quad (a, b) \subset (-2, 2) \quad ??$$

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