Eigenvalues of large sample covariance matrices; Lecture 3

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• Complex Gaussian samples with covariance matrix $T_p \sim diag(t_1, t_2, \ldots, t_p)$

p=population size

n=sample size

• Sample covariance matrix $B = \frac{1}{n} X X^*$

Distribution function of the largest sample eigenvalue

$$\begin{aligned}\mathbb{P}(s_{\max} \le t) &= \det(1 - \chi_{(t,\infty)} \mathcal{K} \chi_{(t,\infty)}) \\ &= \det(1 - \chi_{(t,\infty)} \mathcal{I} \mathcal{J} \chi_{(t,\infty)})\end{aligned}$$

on $L^2((t,\infty)) \to L^2((0,\infty) \to L^2((t,\infty))$ with kernel

$$\int_0^\infty \mathcal{I}(u+y) \mathcal{J}(y+v) dy, \qquad u, v > t.$$

Goal: $n, p \to \infty$, $\frac{p}{n} \to c$, $0 < c \le 1$ (for spiked model)

General theory of Fredholm determinants:

$A^{(n)}, B^{(n)} : L^2(I) \to L^2(I)$

If $A^{(n)} \to A^{(\infty)}$ and $B^{(n)} \to B^{(\infty)}$ in Hilbert-Schmidt norm, i.e.

$$\int_I \int_I |A^{(n)}(u,v) - A^{(\infty)}(u,v))|^2 du dv \to 0,$$

then $\det(1 - A^{(n)}B^{(n)}) \to \det(1 - A^{(\infty)}B^{(\infty)})$.

From Lecture 2,

- **Translation;** $u \mapsto u - t$, ...

$$\mathbb{P}(s_{\max} \le t) = \det(1 - \mathcal{K}_t) = \det(1 - \mathcal{I}_t \mathcal{J}_t)$$

acting on $L^2((0,\infty))$ and

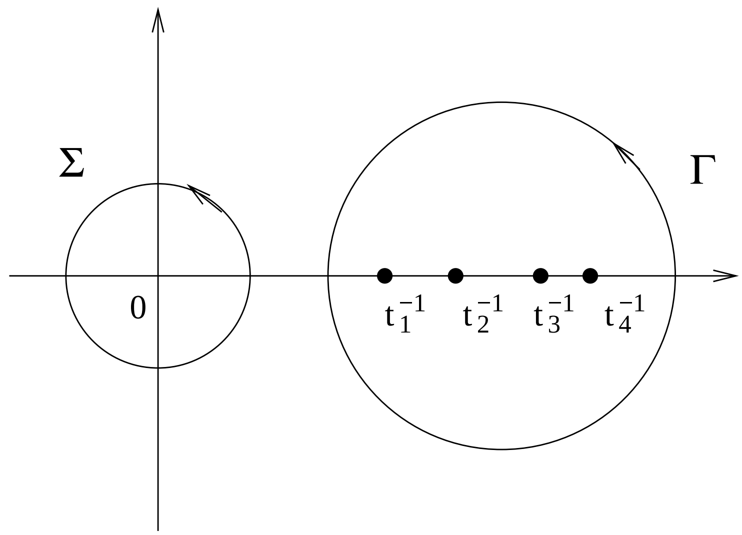
$$\mathcal{K}_t(u,v) = \int_0^\infty \mathcal{I}(t+u+y)\,\mathcal{J}(y+v+t)dy, \quad u,v>0.$$

- **Scale:** with μ, ν, α to be determined,

$$\mathbb{P}\left((s_{\max} - \mu)\nu^{-1} n^{\alpha} \le x\right) = \mathbb{P}\left(s_{\max} \le \mu + \frac{\nu x}{n^{\alpha}}\right)$$
$$= \det(1 - \hat{\mathcal{K}}_x)$$

New kernel: with $t = \mu + \frac{\nu x}{n^{\alpha}}$, for $u, v > 0$,

$$\hat{\mathcal{K}}_x(u,v) := \frac{\nu}{n^{\alpha}} \mathcal{K}_t\left(\frac{\nu u}{n^{\alpha}}, \frac{\nu v}{n^{\alpha}}\right)$$
$$= \frac{\nu^2}{n^{2\alpha}} \int_0^\infty \mathcal{I}\left(\mu + \frac{\nu(x+u+y)}{n^{\alpha}}\right) \mathcal{J}\left(\nu + \frac{\nu(y+v+x)}{n^{\alpha}}\right) dy$$
$$= \int_0^\infty \hat{\mathcal{I}}(x+u+y)\,\hat{\mathcal{J}}(y+v+x)dy$$



Here

$$\hat{\mathcal{I}}(y) := \frac{\nu}{n^\alpha}\mathcal{I}\left(\mu + \frac{\nu y}{n^\alpha}\right) = \frac{\nu n^{1-\alpha}}{2\pi}\int_\Gamma z^n e^{-n(\mu+\frac{\nu y}{n^\alpha})z}\prod_k (z - t_k^{-1})^{-1} dz$$

and

$$\hat{\mathcal{J}}(y) := \frac{\nu}{n^\alpha}\mathcal{J}\left(\mu + \frac{\nu y}{n^\alpha}\right) = \frac{\nu n^{1-\alpha}}{2\pi}\int_\Sigma w^{-n} e^{n(\mu+\frac{\nu y}{n^\alpha})w}\prod_k (w - t_k^{-1}) dw$$

Need $n, p \to \infty$ for all $y \geq x$: Steepest-decent

Need to choose μ, ν, α to have interesting limit

Steepest-descent method

Limit of $Q_n := \int_C g(z)e^{nf(z)}dz$ as $n \to \infty$?

Let $f(z) = R(z) + iI(z)$

Critical points: $f'(z_c) = 0$

Try to change the contour C so that it passes z_c and $R(z) < R(z_c)$ for all $z \in C \setminus \{z_c\}$: if this is possible,

$$Q_n \sim \int_{C_\epsilon} g(z_c)e^{n\{f(z_c)+\frac{1}{2}f''(z_c)(z-z_c)^2+\cdots\}}dz$$
$$\sim g(z_c)e^{nf(z_c)} \int e^{\frac{nf''(z_c)}{2}(z-z_c)^2}ds$$
$$= g(z_c)e^{nf(z_c)}\sqrt{\frac{2\pi}{-nf''(z_c)}}$$

if $f''(z_c) \neq 0$

Cauchy-Riemann eq's:
$\nabla R = (R_x, R_y) = (R_x, -I_x) = \overline{f'(z)}$

Steepest-descent curve:

At each point $x + iy \in \mathbb{C}$, choose the direction such that $R(x, y)$ decreases most rapidly:

If $\nabla R(x, y) \neq 0$, $\vec{v}(x, y) \perp \{(a, b) : R(a, b) = constant\}$; hence the direction of $\nabla R(x, y) = \overline{f'(z)}$

As $\{R = constant\} \perp \{I = constant\}$ $(R_x I_x + R_y I_y = 0)$, it is same as the level curves of I.

Critical points and singular points of f: need local analysis

Assume $\frac{p}{n} = c$ for simplicity. $(0 < c \leq 1)$

Case 1: Null case $T_p = I$

$$\hat{\mathcal{I}}(y) = \frac{\nu n^{1-\alpha}}{2\pi} \int_\Gamma e^{nf(z)} e^{-n^{1-\alpha}\nu y z} dz$$
$$\hat{\mathcal{J}}(y) = \frac{\nu n^{1-\alpha}}{2\pi} \int_\Sigma e^{-nf(w)} e^{n^{1-\alpha}\nu y w} dw$$

where

$$f(z) = -\mu z + \log z - c \log(z-1)$$

(Recall: y, c are given. μ, ν, α are to be determined.)

Critical points of f:

$$f'(z) = -\mu + \frac{1}{z} - \frac{c}{z-1} = \frac{-\mu z^2 + (\mu + 1 - c)z - 1}{z(z-1)} = 0$$

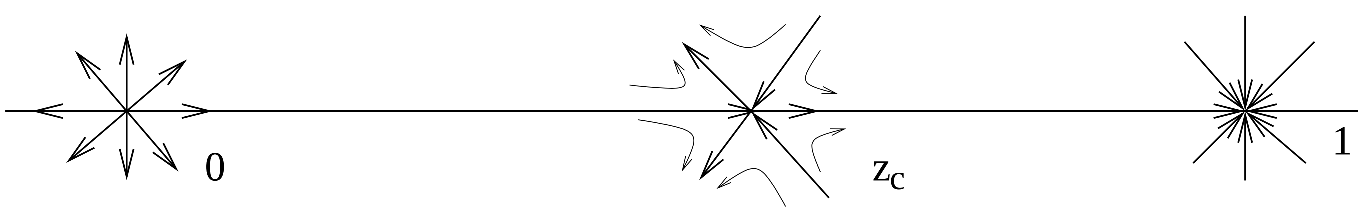
Discriminant $D = (\mu - (1-\sqrt{c})^2)(\mu - (1+\sqrt{c})^2)$

• **Choose $\mu = (1+\sqrt{c})^2$ so that $D = 0$ (double root):** For other choices of μ, steepest-descent analysis yields no interesting limit

Then $z_c = \frac{1}{1+\sqrt{c}}$ and $f'(z) = \frac{-\mu(z-z_c)^2}{z(z-1)}$

$f'(z_c) = f''(z_c) = 0$ and $f^{(3)}(z_c) = \frac{2(1+\sqrt{c})^4}{\sqrt{c}} > 0$

Near $z = z_c$: $f(z) \sim \frac{-\mu}{3z_c(z_c-1)}(z-z_c)^3 = const \cdot r^3 e^{3i\theta}$



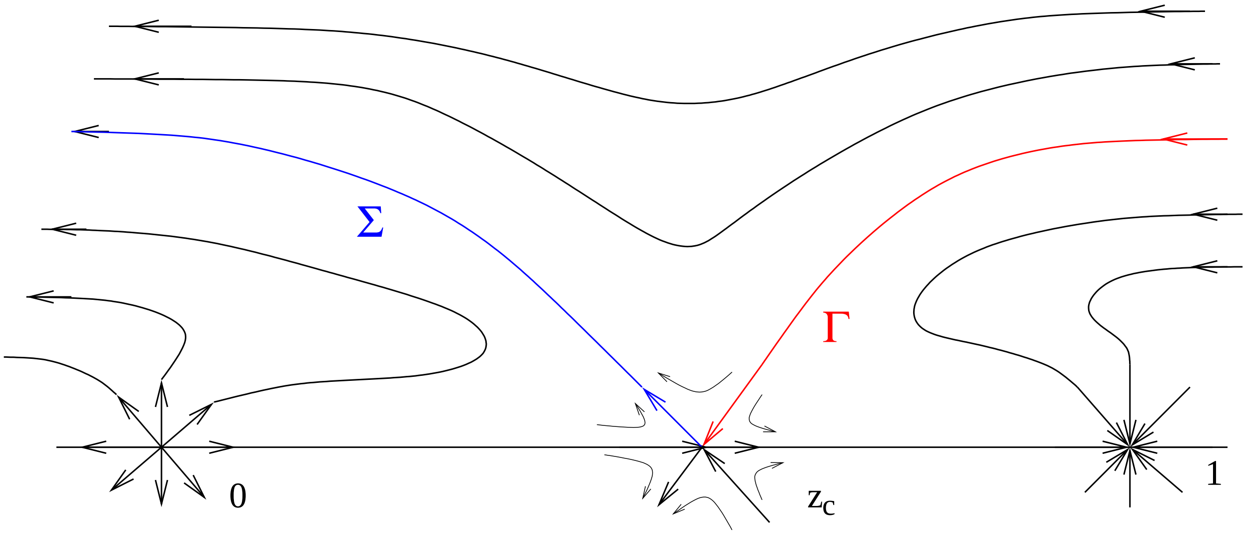
Singular points of f:

$z = 0$: $f(z) \sim \log z = \log r + i\theta$; source

$z = 1$: $f(z) \sim -c\log(z-1) = -c\log\rho - ic\varphi$; sink

$z = \infty$: $f(z) \sim -\mu z = -\mu(x+iy)$: horizontal lines flowing to the left

Steepest-descent curves



It is possible to deform the contours without crossing singular points.

Localize the contour $\Gamma \to \Gamma_\epsilon$ and replace $f(z)$ by the leading two terms

$$\hat{\mathcal{I}}(y) \sim \frac{\nu n^{1-\alpha}}{2\pi} \int_{\Gamma_\epsilon} e^{nf(z_c)+\frac{nf^{(3)}(z_c)}{3!}(z-z_c)^3} e^{-n^{1-\alpha}\nu y z} dz$$

- **Choose** $\alpha = \frac{2}{3}$ **and** $\nu = \left(\frac{1}{2} f^{(3)}(z_c)\right)^{1/3} = \left(\frac{(1+\sqrt{c})^4}{\sqrt{c}}\right)^{1/3}$

$$\hat{\mathcal{I}}(y) \sim \frac{\nu n^{\frac{1}{3}}}{2\pi} e^{nf(z_c) - n^{\frac{1}{3}} \nu y z_c} \int_{\Gamma_\epsilon} e^{\frac{n\nu^3}{3}(z-z_c)^3 - n^{\frac{1}{3}} \nu y (z-z_c)} dz$$

Change of variables $n^{\frac{1}{3}} \nu (z - z_c) =: a$

$$\begin{aligned} \hat{\mathcal{I}}(y) &\sim e^{nf(z_c) - n^{\frac{1}{3}} \nu y z_c} \cdot \frac{1}{2\pi} \int e^{\frac{1}{3}a^3 - ya} da \\ &= e^{nf(z_c) - n^{\frac{1}{3}} \nu y z_c} \cdot \mathrm{Ai}(y) \end{aligned}$$

Airy function

Similarly,

$$\begin{aligned} \hat{\mathcal{J}}(y) &\sim e^{-nf(z_c) + n^{\frac{1}{3}} \nu y z_c} \cdot \frac{1}{2\pi} \int e^{-\frac{1}{3}a^3 + ya} da \\ &= e^{-nf(z_c) + n^{\frac{1}{3}} \nu y z_c} \cdot \mathrm{Ai}(y) \end{aligned}$$

Thus, assuming certain uniformity,

$$\hat{\mathcal{K}}(u,v) = \int_0^\infty \hat{\mathcal{I}}(x+u+y)\, \hat{\mathcal{J}}(y+v+x) dy$$

$$\sim e^{-n^{\frac{1}{3}} \nu u z_c} \left\{ \int_0^\infty \mathrm{Ai}(x+u+y)\, \mathrm{Ai}(y+v+x) dy \right\} e^{n^{\frac{1}{3}} \nu v z_c}$$

Result: When $T_p = I$, with $\frac{p}{n} = c$,

$$\lim_{n \to \infty} \mathbb{P}\left((s_{\text{max}} - \mu) \frac{n^{2/3}}{\nu} \le x \right) = \det(1 - \mathcal{A}_x)$$

i.e. $s_{\text{max}} \sim \mu + \frac{\nu}{n^{2/3}} X$, where

$$\mu = (1 + \sqrt{c})^2, \qquad \nu = \left(\frac{(1+\sqrt{c})^4}{\sqrt{c}} \right)^{1/3}$$

and

$$\begin{aligned} \mathcal{A}_x(u,v) &= \int_0^\infty \text{Ai}(x+u+y)\,\text{Ai}(y+v+x)dy \\ &= \frac{\text{Ai}(x+u)\,\text{Ai}'(x+v) - \text{Ai}'(x+u)\,\text{Ai}(x+v)}{u-v} \end{aligned}$$

acting on $L^2((0,\infty))$.

[Forrester 1993]

Graph of the limiting distribution? ...

Case (2): Spiked model

Fix $r > 0$, independent of n, p.

$T_p \sim diag(t_1, \ldots, t_r, 1, 1, \ldots, 1)$ where $t_1 \geq \cdots \geq t_r$:

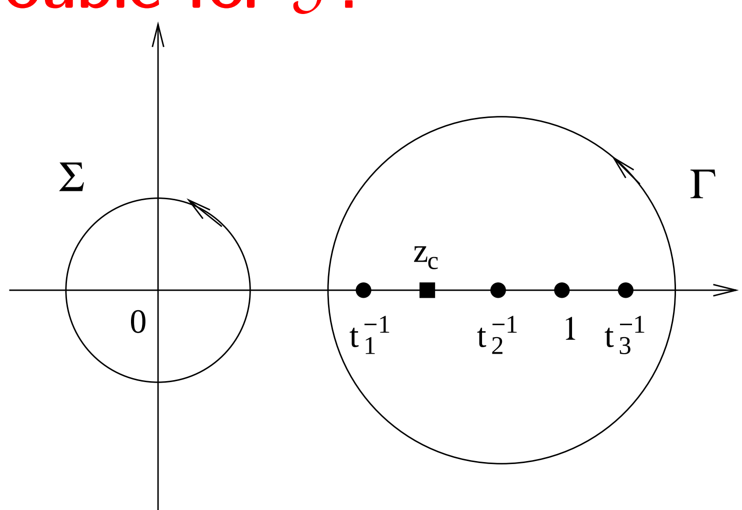
With same f as case (1),

$$\hat{\mathcal{I}}(y) = \frac{\nu n^{1-\alpha}}{2\pi} \int_{\Gamma} e^{nf(z)} e^{-n^{1-\alpha}\nu y z} \left(\prod_{k=1}^{r} \frac{z-1}{z-t_k^{-1}} \right) dz$$

$$\hat{\mathcal{J}}(y) = \frac{\nu n^{1-\alpha}}{2\pi} \int_{\Sigma} e^{-nf(w)} e^{n^{1-\alpha}\nu y w} \left(\prod_{k=1}^{r} \frac{w-t_k^{-1}}{w-1} \right) dw$$

Recall that the poles $z = t_k^{-1}$ and $z = 1$ are inside Γ and outside Σ.

Trouble occurs for $\hat{\mathcal{I}}$ if we need to pass a pole when we deform Γ to the steepest-descent curve. No such trouble for $\hat{\mathcal{J}}$.



$z_c = \frac{1}{1+\sqrt{c}}$. Three cases:

(a) $t_k < 1 + \sqrt{c}$ for all $1 \leq k \leq r$

Deformation possible, no trouble: with same μ, ν, α,

$$\hat{\mathcal{I}}(y) \sim e^{nf(z_c) - n^{\frac{1}{3}}\nu y z_c} \prod_{k=1}^{r} \left(\frac{z_c - 1}{z_c - t_k^{-1}} \right) \cdot \mathrm{Ai}(y)$$

$$\hat{\mathcal{J}}(y) \sim e^{-nf(z_c) + n^{\frac{1}{3}}\nu y z_c} \left(\frac{z_c - t_k^{-1}}{z_c - 1} \right) \cdot \mathrm{Ai}(y)$$

Same limit law as $T_p = I$. Non-unit (true) eigenvalue have no effect.

(b) $t_k = 1 + \sqrt{c}$ for $k = 1, 2, \ldots, d$ (and $t_k < 1 + \sqrt{c}$ for $d + 1 \leq k \leq r$)

Slightly modify Γ so that Γ passes the real axis at the point q such that $q = z_c - \epsilon n^{-1/3}$

Set

$$A_d(y) := \frac{1}{2\pi} \int e^{\frac{1}{3}a^3 - ya} \frac{1}{a^d} da$$
$$B_d(y) := \frac{1}{2\pi} \int e^{-\frac{1}{3}a^3 + ya} a^d da$$

and set

$$\mathcal{A}_x^{(d)}(u, v) = \int_0^\infty A_d(x + u + y) B_d(y + v + x) dy$$

With same μ, ν, α,

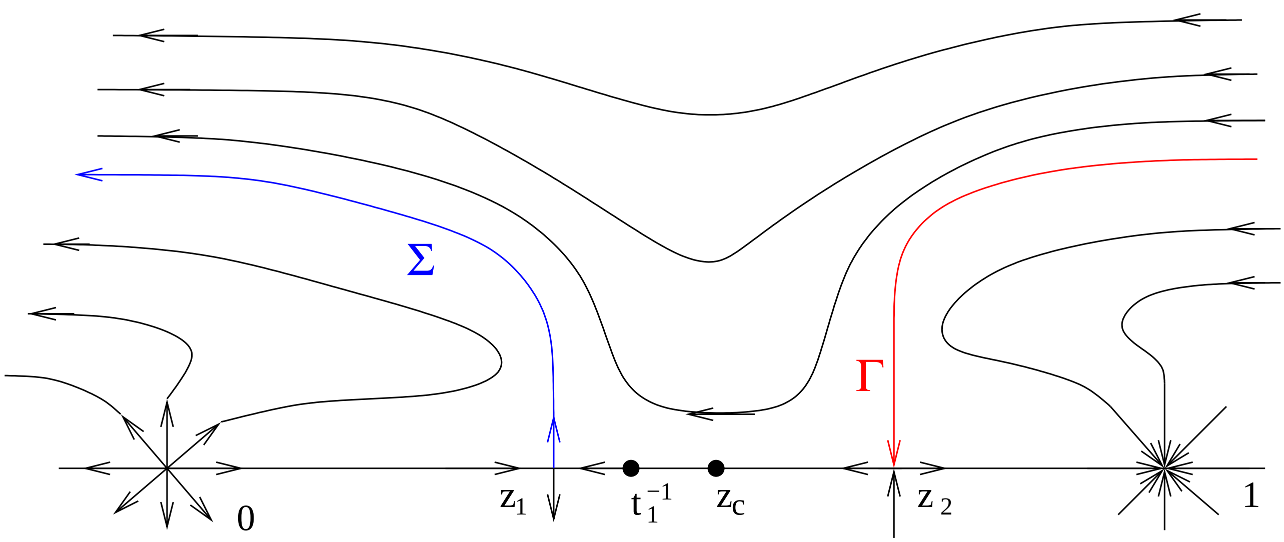
$$\lim_{n \to \infty} \mathbb{P}\left((s_{\max} - \mu) \frac{n^{2/3}}{\nu} \leq x \right) = \det(1 - \mathcal{A}_x^{(d)})$$

Same type of limit law with different limiting distribution: what are these new distribution functions?

(c) $t_1 > 1 + \sqrt{c}$ and $t_1 = t_2 = \cdots = t_d$

With same μ, the contour Γ have to pass the pole $z = t_1^{-1}$ and possibly more poles left of z_c: contributions from poles is bigger than from the steepest-decent curve.

How about if we take different μ?
When $\mu > (1 + \sqrt{c})^2$: **Two critical points $z_1 < z_c < z_2 < 1$.**



We still have to pass the pole $z = t^{-1}$ for Γ and the contribution from the pole turns out to be the leading term of $\hat{\mathcal{I}}$

With $\mu, \nu\, \alpha$ to be determined,

$$\hat{\mathcal{I}}(y) \sim i\nu n^{1-\alpha} Res_{z=t_1^{-1}} \left(e^{nf(z)} e^{-n^{1-\alpha}\nu y z} \frac{g(z)}{(z-t_1^{-1})^d} \right)$$

and hence **the leading term is** $e^{nf(t_1^{-1})}$.

For $\hat{\mathcal{J}}$, the steepest-descent curve passes at z_1, implying that **the leading term is** $e^{-nf(z_1)}$

We want them to cancel! Hence we need $z_1 = t_1^{-1}$. Thus

$$\mu = \frac{t_1(t_1 - (1-c))}{t_1 - 1}$$

Thus $\alpha = \frac{1}{2}$, and

$$\nu = \sqrt{-f''(z_1)} = \frac{t_1 \sqrt{(t_1 - (1-\sqrt{c}))(t_1 - (1+\sqrt{c}))}}{t_1 - 1}$$

With this μ, ν,

$$\lim_{n\to\infty} \mathbb{P}\left((s_{\max} - \mu) \frac{n^{1/2}}{\nu} \leq x \right) = \det(1 - \mathcal{H}_x^{(d)})$$

with some operator $\mathcal{H}_x^{(d)}$

Summary

$T_d \sim diag(t_1, \ldots, t_r, 1, 1, \ldots, 1)$ $(t_1 \geq t_2 \geq \ldots)$

$\mu = (1+\sqrt{c})^2$ and $\nu = \left(\frac{(1+\sqrt{c})^4}{\sqrt{c}}\right)^{1/3}$

$\mu' = \frac{t_1(t_1-(1-c))}{t_1-1}$ and $\nu' = \nu'(t_1)$

(a) $t_k < 1+\sqrt{c}$ for all k

$$\lim_{n\to\infty} \mathbb{P}\left((s_{\max} - \mu)\frac{n^{2/3}}{\nu} \leq x\right) = F_0(x)$$

(b) $t_1 = \cdots = t_d = 1+\sqrt{c}$

$$\lim_{n\to\infty} \mathbb{P}\left((s_{\max} - \mu)\frac{n^{2/3}}{\nu} \leq x\right) = F_d(x)$$

(c) $t_1 > 1+\sqrt{c}$ and $t_1 = \cdots = t_d$

$$\lim_{n\to\infty} \mathbb{P}\left((s_{\max} - \mu')\frac{n^{1/2}}{\nu'} \leq x\right) = G_d(x)$$