Random Unitary Matrices in Wireless Communications

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Random Unitary Matrices in Wireless Communications

General Multiple Input Multiple Output System

Model representing downlink MC-CDMA or Linear Precoded OFDM...

$\mathbf{y}$	$=$	$\mathbf{H}_N$	$\mathbf{U}_{N,K}$	$\mathbf{s}$	$+$	$\mathbf{n}$
Received signal		channel	code matrix	emitted signal		AWGN
$N \times 1$		$N \times N$	$N \times K$	$K \times 1$		$\sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I}_N)$

$$\mathbf{U}_{N,K} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{V}_{N,K-1} \end{bmatrix}, \quad \mathbf{H}_N = \begin{bmatrix} h_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & h_N \end{bmatrix} \quad \mathbf{s} = \begin{bmatrix} s_1 \\ \mathbf{x} \end{bmatrix}$$

The goal is to detect $\mathbf{s}$.

Receiver Structure

Model example :

$$\mathbf{y} = \mathbf{H}_N \mathbf{u}_1 s_1 + \mathbf{H}_N \mathbf{V}_{N,K-1} \mathbf{x} + \mathbf{n}$$

If $\mathbf{H} = \mathbf{I}$,

$$\begin{aligned} \hat{s}_1 &= \mathbf{u}_1^H \mathbf{y} \\ &= s_1 + \mathbf{u}_1^H \mathbf{n} \end{aligned}$$

Signal to Noise Ratio :

$$\begin{aligned} \text{SNR} &= \frac{\mathbb{E}(\mid s_1 \mid^2)}{\mathbb{E}(\mid \mathbf{u}_1^H \mathbf{n} \mid^2)} \\ &= \frac{1}{\sigma^2} \end{aligned}$$

What happens if $\mathbf{H} \neq \mathbf{I}$?

MMSE Receiver

$$\begin{aligned}\mathbf{y} &= \mathbf{H}_N\mathbf{u}_1 s_1 + \mathbf{H}_N\mathbf{V}_{N,K-1}\mathbf{x} + \mathbf{n}\\ &= \mathbf{H}_N\mathbf{u}_1 s_1 + \mathbf{n}'.\end{aligned}$$

$$\mathbb{E}(\mathbf{n}'\mathbf{n}'^H) = (\mathbf{H}_N\mathbf{V}_{N,K-1}\mathbf{V}_{N,K-1}^H\mathbf{H}_N^H + \sigma^2\mathbf{I}) = \mathbf{Q}\Lambda\mathbf{Q}^H$$

Whitening filter :

$$\begin{aligned}\tilde{\mathbf{y}} = \mathbf{\Lambda}^{-\frac{1}{2}}\mathbf{Q}^H\mathbf{y} &= \mathbf{\Lambda}^{-\frac{1}{2}}\mathbf{Q}^H\mathbf{u}s_1 + \mathbf{\Lambda}^{-\frac{1}{2}}\mathbf{Q}^H\mathbf{n}'\\ &= \mathbf{g}s_1 + \mathbf{b}\end{aligned}$$

$\mathbf{b}$ is a white Gaussian noise.

MMSE Receiver

$$\tilde{\mathbf{y}} = \mathbf{g}s_1 + \mathbf{b}$$

$\mathbf{g} = \mathbf{\Lambda}^{-\frac{1}{2}}\mathbf{Q}^H\mathbf{u}_1$

The output SINR is maximized with :

$$\mathbf{g}^H\tilde{\mathbf{y}} = \mathbf{g}^H\mathbf{g}s_1 + \mathbf{g}^H\mathbf{b}$$

Signal to Interference plus Noise Ratio (SINR) :

$$\begin{aligned} \mathrm{SINR}_{\mathbf{u}_1} &= \frac{(\mathbf{g}^H\mathbf{g})^2\mathbb{E}(\mid s_1 \mid^2)}{\mathbf{g}^H\mathbf{g}} \\ &= \mathbf{g}^H\mathbf{g} = \mathbf{u}_1^H\mathbf{H}_N^H\left(\mathbf{H}_N\mathbf{V}_{N,K-1}\mathbf{V}_{N,K-1}^H\mathbf{H}_N^H + \sigma^2\mathbf{I}_N\right)^{-1}\mathbf{H}_N\mathbf{u}_1 \end{aligned}$$

The SINR expression is not tractable and depends strongly on the choice of the orthogonal code sequence of $\mathbf{U}_{N,K}$.

The main trick in the i.i.d case

Suppose the matrix $\mathbf{U}_{N,K} = [U_{ij}]$ has i.i.d elements, $E[U_{ij}] = 0$, $E\left[U_{ij}^2\right] = 1/N$.
Example : IS95.

The important lemma :
$\mathbf{z}_N$ vector $N \times 1$ with i.i.d elements. Each element : zero mean and variance $1/N$.
$\mathbf{X}_N$ matrix $N \times N$ independent of $\mathbf{z}_N$. Then, under some assumptions,

$$\mathbf{z}_N^T \mathbf{X}_N \mathbf{z}_N - \frac{1}{N} \mathrm{trace}\left(\mathbf{X}_N\right) \ \rightarrow \ 0 \ \text{ a.s.}$$

when $N \rightarrow \infty$.

The main trick in the i.i.d case

The SINR can be derived the same way for any symbol :

$$\begin{aligned} \mathrm{SINR}_{\mathbf{u}_k} &= \mathbf{u}_k^H \mathbf{H}_N^H \left(\mathbf{H}_N \mathbf{V}_{N,K-1} \mathbf{V}_{N,K-1}^H \mathbf{H}_N^H + \sigma^2 \mathbf{I}_N \right)^{-1} \mathbf{H}_N \mathbf{u}_k \\ &= \mathbf{u}_k^H (\mathbf{A}_{N,K})^{-1} \mathbf{u}_k^H \end{aligned}$$

with $A_{N,K} = \left(\mathbf{V}_{N,K-1} \mathbf{V}_{N,K-1}^H + \sigma^2 \left(\mathbf{H}_N \mathbf{H}_N^H \right)^{-1} \right)$

Application : $\mathbf{u}_k$ et $\mathbf{V}_{N,K}$ independent, so

$$\mathrm{SINR}_{\mathbf{u}_k} - \frac{1}{N} \mathrm{trace} \left(\mathbf{A}_{N,K} \right)^{-1} \rightarrow 0 \text{ a.s.}$$

The main trick in the i.i.d case

$$\mathbf{u}_k^H \mathbf{A}_{N,K}^{-1} \mathbf{u}_k - \frac{\text{trace}(\mathbf{A}_{N,K}^{-1})}{N} \to 0$$

Depends on the empirical eigenvalue distribution of $d\rho_N(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i(\mathbf{A}_{N,K}))$ of $\mathbf{A}_{N,K}$.

Associated cumulative distribution function $\rho_N(\lambda) = \frac{\text{number of eigenvalues} \leq \lambda}{N}$.

$$\int \phi(\lambda) d\rho_N(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \phi(\lambda_i(\mathbf{A}_{N,K})) \overset{N\to\infty}{\to} \int \phi(\lambda) d\theta(\lambda)$$

θ deterministic, can be calculated (using Silverstein and Bai, 95), $\phi(\lambda) = \frac{1}{\lambda}$

The main trick in the i.i.d case

$$A_{N,K} = \left(\mathbf{V}_{N,K-1}\mathbf{V}_{N,K-1}^H + \sigma^2\left(\mathbf{H}_N\mathbf{H}_N^H\right)^{-1}\right)$$

(Silverstein and Bai, 95)

Consider the $N \times N$ matrix

$$\mathbf{A} = \mathbf{U}\mathbf{T}\mathbf{U}^H + \mathbf{C}$$

- $\mathbf{U}$ of dimension $N \times K$ with iid zero mean elements of variance $1/N$
- $\mathbf{T}$ is a real diagonal matrix. The diagonal elements have a limiting eigenvalue distribution ν.
- $\mathbf{C}$ is hermitian with a limiting eigenvalue distribution μ.
- $\mathbf{U}$, $\mathbf{T}$ et $\mathbf{C}$ are independent.
- $N \to \infty$, $K/N \to \alpha > 0$.

$\mathbf{A}$ has a limiting eigenvalue distribution ρ for which the Cauchy-Stieltjes transform $G_\rho(z)$ is the unique solution of the implicit equation

$$G_\rho(z) = G_\mu\left(z - \alpha \int \frac{t\nu(dt)}{1 + tG_\rho(z)}\right) \tag{1}$$

THE MAIN TRICK IN THE I.I.D CASE

Based on the assumption that :

$$\lim_{N\to+\infty} \frac{1}{N}\sum_{k=1}^{N} f(|h_k|^2) = E(f(|h|^2)) = \int_{\mathbb{R}+} f(t)p(t)\ dt \text{ almost surely.}$$

with $p(t)$ has a compact support,

When $N \to \infty$ and $K/N \to \alpha$, the SINR_k at the k_{th} output of a MMSE equalizer converges almost surely to a value SINR unique solution of the equation :

$$\int_0^{\infty} \frac{t}{\alpha t + \sigma^2 \mathrm{SINR} + \sigma^2}\ p(t)\ dt = \frac{\mathrm{SINR}}{\mathrm{SINR}+1}\ .$$

Our context

$\mathrm{U}_{N\times K}$ **random isometric Precoder :** i.i.d assumption does not fit to our case since orthogonality is generally assumed. Intuitively, isometric codes will perform better than i.i.d ones.

isometric case model : extract K columns from any unitary Haar distributed matrix.

The Haar distribution : For each Q deterministic unitary, $\mathbf{\Theta}$ and $Q\mathbf{\Theta}$ have the same distribution

Why the Haar distribution ? $\rightarrow$ Free probability results can apply

Our context

HAAR DISTRIBUTIONS

Generation :$\boldsymbol{\Theta} = \mathbf{X}(\mathbf{X}^{\mathbf{H}}\mathbf{X})^{-\mathbf{1/2}}$ where $\mathbf{X} = [x_{i,j}]_{1 \leq i,j \leq N}$ has i.i.d complex Gaussian unit variance entries.

To see this, notice that for each constant unitary matrix $\mathbf{U}$,

$$\mathbf{UX}(\mathbf{X}^H\mathbf{X})^{-1/2} = \mathbf{UX}[(\mathbf{UX})^H\mathbf{UX}]^{-1/2} .$$

Since the probability distribution of $\mathbf{X}$ and $\mathbf{UX}$ coincide, matrices $\mathbf{X}(\mathbf{X}^H\mathbf{X})^{-1/2}$ and $\mathbf{UX}((\mathbf{UX})^H\mathbf{UX})^{-1/2}$ have the same distribution. The above equality thus implies that $\mathbf{UX}(\mathbf{X}^H\mathbf{X})^{-1/2}$ and $\mathbf{X}(\mathbf{X}^H\mathbf{X})^{-1/2}$ are identically distributed.

Our context

HAAR DISTRIBUTIONS

There is another way for building Haar distributed unitary matrices that will be useful to our context. Instead of multiplying $\mathbf{X}$ by the inverse of the Hermitian square root of $\mathbf{X}^H\mathbf{X}$, one can introduce the uniquely defined upper triangular matrix with positive diagonal elements $\mathbf{Q}(\mathbf{X})$ defined by

$$\mathbf{X}^H\mathbf{X} = \mathbf{Q}(\mathbf{X})^H\mathbf{Q}(\mathbf{X}) \; .$$

The unitary matrix $\mathbf{V}(\mathbf{X})$ defined by

$$\mathbf{V}(\mathbf{X}) = \mathbf{X}\mathbf{Q}(\mathbf{X})^{-1} \tag{2}$$

is also Haar distributed. To see this, we first remark that for each constant unitary matrix $\mathbf{U}$, the probability distribution of $\mathbf{V}(\mathbf{X})$ and of $\mathbf{V}(\mathbf{UX})$ coincide. But, it is obvious that $\mathbf{Q}(\mathbf{UX}) = \mathbf{Q}(\mathbf{X})$, so that $\mathbf{UV}(\mathbf{X}) = \mathbf{V}(\mathbf{UX})$. Therefore, the probability distribution of $\mathbf{V}(\mathbf{X})$ and of $\mathbf{UV}(\mathbf{X})$ coincide. Remark that the columns of $\mathbf{V}(\mathbf{X})$ are obtained by a Gram-Schmidt orthogonalization of the column space of $\mathbf{X}$.

Our context

HAAR DISTRIBUTIONS

Useful Properties :

$$\sum_{k=1}^{N} |\theta_{ik}|^2 = \sum_{i=1}^{N} |\theta_{ik}|^2 = 1, \quad \text{for all } 1 \leq i, k \leq N$$

$$\sum_{l=1}^{N} \theta_{il}\theta_{lk}^* = 0, \quad \text{for all } i \neq k$$

$$\mathbb{E}\left[|\theta_{1i}|^2 |\theta_{ki}|^2\right] = \frac{1}{N(N+1)}, \; k > 1$$

$$\mathbb{E}\left[\theta_{1i}^*\theta_{ki}\theta_{1l}\theta_{kl}\right] = -\frac{1}{N(N^2-1)}, \; k > 1, \; i \neq l$$

Asymptotic Isometric Performance

Based on the assumption that :

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{k=1}^{N} f(|h_k|^2) = E(f(|h|^2)) = \int_{\mathbb{R}+} f(t)p(t)\ dt \text{ almost surely.}$$

with $p(t)$ has a compact support,

When $N \to \infty$ and $K/N \to \alpha \leq 1$, the SINR SINR_k at the k_{th} output of a MMSE equalizer converges almost surely to a value SINR unique solution of the equation :

$$\int_0^{\infty} \frac{t}{\alpha t + \sigma^2(1-\alpha)\mathrm{SINR} + \sigma^2}\ p(t)\ dt = \frac{\mathrm{SINR}}{\mathrm{SINR}+1}\ .$$

Analysis of communication performances

DOWNLINK CDMA

- **i.i.d codes :**

$$\text{SINR} = \mathbf{E}_{|h|^2}\left(\frac{1}{\frac{\alpha}{\text{SINR}+1} + \frac{\sigma^2}{|h|^2}}\right)$$

- **orthogonal codes :**

$$\text{SINR} = \mathbf{E}_{|h|^2}\left(\frac{1}{\frac{\alpha}{\text{SINR}+1} + \frac{\sigma^2}{|h|^2}(1-\alpha\frac{\text{SINR}}{(1+\text{SINR})})}\right)$$

- **Conclusion :** For a target SINR, performance of isometric precoded system with noise variance σ^2 equivalent to i.i.d. one with noise variance $(1-\alpha\frac{\text{SINR}}{1+\text{SINR}})\sigma^2$

Sketch of Proof

Step 1 : show $\mathrm{SINR}_{\mathbf{u}_k} \rightarrow_{a.s} \mathrm{SINR}$ deterministic.

$$\mathrm{SINR}_{\mathbf{u}_k} = \mathbf{u}_k^H \mathbf{B}_N \mathbf{u}_k$$

,

$$\mathbf{B}_N = \mathbf{H}_N^H \left(\mathbf{H}_N \mathbf{V}_{N,K} \mathbf{V}_{N,K}^H \mathbf{H}_N^H + \sigma^2 \mathbf{I}_N \right)^{-1} \mathbf{H}_N$$

Main difference between i.i.d. and isometric case : $\mathbf{u}_k$ and $\mathbf{B}_N$ are not independent.

$$\mathrm{SINR}_{\mathbf{u}_k} - \frac{1}{N-K} \mathrm{tr}((I - \mathbf{V}\mathbf{V}^H) \mathbf{B}_N (I - \mathbf{V}\mathbf{V}^H) \rightarrow 0$$

$(I - \mathbf{V}_N \mathbf{V}_N^H) \mathbf{B}_N (I - \mathbf{V}_N \mathbf{V}_N^H)$ can be approximated by a non commutative polynomial of $\mathbf{V}_{N,K} \mathbf{V}_{N,K}^H$ and $\mathbf{H}_N^H \mathbf{H}_N$.

$\mathbf{V}_{N,K} \mathbf{V}_{N,K}^H$ and $\mathbf{H}_N^H \mathbf{H}_N$ are almost surely asymptotically free $\Rightarrow \mathrm{SINR}_{\mathbf{u}_k} \rightarrow \mathrm{SINR}$

Sketch of Proof

Step 2 : calculation of SINR $\rightarrow \overline{\eta} = \frac{SINR}{1+SINR}$

$$\begin{aligned}
\overline{\eta} &= \lim_{K\to\infty} \frac{1}{K}\sum_{k=1}^{K} \eta_{\mathbf{u}_k} \\
&= \lim_{K\to\infty} \frac{1}{K}\mathrm{tr}\left((\mathbf{H}_N\mathbf{V}_{N,K}\mathbf{V}_{N,K}^H\mathbf{H}_N^H + \sigma^2\mathbf{I}_N)^{-1}\mathbf{H}_N\mathbf{V}_{N,K}\mathbf{V}_{N,K}^H\mathbf{H}_N^H\right) \\
&= \frac{1}{\alpha}\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\frac{\lambda_i}{\lambda_i+\sigma^2} = \frac{1}{\alpha}\lim_{N\to\infty}\int\frac{\lambda}{\lambda+\sigma^2}d\rho_N(\lambda)
\end{aligned}$$

ρ_N is the empirical eigenvalue distribution of $\mathbf{H}_N\mathbf{U}_{N,K}\mathbf{U}_{N,K}^H\mathbf{H}_N^H$
Step 3 Free probability results show that ρ_N converges almost surely to a compactly supported measure θ, which can be calculated explicitly.
Step 4 $\frac{SINR}{1+SINR} = \frac{1}{\alpha}\int\frac{t}{t+\sigma^2}\, d\theta(t)$

Derivation of Θ

Context : $\mathbf{G}_N = \mathbf{H}_N \mathbf{H}_N^H$, $\mathbf{C}_N = \mathrm{Diag}(I_K, 0)$ and $\mathbf{U}_{N\times K}\mathbf{U}_{N\times K}^H = \mathbf{\Theta}_N \mathbf{C}_N \mathbf{\Theta}_N^H$, $\mathbf{\Theta}_N$ Haar distributed $N \times N$ random unitary matrix, with respective empirical eigenvalue distribution μ_N and ν_N.

Question : Distribution of $\mathbf{G}_N \times \mathbf{\Theta}_N \mathbf{C}_N \mathbf{\Theta}_N^H$?

Answer : $\mathbf{G}_N$ and $\mathbf{\Theta}_N \mathbf{C}_N \mathbf{\Theta}_N^H$ asymptotically free almost everywhere :

When $\mu_N \to_{a.s} \mu$ and $\nu_N \to_{a.s} \nu$, the empirical eigenvalue distribution of $\mathbf{G}_N \times \mathbf{C}_N$ converges to $\mu \boxtimes \nu$, the free multiplicative convolution of the distributions μ and ν.

– $d\mu(t) = p(t)dt$ and $\nu(t) = \alpha\delta(t-1) + (1-\alpha)\delta(t-0)$

Random Unitary Matrices

THE MAIN TRICK IN THE ISOMETRIC CASE

Another important lemma :
Let $\mathbf{U}$ be a $K < N$ columns of an $N \times N$ Haar distributed random matrix and suppose $\mathbf{u}$ is a column of $\mathbf{U}$. Let $\mathbf{B}_N$ be a $N \times N$ random matrix, which is a function of all columns of $\mathbf{U}$ except $\mathbf{u}$ and $\mathbf{B} = \sup_N \| \mathbf{B}_N \| < \infty$, then

$$\mathbb{E} \mid \mathbf{u}^H \mathbf{B}_N \mathbf{u} - \frac{1}{N-K} \mathrm{Tr}(\mathbf{\Pi} \mathbf{B}_N) \mid^4 \leq \frac{C}{N^2}$$

Where $\mathbf{\Pi} = \mathbf{I}_N - \mathbf{U}\mathbf{U}^N - \mathbf{u}\mathbf{u}^H$ and C is a deterministic finite constant which depends only on $\mathbf{B}$ and $\alpha = \frac{K}{N}$.

Remark : We will only prove the result in our case.

The main trick in the isometric case

$\mathbf{U} = [\mathbf{u}_1, ..., \mathbf{u}_K]$ extracted from a Haar matrix.

Let $\mathbf{X} = [\mathbf{x}_1, ..., \mathbf{x}_N]$ be an i.i.d Gaussian matrix.

$$\mathbf{u}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|}$$

$$\mathbf{u}_2 = \frac{\mathbf{\Pi}_1 \mathbf{x}_2}{\|\mathbf{\Pi}_1 \mathbf{x}_2\|}$$

where $\mathbf{\Pi}_1 = \mathbf{I} - \frac{\mathbf{x}_1 \mathbf{x}_1^H}{\|\mathbf{x}_1\|^2}$

$$\mathbf{u}_K = \frac{\mathbf{\Pi}_{K-1} \mathbf{x}_K}{\|\mathbf{\Pi}_{K-1} \mathbf{x}_K\|}$$

Note that $\mathbf{\Pi}_{k-1}$ and $\mathbf{x}_k$ are stochastically independent.

The main trick in the isometric case

Suppose $\mathbf{B}_N$ is a function of $\mathbf{u}_2, ..., \mathbf{u}_K$.

$$\mathbf{u}_1^H \mathbf{B}_N \mathbf{u}_1 - \frac{1}{N-K} \mathrm{Tr}(\mathbf{\Pi} \mathbf{B}_N) = f_1(\mathbf{u}_1, ..., \mathbf{u}_K)$$

with

$$\mathbf{\Pi} = \mathbf{I} - \mathbf{U}\mathbf{U}^H - \mathbf{u}_1 \mathbf{u}_1$$

Let $\mathbf{P}$ be a permutation matrix exchanging column 1 with K.

Define $\tilde{\mathbf{U}} = \mathbf{U}\mathbf{P}$.

It is clear that $f_K(\tilde{\mathbf{u}}_1, ..., \tilde{\mathbf{u}}_K) = f_1(\mathbf{u}_1, ..., \mathbf{u}_K)$

Since $\tilde{\mathbf{U}}$ and $\mathbf{U}$ have the same distribution, random variables $f_1(\mathbf{u}_1, ..., \mathbf{u}_K)$ and $f_K(\mathbf{u}_1, ..., \mathbf{u}_K)$ are identically distributed which implies :

$$\mathbb{E}\| f_1(\mathbf{u}_1, ..., \mathbf{u}_K)\|^4 = \mathbb{E}\| f_K(\mathbf{u}_1, ..., \mathbf{u}_K)\|^4$$

In the following, we focus therefore on column $\mathbf{u}_K$ of $\mathbf{U}$.

The main trick in the isometric case

$\mathbb{E}\|f_1(\mathbf{u}_1, ..., \mathbf{u}_K)\|^4$ does not depend

on the particular way $\mathbf{U}$ is generated provided it is Haar distributed. We can therefore focus on our construction of Haar distributed matrices.

$$\begin{aligned} e_N &= \mathbf{u}_K^H \mathbf{B}_N \mathbf{u}_K - \frac{1}{N-K} \mathrm{Tr}(\mathbf{\Pi} \mathbf{B}_N) \\ &= \frac{x_K^H \Pi_{K-1}^H}{\|x_K^H \Pi_{K-1}^H\|} \mathbf{B}_N \frac{\Pi_{K-1} x_K}{\|\Pi_{K-1} x_K\|} - \frac{1}{N-K} \mathrm{Tr}(\mathbf{\Pi_{K-1}} \mathbf{B}_N) \\ &= e_{1,N} + e_{2,N} \end{aligned}$$

$$e_{1,N} = \frac{x_K^H \Pi_{K-1}^H \mathbf{B}_N \Pi_{K-1} x_K}{\|x_K \Pi_{K-1}\|^2} - \frac{x_K^H \Pi_{K-1}^H \mathbf{B}_N \Pi_{K-1} x_K}{N-K}$$

$$e_{2,N} = \frac{x_K^H \Pi_{K-1}^H \mathbf{B}_N \Pi_{K-1} x_K}{N-K} - \frac{1}{N-K} \mathrm{Tr}(\mathbf{\Pi_{K-1}} \mathbf{B}_N)$$

THE MAIN TRICK IN THE ISOMETRIC CASE

$$e_{2,N} = \frac{x_K^H \Pi_{K-1}^H \mathbf{B}_N \Pi_{K-1} x_K}{N-K} - \frac{1}{N-K} \mathrm{Tr}(\mathbf{\Pi_{K-1}} \mathbf{B}_N)$$

$$\begin{aligned}
\frac{x_K^H \Pi_{K-1}^H \mathbf{B}_N \Pi_{K-1} x_K}{N-K} &\simeq \frac{1}{N-K} \mathrm{Trace}(\Pi_{K-1} \mathbf{B}_N \Pi_{K-1}) \\
&= \frac{1}{N-K} \mathrm{Trace}(\Pi_{K-1}^2 \mathbf{B}_N) \\
&= \frac{1}{N-K} \mathrm{Trace}(\Pi_{K-1} \mathbf{B}_N)
\end{aligned}$$

THE MAIN TRICK IN THE ISOMETRIC CASE

$$e_{1,N} = \frac{x_K^H \Pi_{K-1}^H \mathbf{B}_N \Pi_{K-1} x_K}{\|x_K \Pi_{K-1}\|^2} - \frac{x_K^H \Pi_{K-1}^H \mathbf{B}_N \Pi_{K-1} x_K}{N-K}$$

We rewrite the form :

$$e_{1,N} = \frac{x_K^H \Pi_{K-1}^H \mathbf{B}_N \Pi_{K-1} x_K}{N-K} \left(\frac{N-K}{\|x_K \Pi_{K-1}\|^2} - 1 \right)$$

Let us first show that $\frac{x_K^H \Pi_{K-1}^H \mathbf{B}_N \Pi_{K-1} x_K}{N-K}$ is bounded.

As $e_{2,N}$ converges to 0 almost everywhere,

$$\frac{x_K^H \Pi_{K-1}^H \mathbf{B}_N \Pi_{K-1} x_K}{N-K} < 2 \frac{\mathrm{Trace}(\Pi_{K-1} \mathbf{B}_N)}{N-K} \leq 2 \|\Pi_{K-1} \mathbf{B}_N\|$$

for N large enough (for any matrix $\mathbf{X}$, $\mathrm{Trace}(\mathbf{X}) \leq \|\mathbf{X}\| \mathrm{Rank}(\mathbf{X})$ and the rank of $\Pi_{K-1} \mathbf{B}_N$ does not exceed $N-K$ **in our case**).

As $\sup_{N\in\mathbb{N}}\|\Pi_{K-1}^H\mathbf{B}_N\| < \infty$, $\frac{x_K^H\Pi_{K-1}^H\mathbf{B}_N\Pi_{K-1}x_K}{N-K}$ is bounded almost everywhere.

The main trick in the isometric case

$$e_{1,N} = \frac{x_K^H \Pi_{K-1}^H \mathbf{B}_N \Pi_{K-1} x_K}{N-K} \left(\frac{N-K}{\|\Pi_{K-1} x_K\|^2} - 1 \right)$$

Note that $\mathbf{\Pi_{K-1}}$ is independent of x_K.

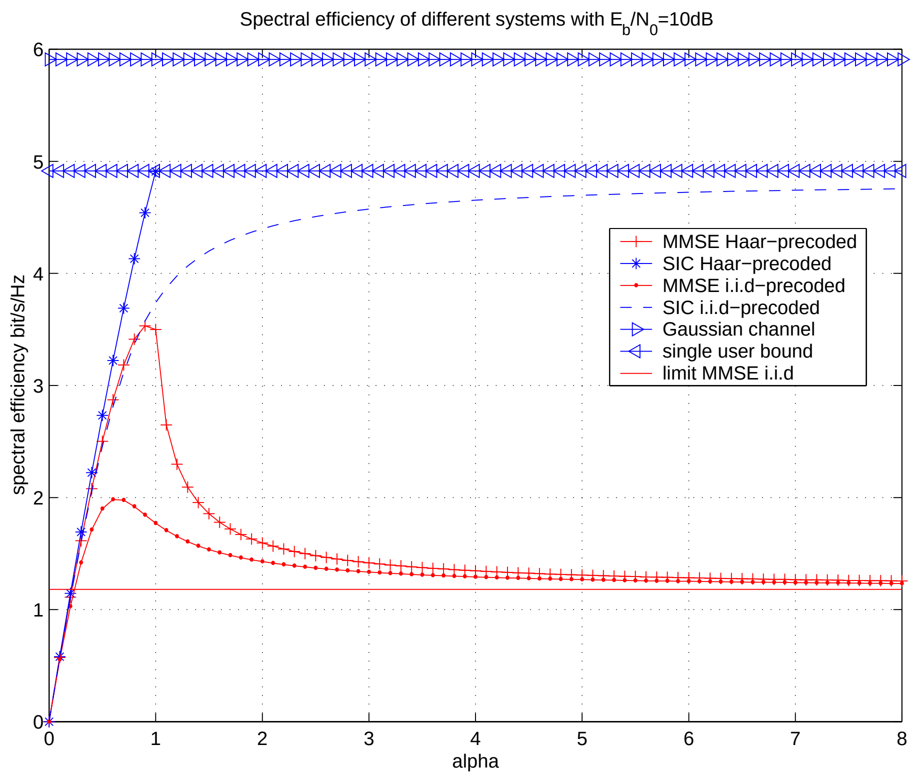
$\|\Pi_{K-1} x_K\|^2$ is χ^2 distributed with $2(N-K)$ degrees of freedom. Its probability density is the function : $\frac{t^{N-K-1}}{(N-k-1)!} e^{-t}$. A direct computation show that :

$$\mathbb{E} \mid \frac{N-K}{\|x_K \Pi_{K-1}\|^2} - 1 \mid^4 = 0((N-K)^{-2})$$

which coincides with $0((N)^{-2})$ if $\frac{K}{N} \to \alpha$.

Analysis of communication performances

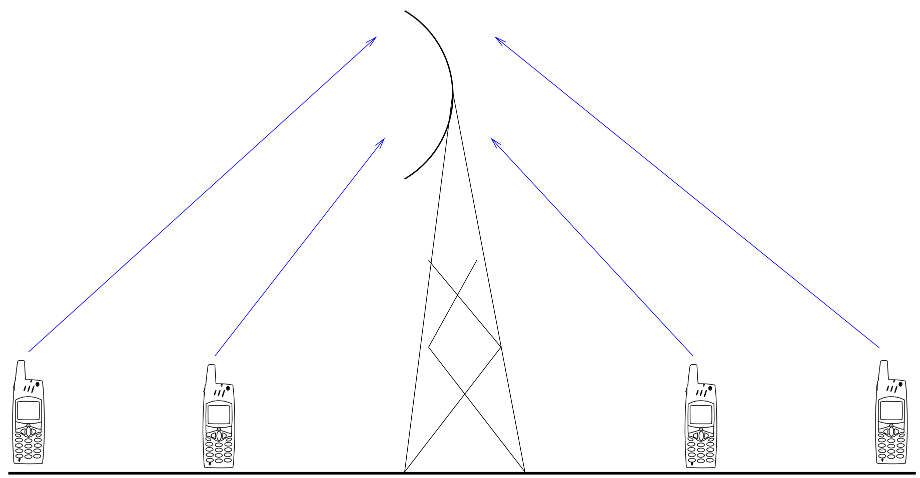
DOWNLINK CDMA



- For low loaded systems, non-linear (very complex) receivers do not make sense.
- Orthogonal codes impact the performance only at high loads.

Analysis of communication performances

UPLINK CDMA



- K users in the cell.
- **C**ode **D**ivision **M**ultiple **A**ccess (CDMA) : Simultaneous communication of all the users to the base station using different codes.

Analysis of communication performances

UPLINK CDMA : FLAT FADING MODEL

- The $N \times 1$ **received signal** vector $\mathbf{y}$ at the base station has the form :

$$\begin{aligned} \mathbf{y} &= h_1 \mathbf{v}_1 s_1 + h_2 \mathbf{v}_2 s_2 + \cdots + h_K \mathbf{v}_K s_K + \mathbf{n} \\ &= \mathbf{VHs} + \mathbf{n} \end{aligned}$$

- $\mathbf{s} = (s_1, \ldots, s_K)$ is the emitted symbol vector.
- $\mathbf{v}_k$ is the $N \times 1$ k-th user code.
- $\mathbf{H} = \mathrm{diag}[h_1, ..., h_K]$ with h_k flat fading of user k.
- N is the spreading length.
- $\mathbf{n}$ is a $N \times 1$ white complex gaussian noise vector of variance σ^2.

Analysis of communication performances

UPLINK CDMA : FLAT FADING MODEL

- SINR of user k :

$$\text{SINR} = \mid h_k \mid^2 \beta$$

where

$$\beta = v_k^H \left(\sum_{i=1, i \neq k}^{K} \mid h_i \mid^2 v_i v_i^H + \sigma^2 \mathbf{I} \right)^{-1} v_k$$

.

- In the i.i.d case,

$$\begin{aligned} \beta & \rightarrow \frac{1}{N} \text{trace} \left(\mathbf{V}\mathbf{V}^H + \sigma^2 \mathbf{I} \right)^{-1} \\ & \rightarrow \int \frac{\rho(\lambda) d\lambda}{\lambda} \\ & \rightarrow G_\rho(0) \end{aligned}$$

Analysis of communication performances

UPLINK CDMA : FLAT FADING MODEL

- Application :

$$G_\rho(z) = G_\mu\left(z - \alpha \int \frac{t\nu(dt)}{1 + tG_\rho(z)}\right) \tag{3}$$

$\mu(\lambda) = \delta(\lambda - \sigma^2)$ and therefore $G_\mu(z) = \int \frac{1}{\sigma^2 - z}$.

$\beta = G_\rho(0)$.

$\nu(dt)$ is the limiting distribution of the fading.

$$\beta = G_\mu\left(-\alpha \int \frac{t\nu(dt)}{1 + t\beta}\right) = \frac{1}{\sigma^2 + \alpha \int \frac{t\nu(dt)}{1+t\beta}} \tag{4}$$

Analysis of communication performances

Uplink CDMA : flat fading model

- Performance measure :

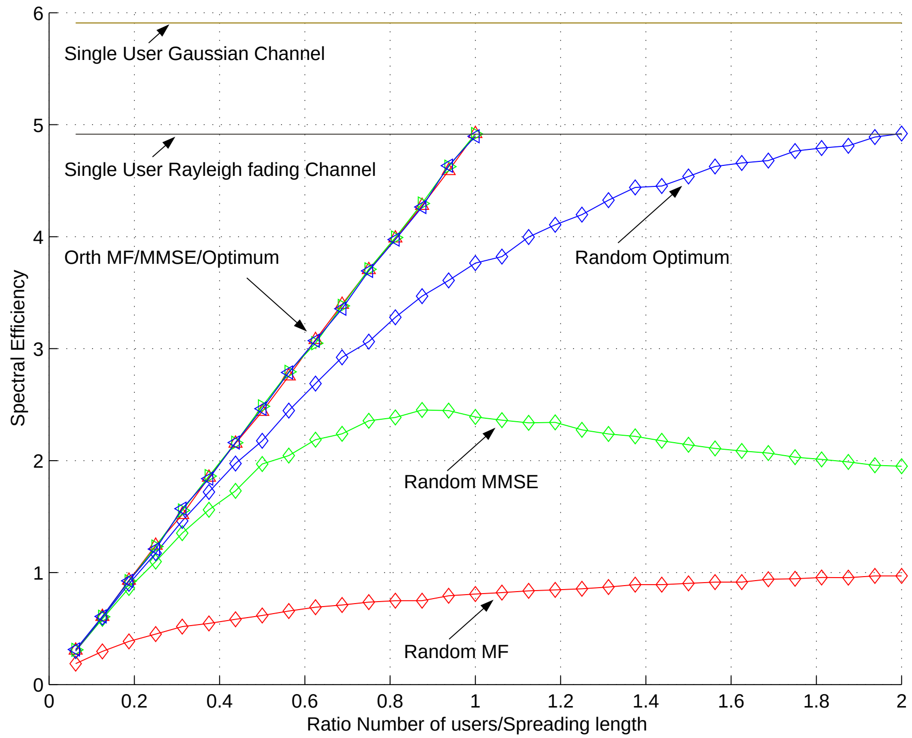
The spectral efficiency of the system is the number of bits/s/Hz that can be reliably transmitted.

$$\begin{aligned}\gamma &= \frac{1}{N}\sum_{i=1}^{K} C_i \\ &= \frac{1}{N}\sum_{i=1}^{K} \log_2(1+ \mid h_i \mid^2 \beta) \\ &= \alpha \mathbb{E}_h \left(\log_2(1+ \mid h \mid^2 \beta)\right)\end{aligned}$$

$$\frac{E_b}{N_0} = (C\sigma^2)^{-1}$$

Analysis of communication performances

Uplink CDMA : flat fading model



- Spectral efficiency is only a function of the load of the system, the statistics of the channel and noise variance.

Analysis of communication performances

Uplink CDMA : to synchronize or not to synchronize ?

- **Question :** When is it useful to use orthogonal uplink signaling ?
- **Answer :** Intuitively,
  - In the case of flat fading, orthogonality is preserved.
  - In the case of frequency-selective fading, orthogonality is destroyed.
- How can we translate this intuition into theoretical assessments ?
- In the uplink case, previous studies limited only to random i.i.d. signaling, with no fading (Shamai-Verdu, Tse-Hanly), flat fading (Shamai-Verdu), and frequency-selective fading (Tulino-Verdu).

Analysis of communication performances

UPLINK CDMA : TO SYNCHRONIZE OR NOT TO SYNCHRONIZE ?

- The $N \times 1$ **received signal** vector $\mathbf{y}$ at the base station has the form :

$$\mathbf{y} = \mathbf{H}_1\mathbf{v}_1 s_1 + \mathbf{H}_2\mathbf{v}_2 s_2 + \cdots + \mathbf{H}_K\mathbf{v}_K s_K + \mathbf{n}$$
$$= \big(\mathbf{H} \odot \mathbf{V}\big)\mathbf{s} + \mathbf{n}$$

– $\mathbf{s} = (s_1, \ldots, s_K)$ is the emitted symbol vector.
– $\mathbf{v}_k$ is the $N \times 1$ k-th user code.
– N is the spreading length.
– $\mathbf{H}_l$: Toeplitz matrix $N \times N$ associated with the channel transfer function $h^l(z) = \sum_{k=0}^{M} {h^l}_k z^{-k}$.
– $\mathbf{n}$ is a $N \times 1$ white complex gaussian noise vector of variance σ^2.

Analysis of communication performances

UPLINK CDMA : TO SYNCHRONIZE OR NOT TO SYNCHRONIZE ?

- **Code structure model :** Codes are orthogonal or i.i.d.
  - **Orthogonal codes :** User codes $\mathbf{v}_k$ are columns extracted from a Haar unitary (random) matrix $\mathbf{V}$.
  - **i.i.d. codes :** Entries of $\mathbf{V}$ are Gaussian i.i.d., with zero mean and variance $1/N$.
- **Channel model :** We will suppose that for the channel model

$$c_k(\tau) = \sum_{p=0}^{L-1} a_p(k)\phi(\tau - \tau_p(k))$$

  - The fading coefficients are i.i.d. Gaussian with

$$\mathbb{E}\left(\mid a_p(k) \mid^2\right) = \frac{\rho}{L}$$

  - The delays are uniformly distributed according to the bandwidth

$$\tau_p(k) = \frac{p}{W}$$

Analysis of communication performances

UPLINK CDMA : TO SYNCHRONIZE OR NOT TO SYNCHRONIZE ?

- In the asymptotic case, the SINR for the Matched filter is

$$\mathrm{SINR}^{\mathsf{orth}} = \frac{\sum_{p=0}^{L-1} \mid a_p \mid^2}{\sigma^2 + \alpha\rho\left(1 - \frac{1}{L}\right)}$$

$$\mathrm{SINR}^{\mathsf{iid}} = \frac{\sum_{p=0}^{L-1} \mid a_p \mid^2}{\sigma^2 + \alpha\rho}$$

- The SINR gain depends only on a few meaningful parameters : σ^2, α, ρ and L :

$$\frac{\mathrm{SINR}^{\mathsf{orth}}}{\mathrm{SINR}^{\mathsf{iid}}} = \frac{\sigma^2 + \alpha\rho}{\sigma^2 + \alpha\rho\left(1 - \frac{1}{L}\right)}$$

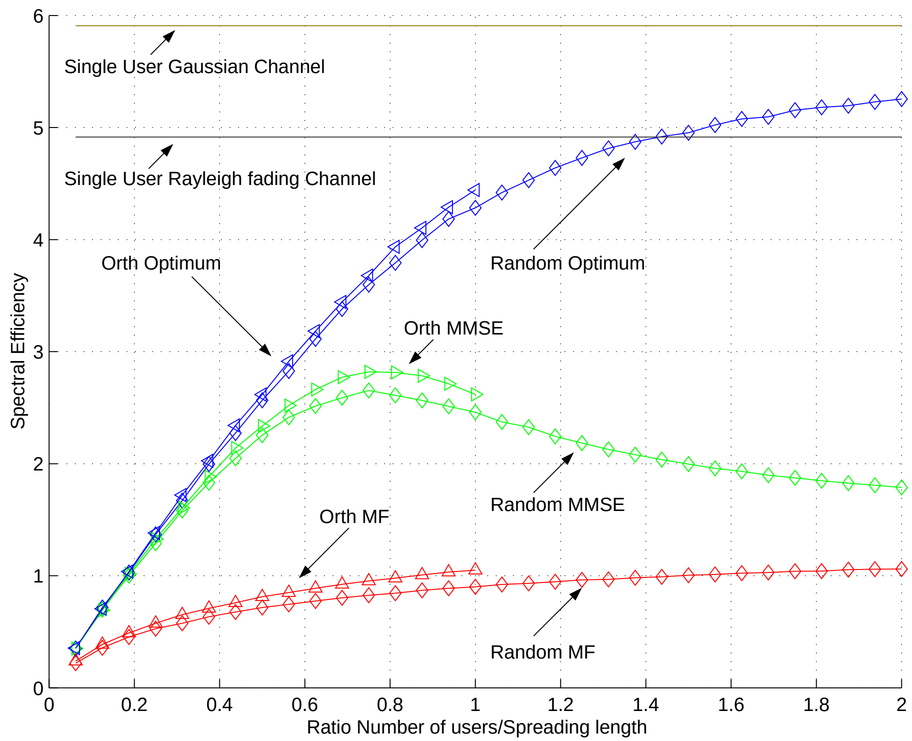
- When $\sigma^2 \to 0$,

$$\frac{\mathrm{SINR}^{\mathsf{orth}}}{\mathrm{SINR}^{\mathsf{iid}}} \to \frac{L}{L-1}$$

– In a two-path channel, gain of 3 dB ; in a 5-path channel, gain of less than 1 dB.

Analysis of communication performances

UPLINK CDMA : SOME RESULTS FOR OTHER RECEIVERS



- As L increases ($L = 1$ and $L = 5$), orthogonality gain decreases for any receiver.

Last slide

THANK YOU!