The Ramanujan Entire Function

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Ramanujan was a selfeducated college drop out who did some of the best mathematics of the twentieth century. He extensively worked on the

$$F(z) = 1 + \sum_{n=1}^{\infty} \frac{(-z)^n q^{n^2}}{(1-q)(1-q^2)\dots(1-q^n)},$$

which we refer to as the Ramanujan entire function. We demonstrate the significance of this function in number theory and analysis and give a new interpretation of the statement

$$1 + \sum_{n=1}^{\infty} \frac{z^n q^{n^2}}{(1-q)(1-q^2)\dots(1-q^n)} = \prod_{n=1}^{\infty} \left(1 + \frac{zq^{2n-1}}{1 - c_1 q^n - c_2 q^{2n} - \dots}\right)$$

in Ramanujan's lost note book.

The coefficients $c_1, c_2, \dots$ turned out to have very interesting paterns and many open problems with be mentioned.