Physical random matrices

Heavy nuclei and quantum mechanics

- Introduced by Wigner in 1950's to explain statistics of highly excited nuclei resonances.
- Global time reversal constrains the elements to real.
- No preferential basis: $P(X) = P(O^T X O)$, e.g. $P(X) \propto e^{-\mathrm{Tr}\, X^2/2}$. Orthogonal invariance.

In the case of no time reversal symmetry, elements will be complex. Unitary invariance.

In the case of a time reversal symmetry $T^2 = -1$ (relevant to a finite dimensional system with an odd number of spin 1/2 particles), $T = \mathbb{Z}_{2N} K$, where K denotes complex conjugation, and

$$\mathbb{Z}_{2N} = \mathbb{I}_N \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

- Matrix commuting with T must have the additional property

$$X = \mathbb{Z}_{2N} \bar{X} \mathbb{Z}_{2N}^{-1}.$$

- Viewed as an $N \times N$ matrix, X must have 2×2 blocks

$$\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}.$$

  Real quaternions.

- Invariance with respect to conjugation by $N \times N$ unitary matrices with real quaternion elements.

Dirac operators and QCD

Leads to random Hermitian matrices with a special block structure.

- Non-zero eigenvalues of the massless Dirac operator occur in pairs $\pm\lambda$.
- In the chiral basis, all eigenfunctions are also eigenfunctions of $i\gamma_5$, with eigenvalue $+1$ or -1. Matrix elements between eigenfunctions with same eigenvalue must vanish. Implies block structure, with zero blocks in top left and bottom right.
- Application to QCD requires Dirac operator has given number ν say of zero eigenvalues.
- Hence the structure

$$\begin{bmatrix} 0_{n\times n} & X_{n\times m} \\ X^\dagger_{m\times n} & 0_{m\times m} \end{bmatrix}$$

  where $n - m = \nu$.
- The positive eigenvalues of this matrix are given by the positive square root of the eigenvalues of $X^\dagger X$.

Question What is the eigenvalue distribution of $X^\dagger X$?

Random scattering matrices

Scattering within an irregular shaped domain, connected to a wave guide.

- Wave guide permits N distinct plane wave states.
- (Complex) amplitudes denoted $\vec{I}$ for incoming, $\vec{O}$ for outgoing states.
- Scattering matrix S,

$$S\vec{I} = \vec{O}.$$

  S must be unitary.

- Time reversal symmetry requires $T^{-1}ST = S^\dagger$.
- For $T^2 = 1$, implies $S = S^T$.
- For $T^2 = -1$ implies $S = \mathbb{Z}_{2N} S^T \mathbb{Z}_{2N}^{-1} =: S^D$.

Seek a measure on $\{S\}$ such that it is invariant under appropriate conjugations.

- For no time reversal symmetry $S \in U(N)$ with Haar measure

$$(d_H S) = (S^\dagger dS).$$

- For $S = S^T$, $S = U_N U_N^T$,

$$(d_H S) = ((U_N^T)^\dagger dS U_N^\dagger).$$

- For $S = S^D$, $S = U_{2N} U_{2N}^D$,

$$(d_H S) = ((U_{2N}^D)^\dagger dS U_{2N}^\dagger).$$

Quantum conductance problems

Scattering within a quasi one-dimensional conductor (lead).

- n available channels at left edge, m at right edge. At each end a reservoir causes current to flow.
- $\vec{I}_n$ left incoming states, $\vec{O}_n$ left outgoing states.
- $\vec{I}'_m$ right incoming states, $\vec{O}'_m$ right outgoing states.
- Scattering matrix S,

$$S \begin{bmatrix} \vec{I} \\ \vec{I}' \end{bmatrix} = \begin{bmatrix} \vec{O} \\ \vec{O}' \end{bmatrix}.$$

- Scattering matrix S,

$$S = \begin{bmatrix} r_{n\times n} & t'_{n\times m} \\ t_{m\times n} & r'_{m\times m} \end{bmatrix}.$$

- Landauer formula

$$G/G_0 = \mathrm{Tr}(t^\dagger t)$$

where $G_0 = 2e^2/h$ is twice the fundamental quantum unit of conductance.

Question What is the eigenvalue distribution of $t^\dagger t$?

Calculation of eigenvalue PDFs

Hermitian matrices

- $H = [x_{jk}]_{j,k=1,\dots,N}$ real symmetric matrix — $N(N+1)/2$ independent variables.
- Diagonalization $H = OLO^T$.
- Want Jacobian for change of variables from independent elements of X to eigenvalues $\lambda_1, \dots, \lambda_N$ and $N(N-1)/2$ linearly independent variables formed out of O.

Wedge products

Define

$$du_1 \wedge \cdots \wedge du_n := \det[du_i(\vec{r}_j)]_{i,j=1,\dots,n}.$$

Suppose we change variables $\{u_1, \dots, u_N\}$ to $\{v_1, \dots, v_N\}$. Since

$$du_i = \sum_{l=1}^{n} \frac{\partial u_i}{\partial v_l} dv_l$$

and

$$\Big[\sum_{l=1}^{n} \frac{\partial u_i}{\partial v_l} dv_l(\vec{r}_j)\Big]_{i,j=1,\dots,n} = \Big[\frac{\partial u_i}{\partial v_j}\Big]_{i,j=1,\dots,n} [dv_i(\vec{r}_j)]_{i,j=1,\dots,n}$$

it follows

$$du_1 \wedge \cdots \wedge du_n = \det\Big[\frac{\partial u_i}{\partial v_j}\Big]_{i,j=1,\dots,n} dv_1 \wedge \cdots \wedge dv_n$$

thus allowing the Jacobian to be read off.

Let H be real symmetric, and let dH denote the matrix of differentials. We have

$$dH = dO\,LO^T + OdL\,O^T + OLdO^T.$$

Noting from $O^T O = \mathbb{I}_N$ that $dO^T\,O = -O^T dO$ it follows from this that

$$O^T dH\,O = O^T dO\,L - LO^T dO + dL =$$
$$\begin{bmatrix} d\lambda_1 & (\lambda_2-\lambda_1)\vec{o}_1^T\cdot d\vec{o}_2 & \cdots & (\lambda_N-\lambda_1)\vec{o}_1^T\cdot d\vec{o}_N \\ (\lambda_2-\lambda_1)\vec{o}_1^T\cdot d\vec{o}_2 & d\lambda_2 & \cdots & (\lambda_N-\lambda_2)\vec{o}_2^T\cdot d\vec{o}_N \\ \vdots & \vdots & & \vdots \\ (\lambda_N-\lambda_1)\vec{o}_1^T\cdot d\vec{o}_N & (\lambda_N-\lambda_2)\vec{o}_2^T\cdot d\vec{o}_N & \cdots & d\lambda_N \end{bmatrix}$$

Require the following result.

Proposition 1. *Let A and M be $N \times N$ matrices, where A is non-singular. For A real ($\beta = 1$), complex ($\beta = 2$) or real quaternion ($\beta = 4$), and M real symmetric ($\beta = 1$), complex Hermitian ($\beta = 2$) or quanternion real Hermitian ($\beta = 4$)*

$$(A^\dagger dM\,A) = \Big(\det A^\dagger A\Big)^{\beta(N-1)/2+1}(dM).$$

Applying the proposition with $\beta = 1$ to the LHS, and taking the wedge product directly on the RHS gives

$$(dH) = \prod_{1 \le j < k \le N} (\lambda_k - \lambda_j) \bigwedge_{j=1}^{N} d\lambda_j (O^T dO).$$

Scaling argument

- There are $N(N+1)/2$ independent differentials in (dX), and so

$$(dcH) = c^{N(N+1)/2}(dH).$$

- Since $cH = OcLO^T$, (dcH) is a homogeneous polynomial of degree $N(N-1)/2$ in $\{\lambda_j\}$.
- Because the probability of repeated eigenvalues occurs with zero probability, (dX) must vanish for $\lambda_j = \lambda_k$.
- According to the last two facts, the dependence on the eigenvalues is proportional to $\prod_{1 \le j < k \le N}(\lambda_k - \lambda_j)$.

For complex Hermitian matrices,

$$(dH) = \prod_{1 \le j < k \le N} (\lambda_k - \lambda_j)^2 \bigwedge_{j=1}^{N} d\lambda_j (U^\dagger dU).$$

For Hermitian matrices with real quaternion elements

$$(dH) = \prod_{1 \le j < k \le N} (\lambda_k - \lambda_j)^4 \bigwedge_{j=1}^{N} d\lambda_j (S^\dagger dS).$$

Degeneracies

Consider a complex Hermitian matrix $[x_{jk} + iy_{jk}]_{j,k=1,\dots,N}$. It has the same eigenvalues as the $2N \times 2N$ real symmetric matrix

$$\left[\begin{bmatrix} x_{jk} & y_{jk} \\ -y_{jk} & x_{jk} \end{bmatrix}\right]_{j,k=1,\dots,N}.$$

Scaling argument applied to a doubly degenerate real symmetric matrix gives that the dependence on the eigenvalues is proportional to $\prod_{1 \le j < k \le N} (\lambda_k - \lambda_j)^2$.

Viewed as a $2N \times 2N$ complex matrix, the $N \times N$ real quaternion matrix $[q_{jk}]_{j,k=1,\dots,N}$ is doubly degenerate. Now viewing this complex matrix as a $4N \times 4N$ real symmetric matrix, we have a four fold degeneracy.

The scaling argument applied to a four fold degenerate real symmetric matrix gives that the dependence on the eigenvalues is proportional to $\prod_{1 \le j < k \le N} (\lambda_k - \lambda_j)^4$.

Wishart matrices

For X a random $n \times m$ rectangular matrix ($n \geq m$), $A := X^\dagger X$ is referred to as a Wishart matrix.

Relevance to multivariate statistics comes from noting

$$\frac{1}{n} X^T X = [\frac{1}{n} \sum_{j=1}^n x_{k_1}^{(j)} x_{k_2}^{(j)}]_{k_1,k_2=1,\dots,m} \approx [\langle x_{k_1} x_{k_2} \rangle]_{k_1,k_2=1,\dots,m}.$$

Require the Jacobian for changing variables from the elements of X to the elements of A (and other associated variables).

Proposition 2. *Let the $n \times m$ matrix X have real ($\beta = 1$), complex ($\beta = 2$) or real quaternion ($\beta = 4$) elements, and suppose it has a PDF of the form $F(X^\dagger X)$. The PDF of $A := X^\dagger X$ is then proportional to*

$$F(A)(\det A)^{(\beta/2)(n-m+1-2/\beta)}.$$

For the proof we will use a scaling argument, due to Olkin. This in turn makes essential use of the result

$$(A^\dagger dM\, A) = \Big(\det A^\dagger A\Big)^{\beta(N-1)/2+1} (dM).$$

noted in Proposition 1 above. Further, for B a $m \times m$ fixed matrix, and $X = YB$, we need the result that

$$(dX) = (\det B^\dagger B)^{\beta n/2} (dY).$$

This follows by noting that the Jacobian for $\vec{x}^T = \vec{y}^T B$ is $(\det B^\dagger B)^{\beta/2}$.

Proof

- The PDF of A must equal $F(A)h(A)$ for some h.
- Write $A = B^\dagger V B$ where V is positive definite. According to Prop. 1, PDF of V equals

$$F(B^\dagger V B)h(B^\dagger V B)\det(B^\dagger B)^{(\beta/2)(m-1+2/\beta)}.$$

- Let $X = YB$, where Y is such that $V = Y^\dagger Y$. As already noted $(dX) = (\det B^\dagger B)^{\beta n/2}(dY)$ and so the PDF of Y is

$$F(B^\dagger Y^\dagger Y B)(\det B^\dagger B)^{\beta n/2}.$$

- This is a function of $Y^\dagger Y$, so the PDF of $V = Y^\dagger Y$ is

$$F(B^\dagger V B)(\det B^\dagger B)^{\beta n/2}h(V).$$

- Equating the two expressions for the PDF of V gives

$$h(B^\dagger V B) = h(V)(\det B^\dagger B)^{(\beta/2)(n-m+1-2/\beta)}.$$

- Set $V = \mathbb{I}$ and note $h(\mathbb{I}) = c$ to get the result.

Eigenvalue PDF for Wishart matrices

- We have shown that the Jacobian for the change of variables $A = X^\dagger X$ is proportional to
$$(\det A)^{(\beta/2)(n-m+1-2/\beta)}.$$
- We have shown that for Hermitian matrices, the Jacobian for the change of variables to its eigenvalues and eigenvectors is proportional to
$$\prod_{1\le j<k\le N} |\lambda_k - \lambda_j|^\beta.$$

Hence, if X is distributed according to
$$\frac{1}{C} e^{-\mathrm{Tr}(V(X^\dagger X))},$$
then the PDF of $A = X^\dagger X$ is equal to
$$\frac{1}{C} e^{-\sum_{j=1}^m V(\lambda_j)} \prod_{j=1}^m \lambda_j^{(\beta/2)(n-m+1-2/\beta)} \prod_{1\le j<k\le m} |\lambda_k - \lambda_j|^\beta,$$
where $0 \le \lambda_j < \infty$.

Recall that the eigenvalues of
$$\begin{bmatrix} 0_{n\times n} & X_{n\times m} \\ X^\dagger_{m\times n} & 0_{m\times m} \end{bmatrix}$$
$\{x_j\}$ say, are related to the eigenvalues $\{\lambda_j\}$ of $X^\dagger X$ by $x_j^2 = \lambda_j$, and so $\{x_j\}$ have PDF
$$\frac{1}{C} e^{-\sum_{j=1}^m V(x_j)} \prod_{j=1}^m |x_j|^{\beta(n-m+1-2/\beta)+1} \prod_{1\le j<k\le m} |x_k^2 - x_j^2|^\beta.$$

Unitary matrices

We seek the eigenvalue PDF corresponding to the Haar volume form $(U^\dagger dU)$.

Our strategy is to make use of the Cayley transform, by parametrizing U in terms of an Hermitian H so that

$$U = \frac{\mathbb{I}_N + iH}{\mathbb{I}_N - iH}.$$

Making use of the general operator identity

$$\frac{d}{da}(1-K)^{-1} = (1-K)^{-1}\frac{dK}{da}(1-K)^{-1},$$

where K is assumed to be a smooth function of a, it follows

$$U^\dagger dU = 2i(\mathbb{I}_N + iH)^{-1} dH (\mathbb{I}_N - iH)^{-1}.$$

Holds with H real (real quaternion) for U symmetric (self dual quaternion).

Consequently, using Proposition 1,

$$(U^\dagger dU) = 2^{N(\beta(N-1)/2+1)} \det(\mathbb{I}_N + H^2)^{-\beta(N-1)/2-1}(dH).$$

Since H is complex Hermitian, the eigenvalue PDF in terms of $\{\lambda_j\}$ is

$$\frac{1}{C}\prod_{l=1}^{N}\frac{1}{(1+\lambda_l^2)^{\beta(N-1)/2+1}}\prod_{j<k}|\lambda_k - \lambda_j|^\beta.$$

But

$$\lambda_j = i\frac{1-e^{i\theta}}{1+e^{i\theta}}.$$

Thus the eigenvalue PDF in terms of $\{\theta_j\}$ is

$$\frac{1}{C}\prod_{j<k}|e^{i\theta_k} - e^{i\theta_j}|^\beta.$$

Blocks of unitary matrices

We seek the distribution of the non-zero eigenvalues of $t^\dagger t$ in the decomposition

$$S = \begin{bmatrix} r_{n\times n} & t'_{n\times m} \\ t_{m\times n} & r'_{m\times m} \end{bmatrix}.$$

- Make use of singular value decompositions of individual blocks, for example

$$t = U_t \Lambda_t V_t^\dagger.$$

- Λ_t is a rectangular diagonal matrix, diagonal entries consisting of the positive square roots of the non-zero eigenvalues of $t^\dagger t$.
- U_t and V_t are $m \times m$ and $n \times n$ unitary matrices.

This leads to the parameterization

$$S = \begin{bmatrix} U_r & 0 \\ 0 & U_{r'} \end{bmatrix} L \begin{bmatrix} V_r^\dagger & 0 \\ 0 & V_{r'}^\dagger \end{bmatrix}$$

where

$$L = \begin{bmatrix} \sqrt{1-\Lambda_t\Lambda_t^T} & i\Lambda_t \\ i\Lambda_t^T & \sqrt{1-\Lambda_t^T\Lambda_t} \end{bmatrix}.$$

For S symmetric

$$V_r^\dagger = U_r^T, \qquad V_{r'}^\dagger = U_{r'}^T,$$

while for S self dual quaternion

$$V_r^\dagger = U_r^D, \qquad V_{r'}^\dagger = U_{r'}^D.$$

Using the method of wedge products, can show that the non-zero elements of Λ_t have the distribution

$$\prod_{j=1}^{m} \lambda_j^{\beta\alpha} \prod_{1\le j<k\le m} |\lambda_k^2 - \lambda_j^2|^{\beta}, \qquad \alpha = n - m + 1 - 2/\beta$$

where $0 < \lambda_j < 1$.

For $\beta = 2$, different approaches also work, and further a more general result holds.

Proposition 3. *Let U be an $N \times N$ random unitary matrix chosen with Haar measure. Decompose U into blocks*

$$U = \begin{bmatrix} A_{n_1\times n_2} & C_{n_1\times(N-n_2)} \\ B_{(N-n_1)\times n_2} & D_{(N-n_1)\times(N-n_2)} \end{bmatrix}$$

where $n_1 \ge n_2$. The eigenvalue PDF of $Y := A^\dagger A$ is proportional to

$$\prod_{j=1}^{n_2} y_j^{(n_1-n_2)} (1-y_j)^{(N-n_1-n_2)} \prod_{j<k} (y_k - y_j)^2.$$

Possible strategies:

- Wedge products.
- Matrix integrals.
- Orthogonal projections relating to the matrix structure $A(A+B)^{-1}$.

Proof using matrix integrals

- Use the Ingham-Seigel type integral

$$\int e^{(i/2)\mathrm{Tr}(HQ)}\Big(\det(H-\mu\mathbb{I}_m)\Big)^{-n}(dH)\propto(\det Q)^{(n-m)}e^{(i/2)\mu\mathrm{Tr}Q},$$

  valid for Q Hermitian, and $\mathrm{Re}(\mu)>0$, and the integration is over the space of $m\times m$ Hermitian matrices.

- Regard A and C as general $n_1\times n_2$ and $n_1\times(N-n_2)$ complex rectangular matrices. Then the distribution of A is

$$\int\delta(AA^\dagger+CC^\dagger-\mathbb{I}_{n_2})(dC).$$

- The delta function can itself be written as a matrix integral

$$\int e^{-i\mathrm{Tr}(H(AA^\dagger+CC^\dagger-\mathbb{I}_{n_2}))}(dH),$$

  where the integration is over the space of $n_2\times n_2$ Hermitian matrices.

- Interchange order of integration, doing integration over C first. For this must regularize $H\mapsto H-i\mu\mathbb{I}_n$, $\mu>0$. Gives

$$\lim_{\mu\to0^+}\int(\det(H-i\mu I_{n_1}))^{-(N-n_2)}e^{i\mathrm{Tr}(H(\mathbb{I}_{n_1}-AA^\dagger)}$$

- Evaluate using Ingham-Seigel type integral to get

$$(\det(\mathbb{I}_{n_1}-AA^\dagger))^{(N-n_1-n_2)}.$$

- Use Propostion 2 above to conclude the distribution of $Y:=A^\dagger A$ is proportional to

$$(\det Y)^{(n_1-n_2)}(\det(\mathbb{I}_{n_2}-Y))^{(N-n_1-n_2)}.$$

Classical random matrix ensembles

Let $\beta = 1, 2$ or 4 according to the elements being real, complex or quaternion real respectively.

In the case of Hermitian matrices, the eigenvalue PDFs derived above all have the general form

$$\frac{1}{C} \prod_{l=1}^{N} g(x_l) \prod_{1 \le j < k \le N} |x_k - x_j|^{\beta}.$$

Choosing the entries to be matrices to be independent Gaussians, when there is a choice, the form of $g(x)$ is, up to scaling $x_l \mapsto c x_l$,

$$g(x) = \begin{cases} e^{-x^2}, & \text{Gaussian} \\ x^a e^{-x} \ (x > 0) & \text{Laguerre} \\ x^a (1-x)^b \ (0 < x < 1) & \text{Jacobi} \\ (1 + x^2)^{-\alpha} & \text{Cauchy} \end{cases}$$

These are the four classical weight functions from orthogonal polynomial theory, which can be characterized by the property that

$$\frac{d}{dx} \log g(x) = \frac{a(x)}{b(x)}$$

where

$$\text{degree}\, a(x) \le 1, \qquad \text{degree}\, b(x) \le 2.$$

The corresponding eigenvalue PDF is said to define a classical random matrix ensemble.

Gaussian β ensemble

So far we've seen that the eigenvalue PDF

$$\frac{1}{C}\prod_{l=1}^{N} e^{-x_l^2} \prod_{1 \le j < k \le N} |x_k - x_j|^\beta$$

corresponds to random Hermitian matrices with probability measure

$$\frac{1}{C} e^{-\mathrm{Tr}\, X^2},$$

where the elements are real ($\beta = 1$), complex ($\beta = 2$) or quaternion real ($\beta = 4$). Said to define the GOE (Gaussian orthogonal ensemble), GUE (Gaussian unitary ensemble) and GSE (Gaussian symplectic ensemble).

We would like to give a meaning in the context of random matrix theory for general $\beta > 0$.

Inductively define a sequence of random matrices $\{M_j\}_{j=1,2,\ldots}$ by $M_1 \sim a$ and

$$M_{N+1} = \begin{bmatrix} D_N & \vec{w} \\ \vec{w}^T & a \end{bmatrix}$$

where $D_N = \mathrm{diag}(a_1, \ldots, a_N)$ where $\{a_j\}$ denotes the eigenvalues of M_N.

Relationship to GOE

Let $a \in \mathrm{N}[0,1]$, and let $w_j \in \mathrm{N}[0,1/\sqrt{2}]$. Analogous to above, define

$$M_{N+1} = \begin{bmatrix} M_N & \vec{w} \\ \vec{w}^T & a \end{bmatrix}$$

Let O_N be the real orthogonal matrix which diagonalizes M_N, so that $M_N = O_N D_N O_N^T$, and observe

$$\begin{bmatrix} O_N & \vec{0} \\ \vec{0}^T & 1 \end{bmatrix} \begin{bmatrix} M_N & \vec{w} \\ \vec{w}^T & a \end{bmatrix} \begin{bmatrix} O_N & \vec{0} \\ \vec{0}^T & 1 \end{bmatrix}^T \sim \begin{bmatrix} D_N & \vec{w} \\ \vec{w}^T & a \end{bmatrix}$$

Question: Can we understand this result using different reasoning?

A random rational function

Recall

$$M_{N+1} = \begin{bmatrix} D_N & \vec{w} \\ \vec{w}^T & a \end{bmatrix}$$

where

$$D_N = \mathrm{diag}(a_1, \ldots, a_N).$$

For given $\{a_j\}$, we would like to compute the eigenvalue distribution of M_{N+1}. We have

$$\begin{aligned}
\det(\lambda \mathbb{I}_{N+1} - M_{N+1}) &= \det \begin{bmatrix} \lambda \mathbb{I}_N - D_N & \vec{w} \\ \vec{w}^T & \lambda - a \end{bmatrix} \\
&= \det \begin{bmatrix} \lambda \mathbb{I}_N - D_N & -\vec{w} \\ \vec{0}^T & \lambda - a - \vec{w}^T (\lambda \mathbb{I}_N - D_N)^{-1} \vec{w} \end{bmatrix} \\
&= p_N(\lambda)(\lambda - a - \vec{w}^T (\lambda \mathbb{I}_N - D_N)^{-1} \vec{w})
\end{aligned}$$

But $\lambda \mathbb{I}_N - D_N$ is diagonal, so its inverse is also diagonal, allowing us to conclude

$$\frac{p_{N+1}(\lambda)}{p_N(\lambda)} = \lambda - a - \sum_{i=1}^{N} \frac{q_i}{\lambda - a_i}, \qquad q_i := w_i^2.$$

Conclusion The eigenvalues of M_{N+1} are given by the zeros of the above rational function.

The corresponding PDF can be computed for a certain choice of the distribution of the q_i, generalizing $q_i \sim (\mathrm{N}[0, 1/\sqrt{2}])^2 \sim \Gamma[1/2, 1]$ which corresponds to the GOE.

Proposition 4. *Let* $w_i^2 \sim \Gamma[\beta/2, 1]$ *where* $\Gamma[s, \sigma]$ *refers to the gamma distribution, specified by the PDF* $\sigma^{-s} x^{s-1} e^{-x/\sigma}/\Gamma(s)$ $(x > 0)$*. Given*

$$a_1 > a_2 > \cdots > a_N$$

the PDF for the zeros of the random rational function

$$\lambda - a - \sum_{i=1}^{N} \frac{q_i}{\lambda - a_i}$$

is equal to

$$\frac{e^{a^2/2}}{(\Gamma(\beta/2))^N} \frac{\prod_{1 \le j < k \le N+1} (\lambda_j - \lambda_k)}{\prod_{1 \le j < k \le N} (a_j - a_k)^{\beta - 1}} \prod_{j=1}^{N+1} \prod_{p=1}^{N} |\lambda_j - a_p|^{\beta/2 - 1}$$
$$\times \exp\Big(-\frac{1}{2} \Big(\sum_{j=1}^{N+1} \lambda_j^2 - \sum_{j=1}^{N} a_j^2 \Big) \Big)$$

where

$$\infty > \lambda_1 > a_1 > \lambda_2 > \cdots > a_N > \lambda_{N+1} > -\infty$$

and

$$\sum_{j=1}^{N+1} \lambda_j = \sum_{j=1}^{N} a_j + a.$$

Proof

- Because the q_i are positive, graphical considerations imply the interlacing condition.
- The summation constraint is equivalent to the statement that $\mathrm{Tr}\, M_{N+1} = \mathrm{Tr}\, D_N + a$.
- To compute the PDF we change variables from $\{q_i\}_{i=1,\dots,N}$ to $\{\lambda_j\}_{j=1,\dots,N}$.
- The translations $\lambda_j \mapsto \lambda_j - a$, $a_j \mapsto a_j - a$ shows it suffices to consider the case $a = 0$

With $a = 0$ we have

$$\lambda - \sum_{i=1}^{N} \frac{q_i}{\lambda - a_i} = \frac{\prod_{j=1}^{N+1}(\lambda - \lambda_j)}{\prod_{l=1}^{N}(\lambda - a_l)}$$

From the residue at $\lambda = a_i$ it follows

$$\frac{\prod_{j=1}^{N+1}(a_i - \lambda_j)}{\prod_{l=1, l\neq i}(a_i - a_l)} = -q_i.$$

We have just shown that

$$\frac{\prod_{j=1}^{N+1}(a_i - \lambda_j)}{\prod_{l=1, l\neq i}(a_i - a_l)} = -q_i.$$

Now use the method of wedge products to compute the Jacobian:

$$\bigwedge_{j=1}^{N} dq_i = \prod_{j=1}^{N} q_j \det\Big[\frac{1}{a_i - \lambda_j}\Big] \bigwedge_{j=1}^{N} d\lambda_j.$$

Making use of the Cauchy double alternant identity, we read off that the Jacobian is equal to

$$\prod_{j=1}^{N} q_j \Big| \frac{\prod_{1\leq i<j\leq N}(a_i - a_j)(\lambda_i - \lambda_j)}{\prod_{i,j=1}^{N}(a_i - \lambda_j)} \Big|.$$

We are given that the distribution of $\{q_i\}$ is equal to

$$\frac{1}{(\Gamma(\beta/2))^N} \prod_{j=1}^{N} q_j^{\beta/2-1}\, e^{-\sum_{j=1}^{N} q_j}.$$

Must multiply these last two equations together, and substitute for q_i, noting

$$\sum_{j=1}^{N} q_j = \frac{1}{2}\Big(\sum_{j=1}^{N+1} \lambda_j^2 - \sum_{j=1}^{N} \mu_j^2\Big).$$

Suppose now that $a \sim \mathrm{N}[0,1]$. Thus we must multiply the PDF of Proposition 4 by $\frac{1}{\sqrt{2\pi}} e^{-a^2/2}$ and integrate over a. Doing this gives the PDF

$$\frac{1}{\sqrt{2\pi}(\Gamma(\beta/2))^N} \frac{\prod_{1 \le j < k \le N+1} (\lambda - \lambda_k)}{\prod_{1 \le j < k \le N} (a_j - a_k)^{\beta - 1}} \prod_{j=1}^{N+1} \prod_{p=1}^{N} |\lambda_j - a_p|^{\beta/2 - 1}$$
$$\times \exp\Big(- \frac{1}{2} \Big(\sum_{j=1}^{N+1} \lambda_j^2 - \sum_{j=1}^{N} a_j^2 \Big) \Big)$$

where

$$\infty > \lambda_1 > a_1 > \lambda_2 > \cdots > a_N > \lambda_{N+1} > -\infty$$

- Denote the above conditional PDF $G_N(\{\lambda_j\}, \{a_k\})$.
- Denote the domain of support by R_N.
- Let $\{a_j\}$ have PDF $p_N(a_1, \ldots, a_N)$.
- Let $\{\lambda_j\}$ have PDF $p_{N+1}(\lambda_1, \ldots, \lambda_{N+1})$.

The PDFs $\{p_n\}$ must satisfy the recurrence

$$p_{N+1}(\lambda_1, \ldots, \lambda_{N+1})$$
$$= \int_R da_1 \cdots da_N \, G_N((\{\lambda_j\}, \{a_k\}) p_N(a_1, \ldots, a_N)$$

We seek the solution with $p_0 := 1$.

Question How can we solve this recurrence.

For $\beta = 1$ we know that the solution of the recurrence is

$$\frac{1}{C}\prod_{j=1}^{N} e^{-\lambda_j^2/2} \prod_{1\le j<k\le N} |\lambda_k - \lambda_j|.$$

It must then be that there is an integration formula implying the recurrence.

New question How can we derive such integration formulas?

We again take inspiration from the case of the GOE, applied to a general formula:

- Let M_N be a general $N \times N$ real symmetric matrix with eigenvalues $\{\lambda_j\}$.
- Let μ_i denote the top entry of the normalized eigenvector corresponding to λ_i.
- We have

$$\sum_{j=1}^{N} \frac{\mu_j^2}{\lambda - \lambda_j} = \Big((\lambda \mathbb{I}_N - M_N)^{-1} \Big)_{11} = \frac{p_{N-1}(\lambda)}{p_N(\lambda)}.$$

Here the first equality follows from a spectral decomposition, while the second follows from Cramer's rule. Note that we must have

$$\sum_{j=1}^{N} \mu_i^2 = 1.$$

In the case $\beta = 1$, $M_N \in$ GOE, and is thus orthogonally invariant.

Consequently

$$\mu_i^2 \sim \frac{w_i^2}{w_1^2 + \cdots + w_N^2} =: \rho_i$$

where each $w_i^2 \sim \Gamma[1/2, 1]$.

Generally, if $w_i^2 \sim \Gamma[\beta/2, 1]$ then the PDF of $\rho_1, \ldots, \rho_N$ is equal to the Dirichlet distribution

$$\frac{\Gamma(N\beta/2)}{(\Gamma(\beta/2))^N} \prod_{j=1}^{N} \rho_j^{\beta/2-1}$$

where each ρ_j is positive and $\sum_{j=1}^{N} \rho_j = 1$.

Question What is the distribution of the roots of the random rational function

$$\sum_{j=1}^{N} \frac{\mu_j^2}{\lambda - \lambda_j}$$

when the $\{\mu_j^2\}$ have a Dirichlet distribution?

Proposition 5. *Let* $\{\rho_i\}$ *have the Dirichlet distribution*

$$\frac{\Gamma(N\beta/2)}{(\Gamma(\beta/2))^N}\prod_{j=1}^{N}\rho_j^{\beta/2-1}$$

and let $\{b_j\}$ *be given. The roots of the random rational function*

$$\sum_{j=1}^{N}\frac{\rho_j}{x-b_j},$$

denoted $\{x_1,\dots,x_{N-1}\}$ *say, have the PDF*

$$\frac{\Gamma(N\beta/2)}{(\Gamma(\beta/2))^N}\frac{\prod_{1\le j<k\le N-1}(x_j-x_k)}{\prod_{1\le j<k\le N}(b_j-b_k)^{\beta-1}}\prod_{j=1}^{N-1}\prod_{p=1}^{N}|x_j-b_p|^{\beta/2-1}$$

where

$$x_1>b_1>x_2>b_2>\cdots>x_{N-1}>b_N.$$

The proof is very similar to that of Proposition 4. We seek to change variables from $\{\rho_j\}$ to $\{x_j\}$.

- Begin by noting

$$\sum_{j=1}^{N} \frac{\rho_j}{x - b_j} = \frac{\prod_{l=1}^{N-1}(x - x_l)}{\prod_{l=1}^{N}(x - b_l)}.$$

- Equating the residue at $x = b_j$ gives

$$\rho_j = \frac{\prod_{l=1}^{N-1}(b_j - x_l)}{\prod_{l=1, l\neq j}^{N}(b_j - b_l)}.$$

- Taking wedge products gives

$$\bigwedge_{j=1}^{N-1} d\rho_j = \prod_{j=1}^{N-1} \rho_j \det\left[\frac{1}{b_j - x_k}\right]_{j,k=1,\dots,N-1} \bigwedge_{j=1}^{N-1} dx_j$$

- Now use the Cauchy double alternant identity to conclude the Jacobian for the change of variables is equal to

$$\prod_{j=1}^{N-1} \rho_j \left| \frac{\prod_{1\leq j<k\leq N-1}(b_k - b_j)(x_k - x_j)}{\prod_{j,k=1}^{N-1}(b_j - x_k)} \right|.$$

- Multiply this Jacobian and the PDF for the distribution of the $\{\rho_j\}$, then substitute for the ρ_j to get the result.

Consequence

Because the expression for the distribution of the roots of the rational function is a PDF for $\{x_j\}$, integrating over the region

$$x_1 > b_1 > x_2 > b_2 > \cdots > x_{N-1} > b_N$$

denoted R'_{N-1} say must give unity. Hence we have the integration formula

$$\int_{R'_{N-1}} dx_1 \cdots dx_{N-1} \prod_{1 \le j < k \le N-1} (x_j - x_k) \prod_{j=1}^{N-1} \prod_{p=1}^{N} |x_j - b_p|^{\beta/2-1}$$
$$= \frac{(\Gamma(\beta/2))^N}{\Gamma(N\beta/2)} \prod_{1 \le j < k \le N} (b_j - b_k)^{\beta-1}.$$

This integration formula allows the solution of the recurrence to be verified.

Proposition 6. *The solution of the recurrence*

$$p_{N+1}(\lambda_1, \dots, \lambda_{N+1}) \\ = \int_R da_1 \cdots da_N \, G_N((\{\lambda_j\}, \{a_k\}) p_N(a_1, \dots, a_N)$$

is given by

$$p_N(x_1, \dots, x_N) = \frac{1}{m_N(\beta)} \prod_{j=1}^{N} e^{-x_j^2/2} \prod_{1 \le j < k \le N} |x_k - x_j|^\beta$$

where

$$N! m_N(\beta) = (2\pi)^{N/2} \prod_{j=0}^{N-1} \frac{\Gamma(1 + (j+1)\beta/2)}{\Gamma(1 + \beta/2)}.$$

This PDF is said to define the Gaussian β ensemble.

Proof

Substituting the stated form for p_N in the recurrence gives

$$\frac{1}{\sqrt{2\pi}} \frac{1}{(\Gamma(\beta/2))^N} \frac{1}{m_N(\beta)} e^{-\frac{1}{2}\sum_{j=1}^{N+1} \lambda_j^2} \prod_{1 \le j < k \le N+1} (\lambda_j - \lambda_k) \\ \times \int_{R_N} da_1 \cdots da_N \prod_{1 \le j < k \le N} (a_j - a_k) \prod_{j=1}^{N+1} \prod_{p=1}^{N} |\lambda_j - a_p|^{\beta/2 - 1}.$$

Evaluating the integral according to the integration formula of the previous page verifies the recurrence.

Three term recurrence

The characteristic polynomial $p_N(x) := \prod_{l=1}^{N}(x - x_l)$, where $\{x_j\}$ is distributed according to the Gaussian β-ensemble, satisfies

$$\frac{p_{N-1}(x)}{p_N(x)} = \sum_{j=1}^{N} \frac{\rho_j}{x - x_j}$$

where

$$\rho_j \sim w_j^2/(w_1^2 + \cdots + w_N^2), \qquad w_j^2 \sim \Gamma[\beta/2, 1]$$

and

$$\frac{p_{N+1}(x)}{p_N(x)} = x - a - \sum_{j=1}^{N} \frac{w_j^2}{x - x_j}, \qquad a \in \mathrm{N}[0, 1].$$

Hence

$$\frac{p_{N+1}(x)}{p_N(x)} = (x - a) - b_N^2 \frac{p_{N-1}(x)}{p_N(x)}$$

where

$$b_N^2 \sim (w_1^2 + \cdots + w_N^2) \sim \Gamma[N\beta/2, 1].$$

Rearranging gives the random three term recurrence

$$p_{N+1}(x) = (x - a)p_N(x) - b_N^2 p_N(x).$$

Relationship to tridiagonal matrices

Consider a general real symmetric tridiagonal matrix

$$T_n = \begin{bmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & b_2 & a_3 & & \\ & & & \vdots & b_{n-1} \\ & & & b_{n-1} & a_n \end{bmatrix}$$

By forming $\lambda \mathbb{I}_n - T_n$ and expanding the determinant along the bottom row one sees

$$\det(\lambda \mathbb{I}_n - T_n) = (\lambda - a_n) \det(\lambda \mathbb{I}_{n-1} - T_{n-1}) - b_{n-1}^2 \det(\lambda \mathbb{I}_{n-2} - T_{n-2}).$$

Conclusion

The Gaussian β-ensemble is realized by random tridiagonal matrices with

$$a_j \sim \mathrm{N}[0,1] \qquad b_j^2 \sim \Gamma[j\beta/2, 1].$$

Laguerre β ensemble

Consider the LOE.

- This is realized by matrices $X_{(n)}^T X_{(n)}$ where $X_{(n)}$ is an $n \times N$ rectangular matrix with Gaussian entries N$[0,1]$.
- Have the recurrence

$$X_{(n+1)}^T X_{(n+1)} = X_{(n)}^T X_{(n)} + \vec{x}_{(1)} \vec{x}_{(1)}^T.$$

- For $n < N$, $X_{(n)}^T X_{(n)}$ has n non-zero eigenvalues, and $N - n$ eigenvalues.
- Suggests an inductive contruction of $N \times N$ positive definite matrices $\{A^{(n)}\}_{n=1,\dots,N}$,

$$A_{(n+1)} = \operatorname{diag} A_{(n)} + \vec{x}_{1\times m} \vec{x}_{1\times m}^T$$

  where $A_{(0)} = [0]_{m\times m}$ and

  $$x_j^2 \sim \Gamma[\beta/2, 1] \ (j = 1, \dots, n), \qquad x_j^2 \sim \Gamma[a\beta/2, 1] \ (j = n+1, \dots, N)$$

For this must study the eigenvalues of the $N \times N$ matrix

$$Y := \operatorname{diag}(a_1, \dots, a_n, \underbrace{a_{n+1}, \dots, a_{n+1}}_{N-n}) + \vec{x}\vec{x}^T$$

Since

$$\det(\lambda \mathbb{I}_N - Y) = \det(\lambda \mathbb{I}_N - A)\det(\mathbb{I}_N - (\lambda \mathbb{I}_N - A)^{-1}\vec{x}\vec{x}^T)$$

it follows

$$\frac{\det(\lambda \mathbb{I}_N - Y)}{\det(\lambda \mathbb{I}_N - A)} = 1 - \sum_{j=1}^{n} \frac{x_j^2}{\lambda - a_j} - \frac{\sum_{j=n+1}^{N} x_j^2}{\lambda - a_{n+1}}$$

Question

What is the density of zeros of the random rational function

$$1 - \sum_{j=1}^{n+1} \frac{w_j}{\lambda - a_j}$$

for $w_j \in \Gamma[s_j, 1]$.

Proposition 7. *The zeros of the above rational function have PDF*

$$\frac{1}{\Gamma(s_1)\cdots\Gamma(s_{n+1})} e^{-\sum_{j=1}^{n+1}(\lambda_j - a_j)} \prod_{1 \le i < j \le n+1} \frac{(\lambda_i - \lambda_j)}{(a_i - a_j)^{s_i + s_j - 1}}$$
$$\times \prod_{i,j=1}^{n+1} |\lambda_i - a_j|^{s_j - 1}$$

where

$$\lambda_1 > a_1 > \lambda_2 > \cdots > a_{n+1}.$$

The case of interest is

$$s_1 = \cdots = s_n = \beta/2, \qquad s_{n+1} = (N-n)a\beta/2, \qquad a_{n+1} = 0.$$

A recurrence relation

- Recall the case of interest is

$$s_1 = \cdots = s_n = \beta/2, \qquad s_{n+1} = (N-n)a\beta/2, \qquad a_{n+1} = 0.$$

- Denote the conditional PDF for $\{\lambda_j\}$ in the case of interest by $G(\{\lambda_j\}_{j=1,\dots,n+1}; \{a_j\}_{j=1,\dots,n})$.
- Let the PDF of $\{\lambda_j\}_{j=1,\dots,n+1}$ be denoted $p_{n+1}(\{\lambda_j\})$.
- Let the PDF of $\{a_j\}_{j=1,\dots,n+1}$ be denoted $p_n(\{a_j\})$.

For $n < N$ the recursive construction of $\{A^{(n)}\}$ gives that

$$\begin{aligned} p_{n+1}(\{\lambda_j\}) &= \int_{\lambda_1 > a_1 > \cdots > \lambda_{n+1} > 0} da_1 \cdots da_n \\ &\qquad \times G(\{\lambda_j\}_{j=1,\dots,n+1}; \{a_j\}_{j=1,\dots,n}) p_n(\{a_j\}) \end{aligned}$$

subject to the initial condition $p_0 = 1$.

A special solution

For $\beta = 1$, the recurrence has solution

$$p_n(\{\lambda_j\}) = \frac{1}{C_n} \prod_{l=1}^{n} \lambda_l^{(N-n-1)/2} e^{-\lambda_l} \prod_{1 \le j < k \le n} |\lambda_k - \lambda_j|$$

since this corresponds to the LOE.

For general $\beta > 0$, want to check that the recurrence has solution

$$p_n(\{\lambda_j\}) = \frac{1}{C_{n,\beta}} \prod_{l=1}^{n} \lambda_l^{(N-n+1)\beta/2-1} e^{-\lambda_l} \prod_{1 \le j < k \le n} |\lambda_k - \lambda_j|^\beta.$$

Since

$$\begin{aligned} &G(\{\lambda_j\}_{j=1,\dots,n+1}; \{a_j\}_{j=1,\dots,n}) \\ &\quad = \frac{1}{(\Gamma(\beta/2))^n \Gamma((N-n)\beta/2)} e^{-\sum_{j=1}^{n}(\lambda_j - a_j) - \lambda_{n+1}} \frac{\prod_{i<j}^{n+1}(\lambda_i - \lambda_j)}{\prod_{i<j}^{n}(a_i - a_j)^{\beta-1}} \\ &\quad \times \frac{\prod_{i=1}^{n+1} \lambda_i^{(N-n)\beta/2+1}}{a_i^{(N-n+1)\beta/2-1}} \prod_{i,j=1}^{n} |\lambda_i - a_j|^{\beta/2-1} \end{aligned}$$

we see we need to evaluate

$$\int_{\lambda_1 > a_1 > \cdots > \lambda_{n+1} > 0} da_1 \cdots da_n \prod_{i<j}^{n} (a_i - a_j) \prod_{i,j=1}^{n} |\lambda_i - a_j|^{\beta/2-1}$$

According to the integration formula which follows from Proposition 5 this equals

$$\frac{(\Gamma(\beta/2))^{n+1}}{\Gamma((n+1)\beta/2)} \prod_{1 \le j < k \le n+1} (\lambda_j - \lambda_k)^{\beta-1}.$$

leaving us with

$$\frac{C_{n+1,\beta}}{C_{n,\beta}} \frac{\Gamma(\beta/2)}{\Gamma((n+1)\beta/2)\Gamma((N-n)/\beta/2)} p_{n+1}(\{\lambda_j\}).$$