Complexity results of path following algorithms for linear programming which take into account the geometry of the central path

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Goals of the talk

- Understand the behavior of the central path and the Mizuno-Todd-Ye predictor-corrector (MTY P-C) algorithm for linear programming from the geometric point of view
- Estimate the iteration complexity of the MTY P-C algorithm in terms of the integral of a certain curvature of the central path
- Relate the above integral to a new iteration complexity bound for the MTY P-C algorithm involving a certain condition number of the constraint matrix A

Talk Outline

- **LP problem and assumptions;**
- **central path and its neighborhood;**
- **Mizuno-Todd-Ye predictor-corrector (MTY P-C) algorithm;**
- **condition number and scale-invariance;**
- **iteration complexity bounds for the MTY P-C alg.**
  - **classical one (1990)**
  - **new one (2003)**
- **illustrative LP instance**
- **curvature of the central path**
- **iteration complexity bounds in terms of a curvature integral**
- **directions for future research**

The LP problem

$$
\begin{array}{lll}
(\mathbf{P}) & \text{minimize}_{\mathbf{x}} & \mathbf{c}^T\mathbf{x} \\
& \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0},
\end{array}
$$

$$
\begin{array}{lll}
(D) & \text{maximize}_{(\mathbf{y},\mathbf{s})} & \mathbf{b}^T\mathbf{y} \\
& \text{subject to} & \mathbf{A}^T\mathbf{y} + \mathbf{s} = \mathbf{c},\ \mathbf{s} \geq \mathbf{0},
\end{array}
$$

Assumptions

1) (**P**) and (**D**) have interior-feasible solutions.
2) the rows of the $\mathbf{m} \times \mathbf{n}$ matrix $\mathbf{A}$ are linearly independent.

Definition: The duality gap of a feasible $\mathbf{w} = (\mathbf{x}, \mathbf{y}, \mathbf{s})$ is given by

$$\mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y} = (\mathbf{A}^T\mathbf{y} + \mathbf{s})^T\mathbf{x} - \mathbf{b}^T\mathbf{y} = \mathbf{x}^T\mathbf{s}.$$

Central path and its neighborhood

For each $\nu > \mathbf{0}$, the system

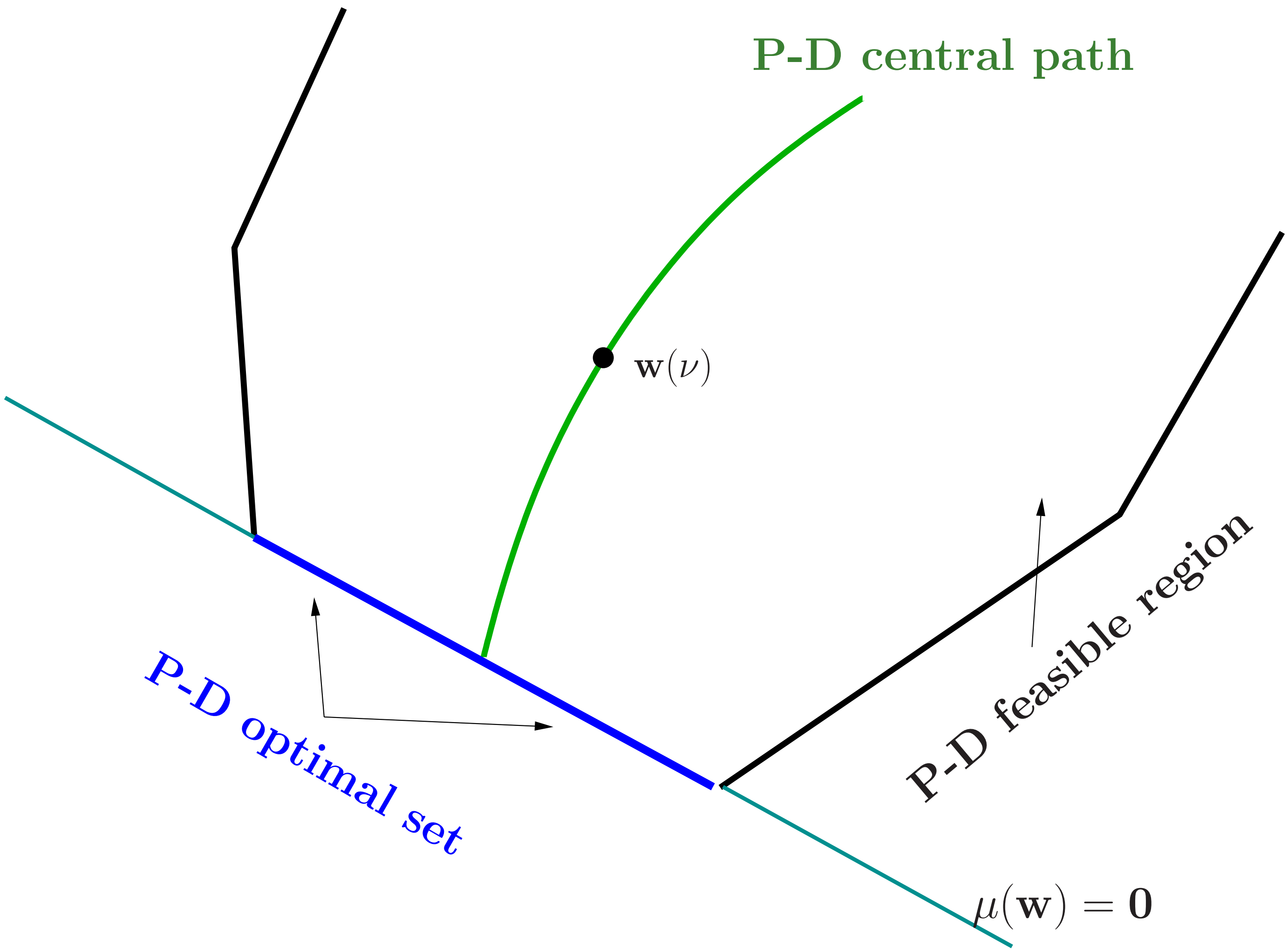
$$\begin{aligned} \mathbf{XS\,e} &= \nu\,\mathbf{e}, \\ \mathbf{Ax} - \mathbf{b} &= \mathbf{0}, \qquad (\mathbf{x}, \mathbf{s}) \geq \mathbf{0}, \\ \mathbf{A}^{\mathbf{T}}\mathbf{y} + \mathbf{s} - \mathbf{c} &= \mathbf{0}, \end{aligned}$$

where $\mathbf{X} = \mathbf{Diag}(\mathbf{x})$, $\mathbf{S} = \mathbf{Diag}(\mathbf{s})$ and $\mathbf{e} = (\mathbf{1}, \ldots, \mathbf{1})^{\mathbf{T}}$, has a unique solution $\mathbf{w}(\nu) = (\mathbf{x}(\nu), \mathbf{y}(\nu), \mathbf{s}(\nu))$, which converges to a primal-dual optimal solution as $\nu \to \mathbf{0}$.

The MTY P-C is based on the 2-norm neighborhood of the central path:

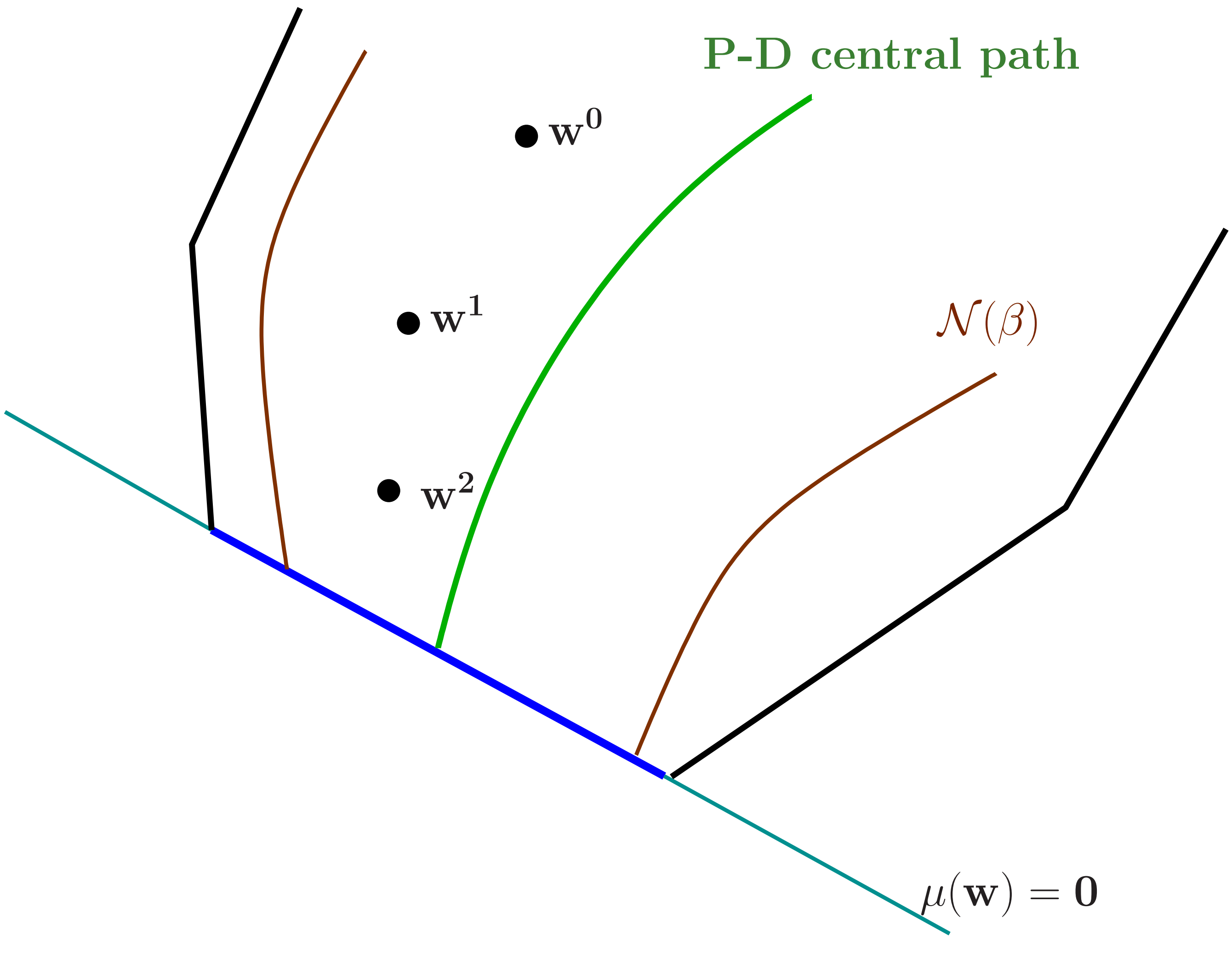
$$\mathcal{N}(\beta) \equiv \{\mathbf{w} = (\mathbf{x}, \mathbf{y}, \mathbf{s}) \textbf{ feasible} : \|\mathbf{Xs} - \mu\mathbf{e}\| \leq \beta\mu\},$$

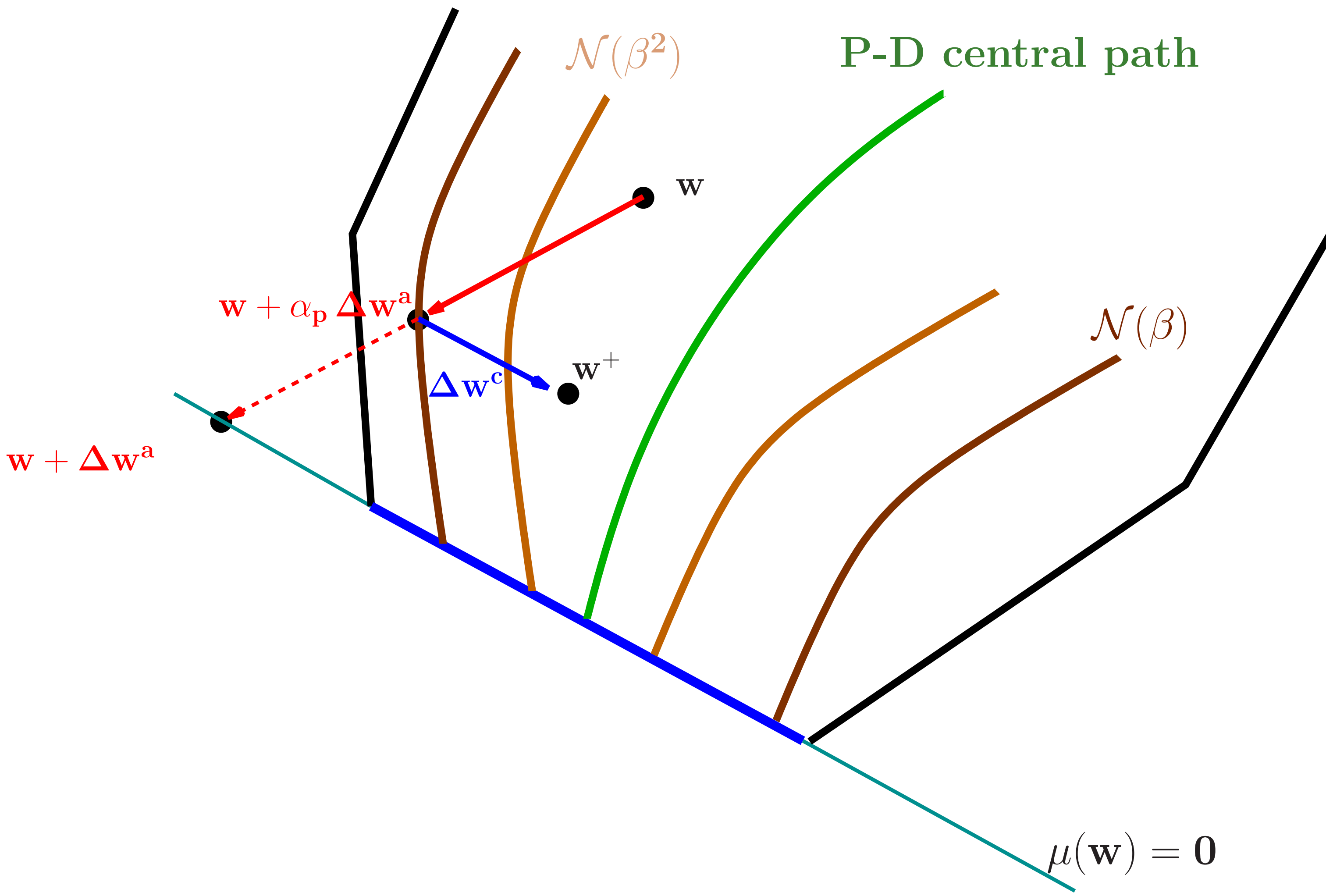
where $\mu = \mu(\mathbf{w}) \equiv (\mathbf{x}^{\mathbf{T}}\mathbf{s})/\mathbf{n}$ and $\beta \in (\mathbf{0}, \mathbf{1})$ is a fixed constant.



$$\mathbf{w}(\nu) \quad = \quad (\mathbf{x}(\nu), \mathbf{s}(\nu), \mathbf{y}(\nu))$$

$$\mu(\mathbf{w}) \quad := \quad \frac{\mathbf{c}^{\mathbf{T}}\mathbf{x} - \mathbf{b}^{\mathbf{T}}\mathbf{y}}{\mathbf{n}} \;=\; \frac{\mathbf{s}^{\mathbf{T}}\mathbf{x}}{\mathbf{n}}$$


P-D central path
$\mathbf{w}^0$
$\mathbf{w}^1$
$\mathbf{w}^2$
$\mathcal{N}(\beta)$
$\mu(\mathbf{w}) = \mathbf{0}$



$\mathcal{N}(\beta^2)$
P-D central path
$\mathbf{w}$
$\mathbf{w} + \alpha_p \, \boldsymbol{\Delta} \mathbf{w}^a$
$\mathcal{N}(\beta)$
$\mathbf{w}^+$
$\boldsymbol{\Delta} \mathbf{w}^c$
$\mathbf{w} + \boldsymbol{\Delta} \mathbf{w}^a$
$\mu(\mathbf{w}) = \mathbf{0}$

Search Directions

For a strictly feasible $\mathbf{w} = (\mathbf{x}, \mathbf{y}, \mathbf{s})$, the Newton direction $\mathbf{\Delta w} = (\mathbf{\Delta x}, \mathbf{\Delta y}, \mathbf{\Delta s})$ towards the point $\mathbf{w}(\nu) = (\mathbf{x}(\nu), \mathbf{y}(\nu), \mathbf{s}(\nu))$ is the solution of

$$\begin{aligned} \mathbf{X\Delta s} + \mathbf{S\Delta x} &= -\mathbf{Xs} + \nu\mathbf{e} \\ \mathbf{A\Delta x} &= \mathbf{0} \\ \mathbf{A^T\Delta y} + \mathbf{\Delta s} &= \mathbf{0} \end{aligned}$$

Setting $\nu = \mathbf{0}$ yields the predictor (or affine scaling) direction at $\mathbf{w}$.

Setting $\nu = \mu(\mathbf{w})$ yields the corrector (or centrality) direction at $\mathbf{w}$.

An iteration of the MTY P-C Alg.

Let $\mathbf{w} = (\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{N}(\beta^2)$ be given, where $\beta \in (0, 1/2]$.

1) **Compute the AS direction $\Delta\mathbf{w^a} = (\Delta\mathbf{x^a}, \Delta\mathbf{y^a}, \Delta\mathbf{s^a})$ at $\mathbf{w}$;**

2) **Let $\alpha_\mathbf{p} > 0$ be the largest $\alpha \in [0, 1]$ such that $\mathbf{w} + \alpha\Delta\mathbf{w^a} \in \mathcal{N}(\beta)$;**

3) **Set $\mathbf{w_p} = \mathbf{w} + \alpha_\mathbf{p}\Delta\mathbf{w^a}$;**

4) **Compute the corrector direction $\Delta\mathbf{w^c} = (\Delta\mathbf{x^c}, \Delta\mathbf{y^c}, \Delta\mathbf{s^c})$ at $\mathbf{w_p}$;**

5) **The next point $\mathbf{w}^+$ is determined as $\mathbf{w}^+ = \mathbf{w_p} + \Delta\mathbf{w^c}$;**

It can be proved that $\mathbf{w}^+ \in \mathcal{N}(\beta^2)$. Hence, a new iteration can be started by setting $\mathbf{w} \leftarrow \mathbf{w}^+$ and going back to 1).

THE CONDITION NUMBER $\bar{\chi}_{\mathbf{A}}$

Define

$$\bar{\chi}_{\mathbf{A}} \equiv \sup\{\|(\mathbf{A}\mathbf{D}\mathbf{A}^{\mathbf{T}})^{-\mathbf{1}}\mathbf{A}\mathbf{D}\| : \mathbf{D} \in \mathcal{D}\},$$

where $\mathcal{D}$ denotes the set of all positive definite diagonal matrices.

Facts:

1) $\bar{\chi}_{\mathbf{A}} = \max\{\|\mathbf{B}^{-\mathbf{1}}\mathbf{A}\| : \mathbf{B}$ **is a basis of A**$\}$.

2) **Finding an upper bound for $\bar{\chi}_{\mathbf{A}}$ is a $\mathcal{NP}$ hard problem.**

3) **If A integral then $\bar{\chi}_{\mathbf{A}} \leq \mathbf{2}^{\mathbf{L}_{\mathbf{A}}}$, where $\mathbf{L}_{\mathbf{A}}$ is the input size of A.**

Scale Invariance

Let $\mathbf{D}$ be a positive diagonal matrix and consider the pair of LPs:

$$
\begin{array}{lll}
(\tilde{\mathbf{P}}) & \text{minimize} & (\mathbf{Dc})^{\mathbf{T}}\tilde{\mathbf{x}} \\
 & \text{subject to} & \mathbf{AD}\tilde{\mathbf{x}} = \mathbf{b},\ \tilde{\mathbf{x}} \geq \mathbf{0}, \\
(\tilde{\mathbf{D}}) & \text{maximize} & \mathbf{b}^{\mathbf{T}}\tilde{\mathbf{y}} \\
 & \text{subject to} & \mathbf{DA}^{\mathbf{T}}\tilde{\mathbf{y}} + \tilde{\mathbf{s}} = \tilde{\mathbf{c}},\ \tilde{\mathbf{s}} \geq \mathbf{0},
\end{array}
$$

obtained from $(\mathbf{P})$ and $(\mathbf{D})$ by performing the change of variables $(\mathbf{x}, \mathbf{y}, \mathbf{s}) = \mathbf{\Phi}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \tilde{\mathbf{s}}) \equiv (\mathbf{D}\tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \mathbf{D}^{-1}\tilde{\mathbf{s}})$.

The MTY P-C algorithm is scaling-invariant, i.e., if $\{\mathbf{w}^{\mathbf{k}}\}$ and $\{\tilde{\mathbf{w}}^{\mathbf{k}}\}$ denote the sequence of iterates generated by the MTY P-C algorithm in the original and the scaled space, then $\mathbf{w}^{\mathbf{k}} = \mathbf{\Phi}(\tilde{\mathbf{w}}^{\mathbf{k}})$ for all $\mathbf{k} \geq \mathbf{1}$, as long as $\mathbf{w}^{\mathbf{0}} = \mathbf{\Phi}(\tilde{\mathbf{w}}^{\mathbf{0}})$.

ITERATION-COMPLEXITY BOUNDS

Given $\mathbf{0} < \nu_\mathbf{f} < \nu_\mathbf{i}$, denote by $\mathbf{N}(\nu_\mathbf{i}, \nu_\mathbf{f}, \beta)$ the largest possible number of iterations required by the MTY P-C algorithm to find an iterate with duality gap $\leq \nu_\mathbf{f}$ when started from any $\mathbf{w^0} \in \mathcal{N}(\beta^\mathbf{2})$ such that $\mu(\mathbf{w^0}) = \nu_\mathbf{i}$.

Classical Result: For any $\beta \in (\mathbf{0}, \mathbf{1}/\mathbf{2}]$,

$$\sqrt{\beta} \cdot \mathbf{N}(\nu_\mathbf{i}, \nu_\mathbf{f}, \beta) \leq \sqrt{\mathbf{n}} \log \left(\frac{\nu_\mathbf{i}}{\nu_\mathbf{f}} \right)$$

Lemma: Suppose $\mathbf{w} \in \mathcal{N}(\beta^\mathbf{2})$, where $\beta \in (\mathbf{0}, \mathbf{1}/\mathbf{2}]$. Then, $\mathbf{w}^+ \in \mathcal{N}(\beta^\mathbf{2})$ and

$$\frac{\mu(\mathbf{w}^+)}{\mu(\mathbf{w})} \leq \mathbf{1} - \sqrt{\frac{\beta}{\mathbf{n}}}$$

Vavasis-Ye Algorithm

Iteration Complexity Bound: The number of iterations to solve a linear program is

$$\mathcal{O}(\mathbf{n}^{\mathbf{3.5}} \log(\mathbf{n} + \bar{\chi}_{\mathbf{A}}))$$

Note: Their bound does not depend on $\nu_{\mathbf{i}}$ and $\nu_{\mathbf{f}}$!

Their algorithm accelerates an ordinary primal-dual path following method (e.g., the MTY P-C algorithm) by using from time to time a step called the layered-least-square step.

V-Y algorithm is not scaling invariant.

New complexity for the MTY method

Theorem (Monteiro and Tsuchiya 2003): For any $\beta \in (\mathbf{0}, \mathbf{1}/\mathbf{2}]$,

$$\mathbf{N}(\nu_{\mathbf{i}}, \nu_{\mathbf{f}}, \beta) = \mathcal{O}\left(\mathbf{T}(\nu_{\mathbf{i}}/\nu_{\mathbf{f}}) + \mathbf{n}^{\mathbf{3.5}} \log(\bar{\chi}^*_{\mathbf{A}} + \mathbf{n})\right)$$

iterations, where $\bar{\chi}^*_{\mathbf{A}} \equiv \inf\{\bar{\chi}_{\mathbf{AD}} : \mathbf{D} \in \mathcal{D}\}$ and

$$\mathbf{T}(\eta) \equiv \min\left\{\mathbf{n}^{\mathbf{2}} \log(\log \eta), \log \eta\right\}$$

Remark: In contrast to $\bar{\chi}_{\mathbf{A}}$, the quantity $\bar{\chi}^*_{\mathbf{A}}$ is scaling invariant. Usually $\bar{\chi}^*_{\mathbf{A}} << \bar{\chi}_{\mathbf{A}}$. Hence, the above complexity is not comparable to the one associated with the V-Y method.

Lemma: For any $\beta \in (\mathbf{0}, \mathbf{1}/\mathbf{2}]$ and $\mathbf{w} \in \mathcal{N}(\beta^{\mathbf{2}})$:

$$\frac{\mu(\mathbf{w}^+)}{\mu(\mathbf{w})} \leq \frac{\kappa(\mathbf{w})^{\mathbf{2}}}{\beta},$$

where

$$\kappa(\mathbf{w}) := \left(\frac{\|\mathbf{\Delta x^a}(\mathbf{w})\mathbf{\Delta s^a}(\mathbf{w})\|}{\mu(\mathbf{w})}\right)^{\mathbf{1}/\mathbf{2}}$$

Consequences

Under the Turing machine model, the iteration-complexity of the MTY P-C algorithm is

$$\begin{aligned} & \mathcal{O}(\mathbf{n^{3.5}L_A} + \min\{\mathbf{L}, \mathbf{n^2} \log \mathbf{L}\}) \\ \leq \; & \mathcal{O}(\mathbf{n^{3.5}L_A} + \mathbf{L}) \end{aligned}$$

Given $\mathbf{A}$, there exist many nontrivial $(\mathbf{b}, \mathbf{c})$ for which the complexity of the MTY P-C algorithm for solving $(\mathbf{P})$ and $(\mathbf{D})$ is $\mathcal{O}(\mathbf{L})$

Example

Consider the LP

$$\max\{\mathbf{b}^{\mathbf{T}}\mathbf{y} : \mathbf{A}^{\mathbf{T}}\mathbf{y} \le \mathbf{c}\},$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & \frac{\sqrt{6}}{3} & -\frac{\sqrt{6}}{3} & \mathbf{0} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{3} & -\frac{2\sqrt{2}}{3} \\ -1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix},$$

$$\mathbf{b} = \begin{pmatrix} -10^{-9} \\ -10^{-5} \\ -1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 0 \\ \frac{2\sqrt{6}}{3} \\ 0 \\ 0 \end{pmatrix}.$$

Example (continued)

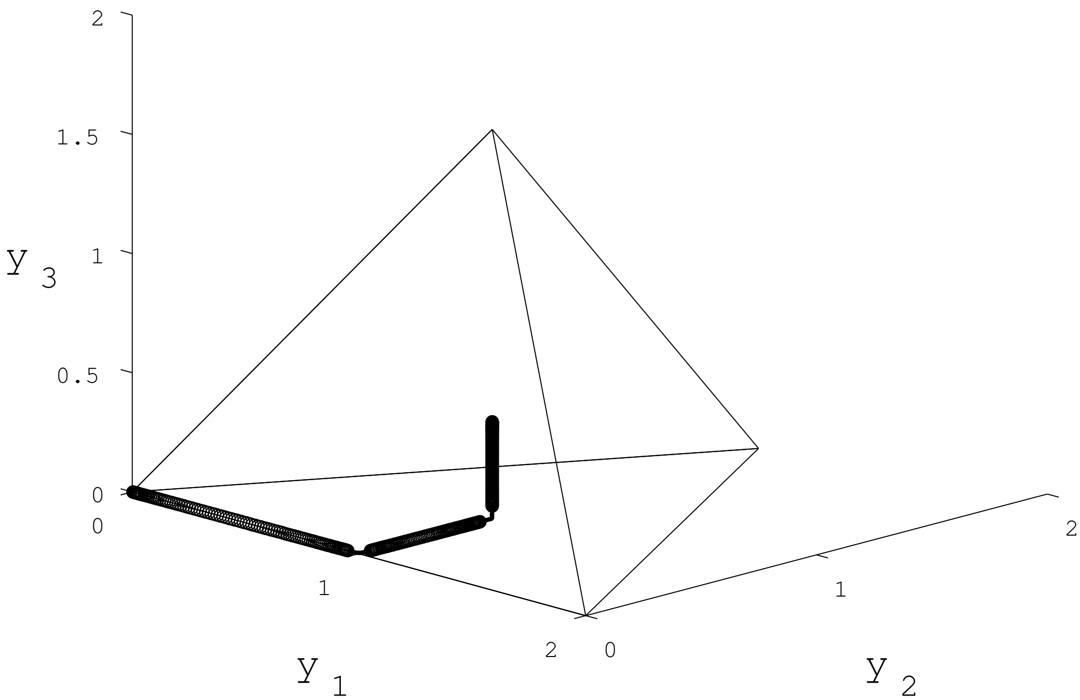


Figure 1: Figure for the LP instance

Example (continued)

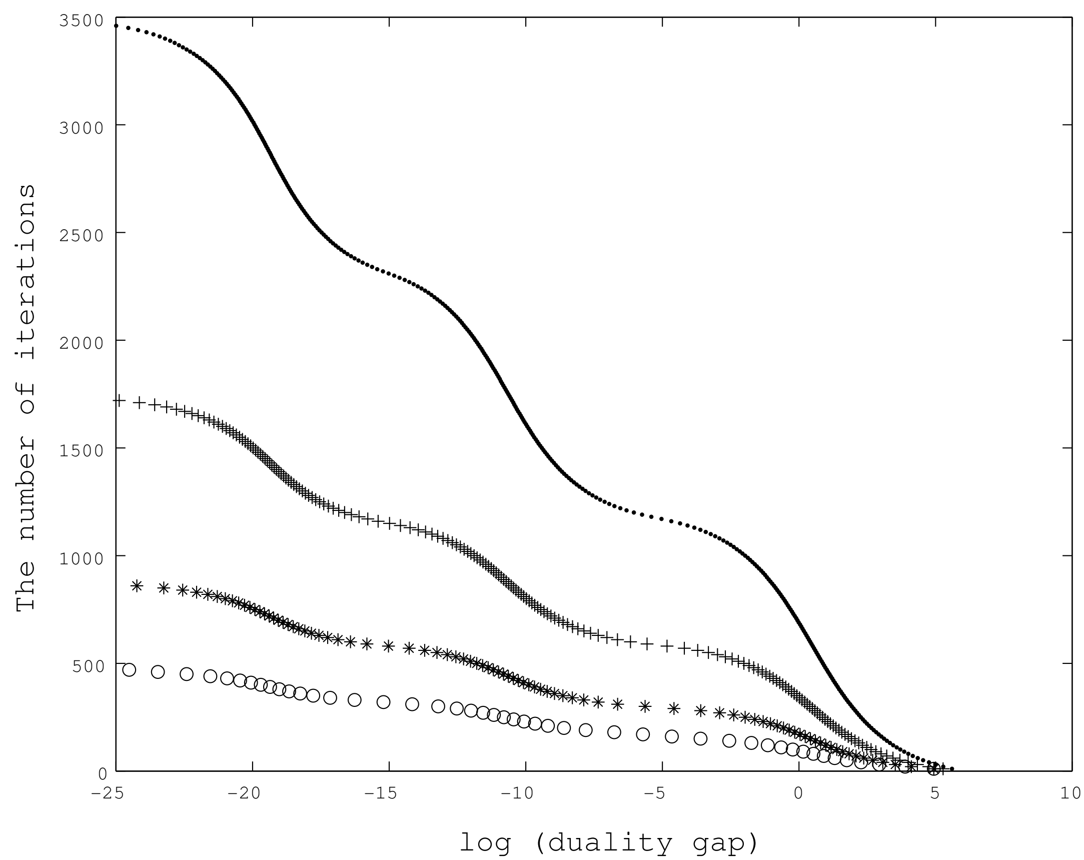


Figure 2: $\log \mu$ **versus** $\mathbf{N}(\nu_{\mathbf{i}}, \mu, \beta)$ **(**$\cdot : \sqrt{\beta} =$ **0.0025**; $+ : \sqrt{\beta} =$ **0.005**; $* : \sqrt{\beta} =$ **0.01**; $\circ : \sqrt{\beta} =$ **0.02)**

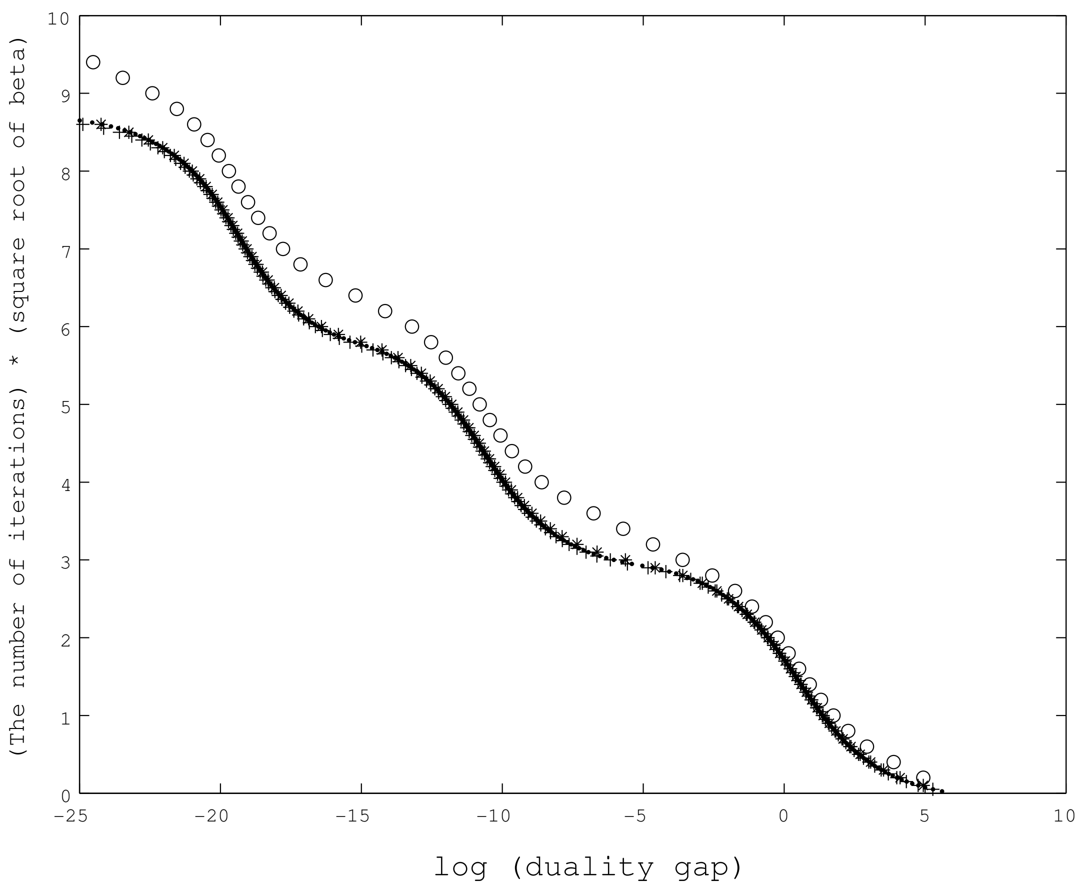


Figure 3: $\log \mu$ **versus** $\sqrt{\beta} \cdot \mathbf{N}(\nu_{\mathbf{i}}, \mu, \beta)$ **(**$\cdot : \sqrt{\beta} =$ **0.0025**; $+ : \sqrt{\beta} =$ **0.005**; $* : \sqrt{\beta} =$ **0.01**; $\circ : \sqrt{\beta} =$ **0.02)**

Question: Does $\sqrt{\beta} \cdot \mathbf{N}(\nu_{\mathbf{i}}, \mu, \beta)$ **always converge as** $\beta \to 0$**?**

Example (continued)

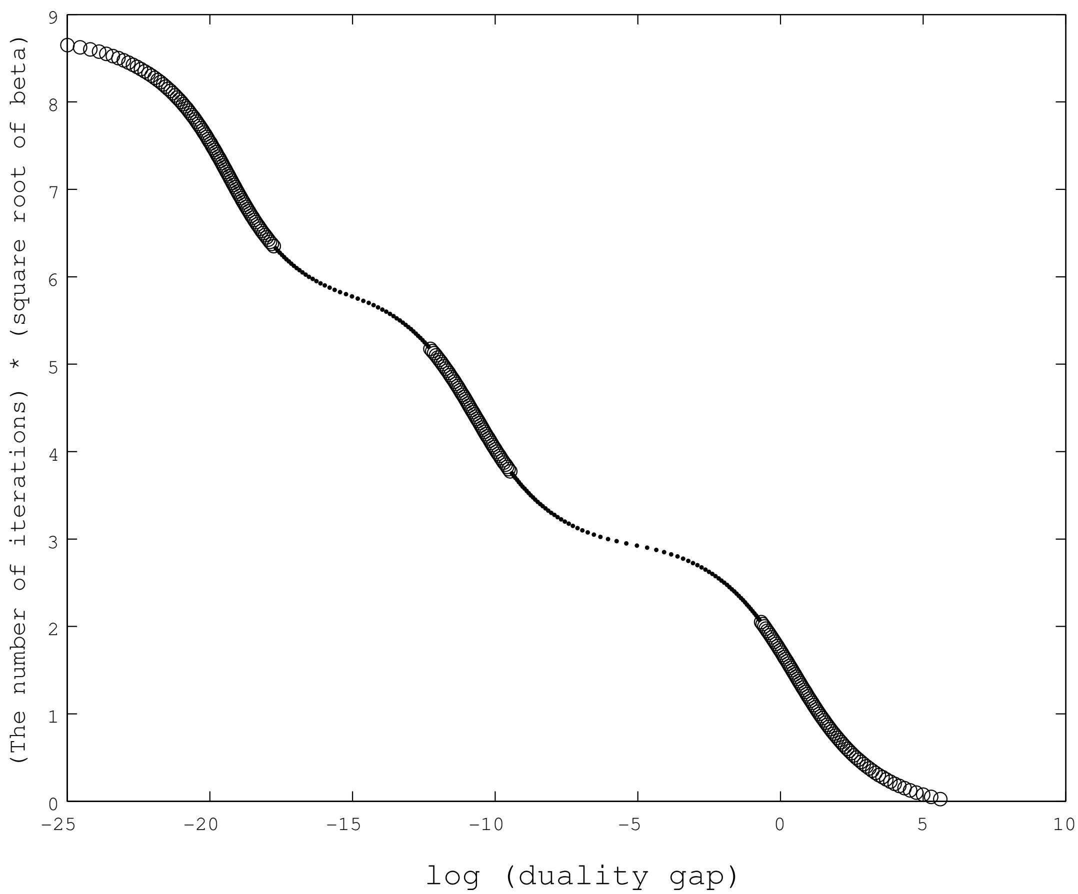


Figure 4: $\log \mu$ **versus** $\sqrt{\beta} \cdot \mathbf{N}(\nu_{\mathbf{i}}, \mu, \beta)$ **(The big dots correspond to the ones in Figure 1.)**

Question: How to define straight and curved parts of the central path?

Curvature of the central path

Definition: **The curvature of the central path is the function** $\kappa : (\mathbf{0}, \infty) \to [\mathbf{0}, \infty)$ **defined as**

$$\kappa(\nu) \equiv \|\nu\dot{\mathbf{x}}(\nu)\dot{\mathbf{s}}(\nu)\|^{\mathbf{1/2}}, \quad \forall \nu > \mathbf{0}.$$

Note: **if** $\mathbf{w} = \mathbf{w}(\nu)$ **then** $\kappa(\mathbf{w}) = \kappa(\nu)$

For a given $\nu > \mathbf{0}$ **and** $\beta \in (\mathbf{0}, \mathbf{1})$, **define**

$$\mathcal{T}(\beta, \nu) \equiv \{\mathbf{t} \in \Re : \mathbf{w}(\nu) - \mathbf{t}\nu\dot{\mathbf{w}}(\nu) \in \mathcal{N}(\beta)\}$$

Note that $\mathbf{w}(\nu) - \mathbf{t}\nu\dot{\mathbf{w}}(\nu) \approx \mathbf{w}((\mathbf{1} - \mathbf{t})\nu)$.

Proposition: $\mathcal{T}(\beta, \nu)$ **is a closed interval and**

$$\lim_{\beta \downarrow \mathbf{0}} \frac{\textbf{length of } \mathcal{T}(\beta, \nu)}{\sqrt{\beta}} = \frac{\mathbf{2}}{\kappa(\nu)}$$

Complexity in terms of the curvature

Theorem (Sonnevend, Stoer and Zhao 1994):

$$\mathbf{N}(\nu_{\mathbf{i}}, \nu_{\mathbf{f}}, \beta) = \mathcal{O}\left(\int_{\nu_{\mathbf{f}}}^{\nu_{\mathbf{i}}} \frac{\kappa(\nu)}{\nu} \mathbf{d}\nu + \log\left(\frac{\nu_{\mathbf{i}}}{\nu_{\mathbf{f}}}\right)\right).$$

Note: **Since $\kappa(\nu) \leq \sqrt{\mathbf{n}/\mathbf{2}}$ for all $\nu > \mathbf{0}$, the classical bound follows from the above bound.**

Theorem 1 (Monteiro and Tsuchiya 2005):

$$\begin{aligned} \lim_{\beta \to \mathbf{0}} \sqrt{\beta} \cdot \mathbf{N}(\nu_{\mathbf{i}}, \nu_{\mathbf{f}}, \beta) &= \int_{\nu_{\mathbf{f}}}^{\nu_{\mathbf{i}}} \frac{\kappa(\nu)}{\nu} \mathbf{d}\nu \\ &\leq \sqrt{\mathbf{n}} \log\left(\frac{\nu_{\mathbf{i}}}{\nu_{\mathbf{f}}}\right) \end{aligned}$$

Recall that one of the M-T bounds is

$$\mathbf{N}(\nu_{\mathbf{i}}, \nu_{\mathbf{f}}, \beta) = \mathcal{O}\left(\mathbf{n}^{\mathbf{3.5}} \log(\bar{\chi}_{\mathbf{A}}^* + \mathbf{n}) + \log\left(\frac{\nu_{\mathbf{i}}}{\nu_{\mathbf{f}}}\right)\right).$$

Bound on the curvature integral

Theorem 2 (Monteiro and Tsuchiya 2005):

For every $0 < \nu_f < \nu_i$, we have:

$$\int_{\nu_f}^{\nu_i} \frac{\kappa(\nu)}{\nu} \mathbf{d}\nu \le \mathcal{O}\left(\mathbf{n}^{3.5} \log(\bar{\chi}^*_{\mathbf{A}} + \mathbf{n})\right)$$

Hence,

$$\int_{\mathbf{0}}^{\infty} \frac{\kappa(\nu)}{\nu} \mathbf{d}\nu \le \mathcal{O}\left(\mathbf{n}^{3.5} \log(\bar{\chi}^*_{\mathbf{A}} + \mathbf{n})\right)$$

Geometry of the central path

Vavasis and Ye 1996: **"The central path consists of $\mathcal{O}(n^2)$ long and straight parts and other curved parts"**

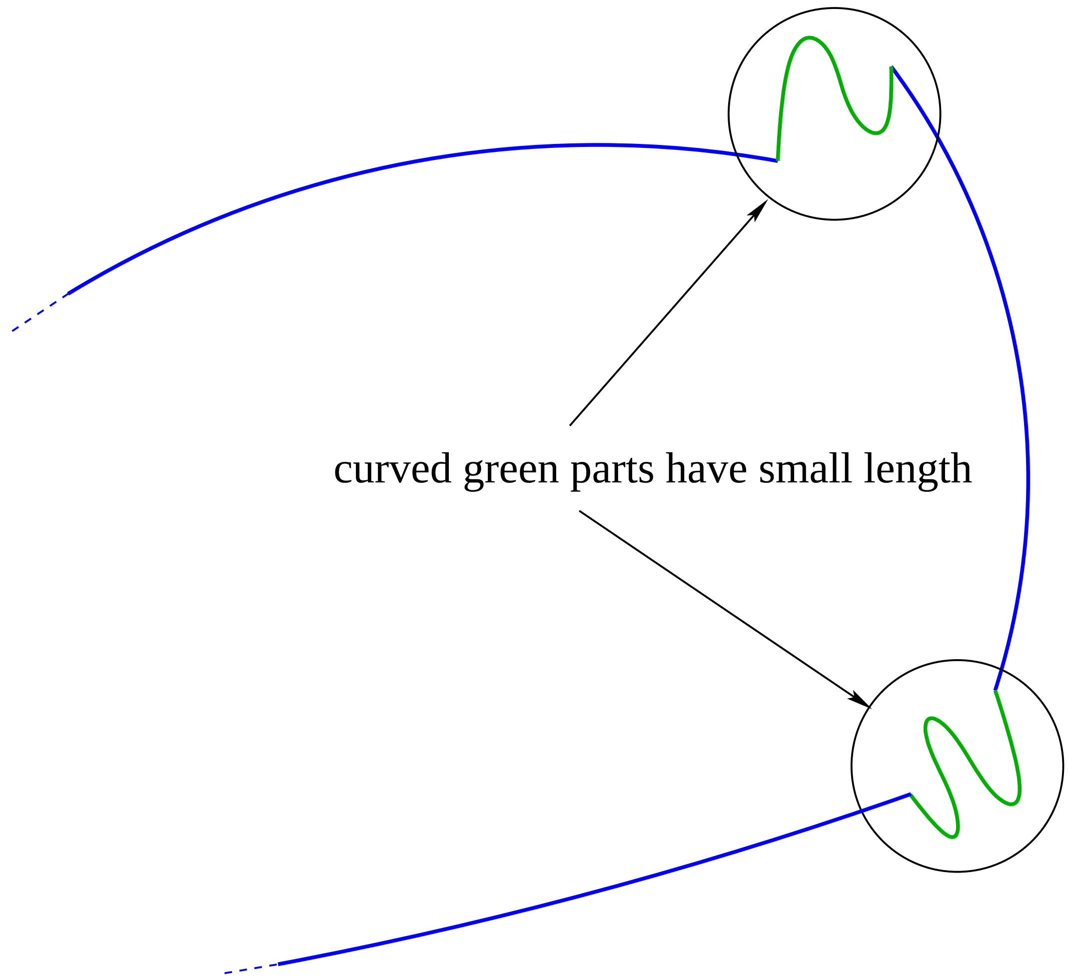
We want to formally establish this statement!

Theorem 3: **For any $\bar{\kappa} \in (\mathbf{0}, \sqrt{\mathbf{n}/\mathbf{2}})$, there exist $\mathbf{l} \leq \mathbf{n}(\mathbf{n}-\mathbf{1})/\mathbf{2}$ closed intervals $\mathbf{I_k}$ such that:**

a) $\{\nu > \mathbf{0} : \kappa(\nu) \geq \bar{\kappa}\} \subseteq \cup_{\mathbf{k}=\mathbf{1}}^{\mathbf{l}} \mathbf{I_k}$
(union of $\mathbf{I_k}$'s covers portion with large curvature)

b) **the logarithmic length of each $\mathbf{I_k}$ is bounded by** $\mathcal{O}\left(\mathbf{n} \log(\bar{\chi}_{\mathbf{A}}^* + \mathbf{n}) + \mathbf{n} \log \bar{\kappa}^{-\mathbf{1}}\right)$
(independent of b and c)

Geometric illustration of the C-P



The blue parts are long but quite straight!

The MTY P-C algorithm converges R-quadratically over the blue parts.

There are at most $\mathcal{O}(n^2)$ blue and green parts.

Directions for future research

- **Generalizations to other cone programming problems such as SOCP and SDP**
- **Are infeasible path following methods ammenable to the same kind of analysis? Can new iteration complexity bounds be obtained for them?**
- **Is it possible to interpret the curvature $\kappa(\nu)$ as the one used in differential geometry? What further insights can be gained through this approach?**
- **Can an iteration complexity bound depending only on $\mathbf{n}$ and $\bar{\chi}^*_{\mathbf{A}}$ be derived for the MTY P-C algorithm?**
- **Is it possible to derive a Zhao and Stoer's type result with $\log\log$, i.e.**

$$\mathbf{N}(\nu_{\mathbf{i}}, \nu_{\mathbf{f}}, \beta) = \mathcal{O}\left(\int_{\nu_{\mathbf{f}}}^{\nu_{\mathbf{i}}} \frac{\kappa(\nu)}{\nu} \mathbf{d}\nu + \mathbf{n^2} \log\log\left(\frac{\nu_{\mathbf{i}}}{\nu_{\mathbf{f}}}\right)\right).$$