Stein's method and stochastic analysis of Rademacher functionals

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A brief history

Stein's method for normal approximation:

First published in *Stein (1972)*; in *Stein (1986)* the method is explained via exchangeable pairs.

In *Barbour (1990)* the so-called generator approach for Stein's method is developed, and applied to diffusion approximation.

In *Götze (1991)* the generator approach is used to obtain multivariate normal approximations.

In *Nourdin and Peccati (2008)* the method is extended to Wiener chaos, using Malliavin calculus.

Nourdin and Peccati (2008) concentrate on functions of Gaussian variables.

Here: extend to functions of centered symmetric Bernoulli (*Rademacher*) variables; in particular functions of infinitely many Rademacher variables.

1. Stein's method for univariate normal approximation

Stein (1972, 1986); Chen and Shao (2005), Daly (2008), Barbour (1990)

$Z \sim \mathcal{N}(0,1)$ if and only if for all smooth functions f,

$$\mathsf{E}Zf(Z) = \mathsf{E}f'(Z).$$

For a random variable W with $\mathsf{E}W = 0, \mathrm{Var}\, W = 1$, if

$$\mathsf{E}f'(W) - \mathsf{E}Wf(W)$$

is close to zero for many functions f, then W should be close to Z in distribution.

Given a test function h, let $Nh = \mathsf{E}h(Z)$, and solve for f in the *Stein equation*

$$f'(w) - wf(w) = h(w) - Nh.$$

Now evaluate the expectation of the r.h.s. by the expectation of the l.h.s.

Can bound

$\| f' \| \leq 4 \| h \|$; and $\| f' \| \leq \| h'' \|$;

$\| f'' \| \leq 2 \| h' \|$;

$\| f''' \| \leq 2 \| h'' \|$.

Example: sum of i.i.d. Rademacher variables

$X, X_1, \ldots, X_n$ i.i.d. with $P(X = 1) = \frac{1}{2} = P(X = -1)$. Then $EX = 0, \mathrm{Var}\, X = 1$. Put $W = W(X_1, \ldots, X_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$.

Then (by Stein's method), for any smooth h,

$$|Eh(W) - Nh| \;\leq\; \frac{3}{\sqrt{n}} \|h'\|.$$

Using the zero-bias transformation (*Goldstein and R. (1997)*) and the symmetry of the distribution of X:

$$|Eh(W) - Nh| \;\leq\; \frac{3}{n} \left(\|h^{(3)}\| + \frac{1}{2} \|h^{(4)}\| \right).$$

Now let $X = \{X_n : n \geq 1\}$ denote an infinite sequence of i.i.d. standard Rademacher variables, so that $P(X_i = 1) = \frac{1}{2} = P(X_i = -1)$.

A (possibly infinite) *Rademacher average* is

$$F = \sum_{i=1}^{\infty} \alpha_i X_i.$$

Here we will present a straightforward framework which gives as easy corollary that for

$$F_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i,$$

we have

$$\Big| E[h(F)] - E[h(Z)] \Big| \leq \frac{20}{3n} \|h''\|.$$

We shall also be able to tackle infinite sums; for $r \geq 2$, we set

$$F_r = \sqrt{r} \sum_{i \geq r} \frac{X_i}{i}$$

and obtain

$$\begin{aligned} &|E[h(F_r)] - E[h(Z)]| \\ &\leq \frac{\min(4\|h\|_\infty, \|h''\|_\infty)}{r} + \frac{20\|h''\|_\infty}{3(r-1)}. \end{aligned}$$

For such and more general results: differential calculus on infinite spaces.

2. Framework: discrete Malliavin calculus

See also *Nourdin and Peccati (2008); Privault (2008), Privault and Schoutens (2002)*

For $X = \{X_n : n \geq 1\}$ a standard Rademacher sequence, on a probability space $(\Omega, \mathcal{F}, P)$, we put $\Omega = \{-1, 1\}^{\mathbb{N}}$ and $P = \left[\frac{1}{2}\{\delta_{-1} + \delta_1\}\right]^{\mathbb{N}}$.

For every $N \geq 1$ define a random signed measure μ_N on $\{1, ..., N\}$: for $A \subset \{1, ..., N\}$

$$\mu_N(A) = \sum_{j \in A} X_j.$$

The *diagonal* of $\mathbb{N}^n$ for $n \geq 2$:

$$\Delta_n = \left\{(i_1, ..., i_n) \in \mathbb{N}^n : \quad \text{the } i_j\text{'s all different}\right\},$$

and, for $N, n \geq 2$,

$$\Delta_n^N = \Delta_n \cap \{1, ..., N\}^n .$$

On the diagonal,

$$\begin{aligned}\mu_N^{\otimes 2}\left([A \times B] \cap D\right) &= \sum_j X_j^2 \mathbf{1}_{\{j \in A\}} \mathbf{1}_{\{j \in B\}} \\ &= \sharp\{j : j \in A \cap B\} \\ &= \kappa\left(A \cap B\right);\end{aligned}$$

the diagonal has non-zero measure.

Here

$$\kappa\left(A \cap B\right) = \sharp\{j : j \in A \cap B\},$$

is the counting measure, where $A, B \subset \{1, ..., N\}$; denote its product measures by $\kappa^{\otimes n}$ for $n \geq 2$.

Classes of functions

For $n \geq 1$, we denote by $\ell^2(\mathbb{N})^n$ the class of functions on $\mathbb{N}^n$ that are square integrable with respect to $\kappa^{\otimes n}$;

$\ell^2(\mathbb{N})^{\circ n}$ is the subset of $\ell^2(\mathbb{N})^n$ composed of symmetric functions;

$\ell_0^2(\mathbb{N})^n$ is the subset of $\ell^2(\mathbb{N})^n$ composed of functions vanishing on diagonals;

$\ell_0^2(\mathbb{N})^{\circ n}$ is the subset of $\ell_0^2(\mathbb{N})^n$ composed of symmetric functions.

Multiple integrals

For every $q \geq 1$ and every $f \in \ell_0^2(\mathbb{N})^{\circ q}$ define the *multiple integral* (of order q) of f with respect to X:

$$\begin{aligned} J_q(f) &= \sum_{(i_1,...,i_q)\in\mathbb{N}^q} f(i_1,...,i_q) X_{i_1}\cdots X_{i_q} \\ &= \sum_{(i_1,...,i_q)\in\Delta^q} f(i_1,...,i_q) X_{i_1}\cdots X_{i_q} \\ &= q! \sum_{i_1<...<i_q} f(i_1,...,i_q) X_{i_1}\cdots X_{i_q}, \end{aligned}$$

where the possibly infinite sum converges in $L^2(\Omega)$.

Set $\ell^2(\mathbb{N})^{\circ 0} = \mathbb{R}$, and $J_0(c) = c$, $\forall c \in \mathbb{R}$.

Isometry and Chaos

Isometry: if $f \in \ell_0^2(\mathbb{N})^{\circ q}$ and $g \in \ell_0^2(\mathbb{N})^{\circ p}$, then

$$E[J_q(f)J_p(g)] = \mathbf{1}_{\{q=p\}} q! \langle f, g \rangle_{\ell^2(\mathbb{N})^{\otimes q}}.$$

The collection of all random variables of the type $J_n(f)$, where $f \in \ell_0^2(\mathbb{N})^{\circ q}$, is called the qth *chaos* associated with X; it is also called *Walsh chaos* and *Rademacher chaos*.

Chaotic decomposition

For every $F \in L^2(\sigma\{X\})$ there exists a unique sequence of functions $f_n \in \ell_0^2(\mathbb{N})^{\circ n}$, $n \geq 1$, such that

$$\begin{aligned} F &= E(F) + \sum_{n\geq 1} J_n(f_n) \\ &= E(F) + \\ &\quad \sum_{n\geq 1} n! \sum_{i_1<i_2<\ldots<i_n} f_n(i_1,\ldots,i_n) X_{i_1} \cdots X_{i_n}, \end{aligned}$$

where the series converge in L^2.

Discrete Malliavin operators

The *gradient operator* D: The domain dom$D$ is the class of random variables $F \in L^2(\sigma\{X\})$ such that the functions $f_n \in \ell_0^2(\mathbb{N})^{\circ n}$ in the chaotic expansion $F = E(F) + \sum_{n\geq 1} J_n(f_n)$ satisfy

$$\sum_{n\geq 1} n n! \|f_n\|^2_{\ell^2(\mathbb{N})^{\otimes n}} < \infty.$$

For such F define

$$D_k F = \sum_{n\geq 1} n J_{n-1}(f_n(\cdot, k)), \quad k \geq 1,$$

where the symbol $f_n(\cdot, k)$ indicates that the integration is performed with respect to $n-1$ variables.

Alternative representation:

Let $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$, and set

$$\omega_+^k = (\omega_1, \omega_2, \ldots, \omega_{k-1}, +1, \omega_{k+1}, \ldots)$$

and

$$\omega_-^k = (\omega_1, \omega_2, \ldots, \omega_{k-1}, -1, \omega_{k+1}, \ldots).$$

Write $F_k^{\pm}$ instead of $F(\omega_{\pm}^k)$ for simplicity. Then

$$D_k F(\omega) = \frac{1}{2}\Big(F_k^+ - F_k^-\Big), \; k \geq 1.$$

The random variables $D_k F$, F_k^+ and F_k^- are independent of X_k.

Write δ for the adjoint of D, also called the *divergence operator*, defined via the following *integration by parts formula*:

for every $F \in \mathrm{dom}D$ and every $u \in \mathrm{dom}\delta \subset L^2(\Omega \times \mathbb{N}, P \otimes \kappa)$

$$\begin{aligned} E[F\delta(u)] &= E[\langle DF, u\rangle_{\ell^2(\mathbb{N})}] \\ &= \langle DF, u\rangle_{L^2(\Omega\times\mathbb{N},P\otimes\kappa)}. \end{aligned}$$

Ornstein-Uhlenbeck operator

Let $L^2_0(\sigma\{X\})$ be the subspace of $L^2(\sigma\{X\})$ of centered random variables. Define the *Ornstein-Uhlenbeck operator*: The domain dom$L$ are all random variables $F = E(F) + \sum_{n\geq 1} J_n(f_n) \in L^2(\sigma\{X\})$ such that

$$\sum_{n\geq 1} n^2 n! \|f_n\|^2_{\ell^2(\mathbb{N})^{\otimes n}} < \infty,$$

and, for $F \in$ dom$L$, we set

$$LF = -\sum_{n\geq 1} n J_n(f_n).$$

Crucial relation:

$$\delta D = -L.$$

The inverse of L is

$$L^{-1}F = -\sum_{n\geq 1} \frac{1}{n} J_n(f_n).$$

Note that the random variable $D_k L^{-1} F$ is independent of X_k.

The chain rule

Unfortunately the chain rule is not as straightforward as in the continuous case, but with Taylor expansion we can show:

Proposition 1. (Chain Rule). *Let* $F \in \mathrm{dom}D$ *and* $f : \mathbb{R} \to \mathbb{R}$ *be three times differentiable with bounded third derivative. Assume moreover that* $f(F) \in \mathrm{dom}D$. *Then, for any integer* k, P*-a.s.:*

$$\begin{aligned} &\Big| D_k f(F) - f'(F) D_k F \\ &\qquad + \frac{1}{2}\Big(f''(F_k^+) + f''(F_k^-)\Big)(D_k F)^2 X_k \Big| \\ &\leq \frac{10}{3} |f'''|_\infty |D_k F|^3. \end{aligned}$$

Sketch of proof

By Taylor expansion,

$$\begin{aligned}
& D_k f(F) \\
&= \frac{1}{2}\Big(f(F_k^+) - f(F_k^-)\Big) \\
&= \frac{1}{2}\Big(f(F_k^+) - f(F)\Big) - \frac{1}{2}\Big(f(F_k^-) - f(F)\Big) \\
&\approx \frac{1}{2}f'(F)(F_k^+ - F) + \frac{1}{4}f''(F)(F_k^+ - F)^2 \\
&\quad -\frac{1}{2}f'(F)(F_k^- - F) - \frac{1}{4}f''(F)(F_k^- - F)^2 \\
&= f'(F)D_kF \\
&\quad +\frac{1}{4}f''(F)\Big((F_k^+ - F)^2 - (F_k^- - F)^2\Big).
\end{aligned}$$

Now

$$
\begin{aligned}
&(F_k^+ - F)^2 - (F_k^- - F)^2 \\
&\quad = (F_k^+ - F)^2 \mathbf{1}_{X_k=-1} + (F_k^+ - F)^2 \mathbf{1}_{X_k=1} \\
&\qquad -(F_k^- - F)^2 \mathbf{1}_{X_k=-1} - (F_k^- - F)^2 \mathbf{1}_{X_k=1} \\
&\quad = (F_k^+ - F_k^-)^2 \mathbf{1}_{X_k=-1} - (F_k^- - F^+)^2 \mathbf{1}_{X_k=1} \\
&\quad = -X_k (F_k^+ - F_k^-)^2 \\
&\quad = -4 X_k (D_k F)^2.
\end{aligned}
$$

Using the approximation

$$
f''(F) \approx \frac{1}{2}\Big(f''(F_k^+) - f''(F_k^-) \Big)
$$

we obtain that

$$
\begin{aligned}
&D_k f(F) \\
&\quad \approx f'(F) D_k F \\
&\qquad + \frac{1}{8}\Big(f''(F_k^+) + f''(F_k^-) \Big) \\
&\qquad \times \Big((F_k^+ - F)^2 - (F_k^- - F)^2 \Big) \\
&\quad = f'(F) D_k F - \frac{1}{2}\Big(f''(F_k^+) + f''(F_k^-) \Big) (D_k F)^2 X_k.
\end{aligned}
$$

Bounding the remainder terms in the approximation gives the result. □

3. Bounds to the normal for functions of Rademacher sequences

Our main result is

Theorem 1. *Let* $F \in \text{dom}D$ *be centered and such that* $\sum_k E\Big|D_kF\Big|^4 < \infty$. *Let* $h \in \mathcal{C}_b^2$ *and* $Z \sim \mathcal{N}(0,1)$. *Then*

$$\begin{aligned} &|E[h(F)] - E[h(Z)]| \\ &\quad \leq \min(4\|h\|_\infty, \|h''\|_\infty)B_1 + \|h''\|_\infty B_2, \end{aligned}$$

where

$$\begin{aligned} B_1 &= E\Big|1 - \langle DF, -DL^{-1}F\rangle_{\ell^2(\mathbb{N})}\Big| \\ &\leq \sqrt{E\Big[(1 - \langle DF, -DL^{-1}F\rangle_{\ell^2(\mathbb{N})})^2\Big]} \end{aligned}$$

and

$$B_2 = \frac{20}{3} E\left[\Big\langle |DL^{-1}F|, |DF|^3 \Big\rangle_{\ell^2(\mathbb{N})}\right].$$

Comparison with *Nourdin and Peccati (2008)*

Nourdin and Peccati (2008) derive a normal approximation of random variables based on a centered Gaussian family X on a real separable Hilbert space $\mathfrak{H}$, with $E\left[X(h)X(g)\right] = \langle h, g\rangle_{\mathfrak{H}}$ (an *isonormal Gaussian process*). For

$$F = g\left(X(\phi_1), \ldots, X(\phi_n)\right),$$

where $n \geq 1$, $g : \mathbb{R}^n \to \mathbb{R} \in \mathcal{C}_c^\infty$ and $\phi_i \in \mathfrak{H}$, define the *Malliavin derivative*

$$DF \;=\; \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}\left(X(\phi_1), \ldots, X(\phi_n)\right)\phi_i.$$

The Malliavin derivative D verifies the usual *chain rule*:

$$D\,\varphi(F) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(F) DF_i.$$

Then *Nourdin and Peccati (2008)* prove that for Wasserstein distance to Z, standard normal, and centered F,

$$d_W(F, Z) \leq E|1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}|.$$

This bound corresponds to our term B_1; the term B_2 arises from the underlying process being a Rademacher sequence, rather than Gaussian.

Proof of Theorem 1.

With Stein's method, for $h \in \mathcal{C}_b^2$,

$$\Big| E[h(F)] - E[h(Z)] \Big| = \Big| E[f'(F) - Ff(F)] \Big|$$

with $f = f_h$ the solution of the Stein equation for h. Then $f(F) \in \mathrm{dom} D$ and

$$\begin{aligned} E\Big[Ff(F)\Big] &= E\Big[LL^{-1}Ff(F)\Big] \\ &= -E\Big[\delta DL^{-1}Ff(F)\Big] \\ &= E\Big[\langle Df(F), -DL^{-1}F\rangle_{\ell^2(\mathbb{N})}\Big]. \end{aligned}$$

Hence

$$\begin{aligned} & E\Big[f'(F) - Ff(F)\Big] \\ &= E\Big[f'(F) - \langle Df(F), -DL^{-1}F\rangle_{\ell^2(\mathbb{N})}\Big]. \end{aligned}$$

By the chain rule,

$$
\begin{aligned}
&\langle Df(F), -DL^{-1}F\rangle_{\ell^2(\mathbb{N})} \\
&= -\sum_k D_k f(F) D_k L^{-1} F \\
&\approx -f'(F) \sum_k D_k F D_k L^{-1} F \\
&\quad + \frac{1}{2} \sum_k \Big(f''(F_k^+) + f''(F_k^-) \Big) (D_k F)^2 X_k D_k L^{-1} F \\
&= f'(F) \langle DF, -DL^{-1}F\rangle_{\ell^2(\mathbb{N})} \\
&\quad + \frac{1}{2} \sum_k \Big(f''(F_k^+) + f''(F_k^-) \Big) (D_k F)^2 X_k D_k L^{-1} F
\end{aligned}
$$

Recall that the random variables $D_k F$, $D_k L^{-1} F$, F_k^+ and F_k^- are independent of X_k; and so

$$E\left[D_k L^{-1} F \times \Big(f''(F_k^+) + f''(F_k^-)\Big)(D_k F)^2 X_k\right] = 0.$$

Hence

$$\begin{aligned} & E\Big[f'(F) - Ff(F)\Big] \\ \approx\ & E\Big[f'(F)(1 - \langle DF, -DL^{-1}F\rangle_{\ell^2(\mathbb{N})}\Big]. \end{aligned}$$

Now apply the bounds on the solution of the Stein equation. □

In the special case that F has the form of a multiple integral of the type $F = J_q(f)$, where $f \in \ell_0^2(\mathbb{N})^{\circ q}$, the terms in the bounds simplify;

$$\langle DF, -DL^{-1}F\rangle_{\ell^2(\mathbb{N})} = \frac{1}{q}\|DF\|^2_{\ell^2(\mathbb{N})},$$
$$\langle |DL^{-1}F|, |DF|^3\rangle_{\ell^2(\mathbb{N})} = \frac{1}{q}\|DF\|^4_{\ell^4(\mathbb{N})}.$$

For a *Rademacher average*

$$F = \sum_{i=1}^{\infty} \alpha_i X_i$$

we have $q = 1$ and $D_i F = \alpha_i$ for all $i \geq 1$, which gives the bounds mentioned above.

4. Connection with exchangeable pairs

Assume that

$$\begin{aligned} F &= \sum_{n=1}^{d} \sum_{1 \le i_1 < \ldots < i_n \le d} n! f_n(i_1, ..., i_n) X_{i_1} \cdots X_{i_n} \\ &= \sum_{n=1}^{d} J_n(f_n) \qquad (1) \end{aligned}$$

and $E(F) = 0$ and $E(F^2) = 1$.

Exchangeable pair: Pick an index I so that $P(I = i) = \frac{1}{d}$ for $i = 1, \ldots, d$, independently of $X_1, ..., X_d$, and if $I = i$ replace X_i by an independent copy X_i^* in all sums in the decomposition (1) which involve X_i. Call the resulting expression F'. Also denote the vector of Rademacher variables with the exchanged component by $\mathbf{X}'_d$. Then (F, F') forms an exchangeable pair.

Note that

$$
\begin{aligned}
&E(J_n'(f_n) - J_n(f_n)|\mathbf{W}) \\
&= -\frac{1}{d}\sum_{i=1}^{d} \sum_{1\le i_1<\ldots<i_n\le d} \mathbf{1}_{\{i_1,\ldots,i_n\}}(i)\; n! \\
&\qquad \times f_n(i_1,\ldots,i_n)E(X_{i_1}\cdots X_{i_n}|\mathbf{W}) \\
&= -\frac{n}{d}\, J_n(f_n)
\end{aligned}
$$

and we obtain the simple expression

$$E(F' - F|\mathbf{W}) = \frac{1}{d}LF = -\frac{1}{d}\delta DF.$$

Coupling bound

Theorem 2. *Denote by L' the Ornstein-Uhlenbeck operator for the exchanged Rademacher sequence $\mathbf{X}'_d$; Z is standard normal. Then*

$$
\begin{aligned}
&|E[h(F)] - E[h(Z)]| \\
\leq\ & 4\|h\|_\infty \sqrt{\mathrm{Var}\left[\frac{d}{2} E\left((F'-F)\big((L')^{-1}F' - L^{-1}F\big)\big|\mathbf{W}\right)\right]} \\
&+\frac{d}{2}\|h'\|_\infty E\left[(F'-F)^2|(L')^{-1}F' - L^{-1}F|\right].
\end{aligned}
$$

5. Example: Bounds for infinite 2-runs

Let $\xi = \{\xi_n : n \in \mathbb{Z}\}$ be a standard *Bernoulli* sequence; $P(\xi_i = 0) = \frac{1}{2} = P(\xi_i = 1)$. Put

$$G_n = \sum_{i \in \mathbb{Z}} \alpha_i^{(n)} \xi_i \xi_{i+1},$$

where $\{\alpha^{(n)} : n \geq 1\} \in \ell^2(\mathbb{Z})$.

Put $F_n = \frac{G_n - E(G_n)}{\sqrt{\mathrm{Var} G_n}}$.

Proposition 2. *Let* $h \in \mathcal{C}_b^2$. *Then, for* $Z \sim \mathcal{N}(0,1)$,

$$\begin{aligned}
&\left|E[h(F)] - E[h(Z)]\right| \\
&\leq \frac{7}{16}\frac{\min(4\|h\|_\infty, \|h''\|_\infty)}{\mathrm{Var}G_n}\sqrt{\sum_{i\in\mathbb{Z}}(\alpha_i^{(n)})^4} \\
&\quad + \frac{35}{24}\,\frac{\|h''\|_\infty}{(\mathrm{Var}G_n)^2}\,\sum_{i\in\mathbb{Z}}(\alpha_i^{(n)})^4
\end{aligned}$$

with

$$\mathrm{Var}G_n = \frac{3}{16}\sum_{i\in\mathbb{Z}}(\alpha_i^{(n)})^2 + \frac{1}{8}\sum_{i\in\mathbb{Z}}\alpha_i^{(n)}\alpha_{i+1}^{(n)}.$$

It follows that a sufficient condition to have $F_n \overset{\mathrm{Law}}{\to} Z$ is that

$$\sum_{i\in\mathbb{Z}}(\alpha_i^{(n)})^4 = o\Big((\mathrm{Var}G_n)^2\Big) \quad \text{as } n \to \infty.$$

Sketch of the argument

Note that

$$F_n = \frac{G_n - E(G_n)}{\sqrt{\text{Var} G_n}} = J_1(f) + J_2(g),$$

with

$$f = \frac{1}{4\sqrt{\text{Var} G_n}} \sum_{a \in \mathbb{Z}} \alpha_a^{(n)} \Big(\mathbf{1}_{\{a\}} + \mathbf{1}_{\{a+1\}} \Big)$$

and

$$g = \frac{1}{8\sqrt{\text{Var} G_n}} \sum_{a \in \mathbb{Z}} \alpha_a^{(n)} \Big(\mathbf{1}_{\{a\}} \otimes \mathbf{1}_{\{a+1\}} + \mathbf{1}_{\{a+1\}} \otimes \mathbf{1}_{\{a\}} \Big).$$

Thus we can write F_n as sum of a single and a double integral and apply our main result.

6. Example: Fractional Cartesian products

Blei and Janson (2004)

Fix integers $d \geq 3$ and $2 \leq m \leq d-1$, and consider a collection $\{S_1, ..., S_d\}$ of distinct non-empty subsets of $[d] = \{1, ..., d\}$ such that:

(i) $S_i \neq \emptyset$,

(ii) $\bigcup_i S_i = [d]$,

(iii) $|S_i| = m$ for every i,

(iv) each index $j \in [d]$ appears in exactly m of the sets S_i, and

(v) the cover $\{S_1, ..., S_d\}$ is connected (i.e., it cannot be partitioned into two disjoint partial covers).

For $\mathbf{y}_d = (y_1, ..., y_d) \in \mathbb{N}^d$, put

$$\pi_{S_i}\mathbf{y} = (y_j : j \in S_i).$$

Select a one-to-one map φ from $[n]^m$ into $[N]$, and define

$$F_N^* = \{(\varphi(\pi_{S_1}\mathbf{k}_d), ..., \varphi(\pi_{S_d}\mathbf{k}_d)) : \mathbf{k}_d \in [n]^d\} \subset [N]^d,$$

put $F_N^{**} = F_N^* \cap \Delta_N^d$, and also

$$F_N = \mathbf{sym}(F_N^{**}),$$

where $\mathbf{sym}(F_N^{**})$ is the collections of all vectors $\mathbf{y}_d = (y_1, ..., y_d) \in \mathbb{N}^d$ such that

$$(y_{\sigma(1)}, ..., y_{\sigma(d)}) \in F_N^{**}$$

for some permutation σ.

Put

$$\widetilde{S}_N = [d! \times |F_N|]^{-\frac{1}{2}} \sum_{(i_1,...,i_d) \in F_N} X_{i_1} \cdots X_{i_d} = J_d(f_N).$$

Then we can show the following result.

Proposition 3. *Let* $Z \sim \mathcal{N}(0,1)$*, then, for every* $h \in \mathcal{C}_b^2$*, there exists a constant* $K > 0$*, independent of* N*, such that*

$$\left| E[h(\widetilde{S}_N)] - E[h(Z)] \right| \leq \frac{K}{N^{1/4}}.$$

Note:

explicit bound (in contrast to Blei and Janson);

can generalise to multiple integrals defined over infinite sets.

7. Final remarks

Bound in Wasserstein distance for non-smooth functions: are available;

Generalisation to other centered Bernoulli variables: should be possible;

Connection with Poisson case: *Peccati and Taqqu (2008)*.