Bounds on the Constant in the Mean Central Limit Theorem

Larry Goldstein
University of Southern California

Abstract

Bounds in the mean central limit theorem, where the L^1 distance is used to measure the discrepancy of the distribution F_n of a standardized sum of i.i.d. random variables with distribution G from the normal, is of some interest when the normal approximation is to be applied over some wide, and perhaps unspecified range of values. Esseen (1958) showed that the limiting value

$$\lim_{n\to\infty} n^{1/2}||F_n - \Phi||_1 = A(G)$$

exists and Zolotarav (1964) provided an explicit representation of $A(G)$ which allowed for the computation of an asymptotic L^1 Berry Esseen constant of $1/2$.

When F_n is the standardized distribution of a sum of independent random variables $X_1, \ldots, X_n$ with distributions $G_1, \ldots, G_n$ in $\mathcal{F}$, the collection of non-degenerate mean zero distributions with finite absolute third moments, we show that for all finite $n \in \mathbb{N}$, with $\sigma^2 = \mathrm{Var}(X_1 + \cdots + X_n)$ and G^* the G-zero biased distribution,

$$||F_n - \Phi||_1 \le \frac{1}{\sigma^3}\sum_{i=1}^n B(G_i)E|X_i|^3 \quad \text{where} \quad B(G_i) = \frac{2EX_i^2||G^* - G||_1}{E|X_i|^3}$$

and calculate the supremum of the functional $B(G)$ as

$$\sup_{G\in\mathcal{F}} B(G) = 1,$$

thus providing the non-asymptotic bound on the mean central limit theorem Berry Esseen constant of 1. A lower bound of

$$\frac{2\sqrt{\pi}(2\Phi(1)-1) - (\sqrt{\pi}+\sqrt{2}) + 2e^{-1/2}\sqrt{2}}{\sqrt{\pi}} = 0.535377\ldots$$

on the smallest possible constant is also demonstrated.