THE COMPARATIVE STATICS OF

CONSTRAINED OPTIMIZATION PROBLEMS

By John K.-H. Quah[1]

Abstract: The objective of this paper is to develop and apply some new results in the theory of monotone comparative statics. Let f be a real-valued function defined on R^l, and consider the problem of maximizing $f(x)$ when x is constrained to lie in some subset C of R^l. We develop a natural way of ordering constraint sets and identify the conditions on f which guarantee that the solution to the maximization problem increases as the constraint set changes. We apply this to a variety of problems in economic theory.

Keywords: lattices, concavity, supermodularity, comparative statics, LeChatelier principle, normality.

Author's Email: john.quah@economics.ox.ac.uk

Author's Address: St Hugh's College, Oxford. OX2 6LE. United Kingdom.

[1]I visited UC Berkeley in the academic year 2002-03 and my thinking on the issues addressed in this paper began there. I am grateful to the economics department at UC Berkeley, and in particular to Robert Anderson, for their hospitality. Chris Shannon gave a very interesting course on comparative statics in the Fall 2002 which introduced me to the subject. The first version of this paper was written at the Singapore Management University, which I visited in 2003. I am grateful to them for their hospitality. I would also like to thank R. Amir, E. Antoniadou, K. Border, F. Echenique, I. Jewitt, P. Milgrom, M. Meyer, R. Raimondo, J. Roberts, C. Shannon, I. Schwarz, and especially K. Reffett for their helpful comments and the ESRC for financial support under their Research Fellowship Scheme.

1. Introduction

In recent years, the methods of lattice programming have been used widely and with considerable success to deal with problems in economic theory.[2] The contribution of these methods are twofold. First, they have turned out be very useful in addressing comparative statics problems which arise in many optimization or game theoretic models. Second, they have contributed to our understanding of these problems because they have helped us to identify the key mathematical features which permit their solution. The success of these methods have highlighted the underlying structural similarity of many of the seemingly different comparative statics problems which arise in economic theory.

The general theory of monotone comparative statics can be applied on abstract lattices, but we will confine our discussion in this introduction, and indeed in this paper, to the Euclidean space R^l with the standard product order, i.e, $x \geq y$ if $x_i \geq y_i$ for all $i = 1, 2, .., l$. In this case, the supremum of two points x and x', denoted by $x \vee x'$, has $(x \vee x')_i = \max\{x_i, x'_i\}$ and their infimum, $x \wedge x'$, is given by $(x \wedge x')_i = \min\{x_i, x'_i\}$. There are essentially two types of comparative statics results in this theory. One concerns the change in the solution to a maximization problem as the objective function changes; the other concerns the change in the solution to a maximization problem when the constraint set changes.

The first type of results can be loosely described thus. Let X be a sublattice of R^l and consider the problem of maximizing the function $f(\cdot, t) : X \to R$ in the sublattice X. To keep things simple, let the parameter t be some scalar. When will the solution of this problem increase (in the product order) with t; in other words, when is there *monotone comparative statics*? One answer, which, in a certain sense, is the best that can be given says the following: the optimal solution increases with t if f is a quasisupermodular function of x and has the single crossing property in (x, t) (see Milgrom and Shannon (1992)).

[2] For a textbook introduction to this theory see Topkis (1998).

For a result of the second type, let X be a sublattice and $f : X \to R$ an objective function; we are interested in comparing the solution obtained from maximizing f in some constraint set C with the solution obtained from maximizing f in another constraint set C'. A standard result on this problem says the following: when C' is greater than C in the strong set order (induced by the product order on R^l), and if f is quasisupermodular, then the optimal solution will increase when the constraint set changes from C to C' (see Milgrom and Shannon (1992)).

If one looks at the applications of these monotone comparative statics results in the economics literature, one sees that the vast majority of applications are in fact applications of results of the first, rather than the second, type. The reason why the first type of results have found such broad application is because, in many settings, the assumptions required of the objective function - quasisupermodularity and the single crossing property - have turned out to be economically interpretable and intuitive.

On the other hand, comparative statics results of the second type have not generally found much application. This is certainly not because *problems* of the second type do not arise naturally in economic settings - in fact, they are very common. For example, we may wish to know what happens to a firm's output when it's production set changes; or to a consumer's demand when an increase in income causes her budget set to grow; or to a firm's profit maximizing output or price when its demand curve (which can be interpreted as a constraint) changes. The reason why monotone comparative statics results of the second type are less often used is simply because in many situations involving constraint set changes the standard results are just not applicable. By definition, the constraint set C' is greater than C in the strong set order (induced by the product order $\geq$) if for any x' in C' and x in C, the supremum $x' \vee x$ is in C' and the infimum $x' \wedge x$ is in C. In many settings, including the examples we have just listed, this condition is very restrictive. For example, it is quite clear that, keeping prices fixed, the budget set at a high income will not be greater in the strong set order than a budget set at a low income.[3]

The objective of this paper is to propose a new comparative statics theorem applicable to comparative statics problems of the second type. The theorem enlarges the scope of application of the existing results by adding to the types of constraint set changes which are comparable. Specifically, we define a new type of ordering on sets called the *generalized strong set order*; if C' is greater than C in the strong set order then C' will be greater than C in the generalized strong set order, but sets can be comparable with respect to the generalized strong set order which are not comparable in the strong set order. The penalty of being more permissive in this respect is that stronger conditions will now have to be imposed on the objective function before one can guarantee monotone comparative statics. In particular, the objective function must satisfy a property stronger than quasisupermodularity which we call *quasiconcavemodularity*. A sufficient condition for f to be quasisupermodular is that it is supermodular; a sufficient condition for f to be quasiconcavemodular is that it is supermodular and concave. In many applications, an objective function can reasonably be expected to satisfy both conditions.

The paper is organized as follows. The next section develops the theory of comparative statics for constrained optimization problems. Following that, we have two sections dealing with applications to consumer and producer theory respectively. Section 5 specializes the theory to the two variable case; the geometry of this case is particularly simple and there are also many applications. In Section 6, we make a few observations on the robustness of our results, before concluding in Section 7.

2. The Theory

We endow R^l with the *product order*, which says that $x \geq y$ if $x_i \geq y_i$ for $i =$

[3]Formally, we define the budget set $B(p,w) = \{x \in R^l_+ : p\cdot x \leq w\}$, where p is the vector of prices and w is income. It is clear that when $w'' > w'$, $B(p,w'')$ does not dominate $B(p,w')$ in the strong set order. For x'' in $B(p,w'')$ and x' in $B(p,w')$, their infimum is in $B(p,w')$ but the supremum will not be in $B(p,w'')$.

$1, 2, ...l$. With this order, R^l becomes a lattice, i.e., it is a partially ordered set where there is a supremum and an infimum to every pair of points in R^l. We denote the supremum and infimum of x and y by $x \vee y$ and $x \wedge y$ respectively; it is not hard to see that

$$\begin{aligned} x \vee y &= (\max\{x_1, y_1\}, \max\{x_2, y_2\}, ..., \max\{x_l, y_l\}) \text{ and} \\ x \wedge y &= (\min\{x_1, y_1\}, \min\{x_2, y_2\}, ..., \min\{x_l, y_l\}) \end{aligned}$$

A subset X of R^l is a *sublattice* (of R^l) if for every pair of points x and y in X, both $x \vee y$ and $x \wedge y$ are also contained in X. A function $f : X \to R$ is *supermodular* if for any x' and y in X,

$$f(x' \vee y) - f(y) \geq f(x') - f(x' \wedge y). \tag{1}$$

When f is a C^2 function defined on R^l, the supermodularity of f is equivalent to $\partial^2 f/\partial x_i \partial x_j \geq 0$ for all $i \neq j$ (see Topkis (1998)). For our purposes, it is important that one has a good geometrical picture of supermodularity. When x' and y are ordered, the inequality holds trivially, so let us assume that they are not ordered. In that case, it is not hard to check that the four points x', y, $x' \vee y$ and $x' \wedge y$ lie on a two dimensional plane and form a rectangle in the following sense: $x' - x' \wedge y = x' \vee y - y$, $x' - x' \vee y = x' \wedge y - y$, and $x' - x' \wedge y$ is orthogonal to $y - x' \wedge y$ (see Figure 1). In essence, supermodularity requires that the difference in the function's value on the right side of the rectangle, $f(x' \vee y) - f(y)$ be bigger than the difference on the left side, which is $f(x') - f(x' \wedge y)$.

Concavemodular functions

For the comparative statics results we have in mind, a property stronger than supermodularity is needed of the objective function. We now assume that X, in addition to being a sublattice, is also a convex set. The function f is *i-concavemodular* if for any x' and y in X with $x'_i > y_i$, and for all λ in $[0, 1]$,

$$f(x' \vee y - \lambda v_{x'}) - f(y) \geq f(x') - f(x' \wedge y + \lambda v_{x'}), \tag{2}$$

where $v_{x'} = x' \vee y - x'$. Note that this inequality holds trivially if x' and y are ordered, so let us assume that it is not. In that case, Figure 1 gives us a good geometrical appreciation of the property. The vector $v_{x'}$ is just the horizontal side of the rectangle; the points x', y, $x' \wedge y + \lambda v_{x'}$ and $x' \vee y - \lambda v_{x'}$ form a backward bending parallelogram and they are all in X because X is a convex sublattice. For f to be i-concavemodular, we require that the difference in the function's value along the right side of the parallelogram be greater than the difference along the left side. Note that (2) is required to hold for all values of λ in $[0, 1]$; in other words, for all the parallelograms formed as λ varies. When $\lambda = 0$, (2) is just the 'rectangular inequality' (1). Thus, it is clear that if f is i-concavemodular for all i (in which case we will refer to f as a *concavemodular* function), then it must also be supermodular.

Comparative statics results which rely on concavemodularity are only useful to the extent that we can show that this property holds under reasonable conditions. For this reason, our next result is important because it shows that concavemodularity arises from the marriage of two conditions which, in many situations, can both be expected to hold: one is supermodularity and the other is a concavity-type assumption. The function $f : X \to R$ is said to be *concave (convex) in direction* v if for all x in X, the map from the scalar t to $f(x + tv)$ is concave (convex). The domain of this map is taken to be the largest possible interval such that $x + tv$ lies in X. We say that f is *i-concave (i-convex)* if it is concave (convex) in all directions $v > 0$ with $v_i = 0$.

PROPOSITION 1: *Let $X \subset R^l$ be a convex sublattice. Then $f : X \to R$ is i-concavemodular if it is supermodular and i-concave.*

Proof: Let x' and y be two elements in X with $x'_i > y_i$ (as in Figure 1). The expression $f(x' \vee y - \lambda v_{x'}) - f(y)$ may be decomposed into $[f(x' \vee y - \lambda v_{x'}) - f(x' \vee y)] + [f(x' \vee y) - f(y)]$. Note that $v_{x'\,i} = 0$, so by the fact that f is i-concave,

$$\begin{aligned} f(x' \vee y - \lambda v_{x'}) - f(x' \vee y) &\geq f(x' \vee y - \lambda v_{x'} - (1-\lambda)v_{x'}) - f(x' \vee y - (1-\lambda)v_{x'}) \\ &= f(x') - f(x' + \lambda v_{x'}). \end{aligned}$$

By the supermodularity of f, we have $f(x' \vee y) - f(y) \geq f(x' + \lambda v_{x'}) - f(x' \wedge y + \lambda v_{x'})$. Together these two inequalities imply that

$$\begin{aligned} f(x' \vee y - \lambda v_{x'}) - f(y) &\geq [f(x') - f(x' + \lambda v_{x'})] + [f(x' + \lambda v_{x'}) - f(x' \wedge y + \lambda v_{x'})] \\ &= f(x') - f(x' \wedge y + \lambda v_{x'}), \end{aligned}$$

as we require. QED

We refer to a function as *partially concave* if it is i-concave for all i. Clearly this property is implied by the concavity of f, but it is a weaker property; for example, the function $f : R^l_{++} \to R$ given by $f(x_1, x_2) = x_1 x_2$ is partially concave but not concave. It follows from Proposition 1 that a function is concavemodular if it is supermodular and partially concave. A natural question then is whether concavemodularity *implies* partial concavity (that it implies supermodularity is obvious). In essence the answer to this question is 'yes'; we deal with this and related issues in Appendix B.

Quasiconcavemodular functions

It has been emphasized by Milgrom and Shannon (1994) in their wide ranging and influential study of comparative statics that comparative statics results rely on the ordinal, rather than the cardinal, properties of the objective function. For that reason, they introduced an ordinal version of supermodularity: the function $f : X \to R$ is *quasisupermodular* if $f(x') \geq (>) f(x' \wedge y)$ implies $f(x' \vee y) \geq (>) f(y)$. Analogously, we say that f is *i-quasiconcavemodular* if for any x' and y in X with $x_i^{'} > y_i$, and for any λ in $[0, 1]$,

$$f(x') \geq (>) f(x' \wedge y + \lambda v_{x'}) \Longrightarrow f(x' \vee y - \lambda v_{x'}) \geq (>) f(y) \tag{3}$$

(Recall that $v_{x'} = x' \vee y - x'$.) We call a function *quasiconcavemodular* if it is i-quasiconcavemodular for $i = 1, 2, ...l$. The property of i-quasiconcavemodularity is ordinal in the sense that if f is i-quasiconcavemodular then so is $\phi \circ f$, for any strictly increasing function $\phi : R \to R$. It is clear that any i-concavemodular function is also

i-quasiconcavemodular and that the latter is a strictly weaker property. In certain applications, this distinction is crucial in the sense that the objective function can be shown to satisfy one property but not the other (see Example 8 in Section 5).

Since i-concavemodularity is preserved by addition, we know that, for any w in R^l, the map $g_w : X \to R$ given by $g_w(x) = f(x) - w \cdot x$ is also an i-concavemodular function provided f is i-concavemodular. The next result shows that i-quasiconcavemodularity of the functions g_w imply the i-concavemodularity of f. This result is analogous to Theorem 10 in Milgrom and Shannon (1994); we prove it in Appendix A.

PROPOSITION 2: *Let $X \subseteq R^l$ be a convex sublattice of R^l. (i) The function $f : X \to R$ is i-concavemodular if for all w_i in R, the map g_{w_i}, bringing x in X to $f(x) - w_i x_i$ is i-quasiconcavemodular. (ii) Provided f is increasing, f is i-concavemodular if for all w in R^l_+, the map g_w, bring x in X to $f(x) - w \cdot x$ is i-quasiconcavemodular.*

The significance of this proposition is that in those situations where we require quasiconcavemodularity for all functions in the class $\{g_w\}_{w \in R^l}$ or $\{g_w\}_{w \in R^l_+}$, we must necessarily impose concavemodularity on f. Of course these classes of functions do indeed arise naturally in comparative statics problems, since it can be interpreted as a profit function, with $f(x)$ as the revenue of the firm when it produces the output vector x and with w_i as the unit cost of producing good i (so $w \cdot x$ is the total cost of producing x).

The Generalized Strong Set Order

Given that our ultimate goal is to obtain results which say how optimal solutions vary with parameters and constraints, we must first develop some way of comparing constraint sets. In standard monotone comparative statics, the order typically used is the *strong set order* introduced by A. Veinott (see Topkis (1998)). In this order, a set V'' is greater than V' if for any y in V'' and x' in V', $x' \vee y$ is in V'' and $x' \wedge y$ is in V'. As we had indicated in the introduction, the strong set order is, in a sense, too strong because it does not always successfully order pairs of constraint sets whose optimal solutions we wish to compare. What we need is a weaker notion of order, which we now define.

Let C' and C'' be subsets of the convex sublattice X. We say that C'' *is i-greater than (or i-dominates) C' in the generalized strong set order* (and write $C'' \geq_i C'$) if for any x' be in C' and y in C'', with $x'_i > y_i$, there is some λ in $[0,1]$ such that $x' \wedge y + \lambda v_{x'}$ is in C' and $x' \vee y - \lambda v_{x'}$ is in C''. Pictorially, this condition just means that one can find two other points, in addition to x' and y, one in C' and one in C'' such that the four points form a backward bending parallelogram. For the special case of $x' > y$, $v_{x'} = 0$, so this condition requires that y be in C' and x' be in C''. We say that C'' *is greater than (or dominates) C' in the generalized strong order* (and write $C'' \geq C'$) if $C'' \geq_i C'$ for all $i = 1, 2, ...l$.

Notice that the point $x' \vee y - \lambda v_{x'}$ which lies in C'' is greater than x', and that the point $x' \wedge y + \lambda v_{x'}$ which lies in C' is smaller than y. Our next claim is then obvious.

PROPOSITION 3: *Let C' and C'' be nonempty subsets of a convex sublattice X in R^l. (i) If $C'' \geq_i C'$, then for any x' in C', there is x'' in C'' such that $x''_i \geq x'_i$ and for any x'' in C'' there is x' in C' such that $x''_i \geq x'_i$. (ii) If $C'' \geq C'$, then for any x' in C', there is x'' in C'' such that $x'' \geq x'$ and for any x'' in C'' there is x' in C' such that $x'' \geq x'$.* [4]

As a simple illustration, let $C'' = \{(1+t, 2), (2+t, 1)\}$ and $C' = \{(1, 2), (2, 1)\}$. For any $t > 0$ it is easy to see that $C'' >_2 C'$, though for t in $(0, 1)$, $C'' \not>_1 C'$. We do have $C'' >_1 C'$ if $t \geq 1$ so in this case $C'' > C'$. Note that C'' is certainly not a superset of C', so a set can be greater than another in the generalized strong set order without it being a superset of the other set. That said, the constraint sets one encounters in applications are often ordered in the set-theoretic sense. Indeed they often obey *free disposal* as well; a subset C of X has this property if whenever x is

[4] Beware of the tricky logic in this proposition. Parts (i) and (ii) both follow immediately from the definitions, but (ii) does not follow logically from (i).

in C and y in X satisfies $y < x$ then y is in C. The next result, which we prove in Appendix A, gives a characterization of the generalized strong set order for such sets.

PROPOSITION 4: *Let C' and C'' be subsets of a convex sublattice X of R^l which are both closed, obey free disposal and satisfy $C' \subseteq C''$. Then $C'' >_i C'$ if and only if the following property $(\star)$ holds:*
whenever x and u are vectors with $u > 0$, $u_i = 0$, x in C', $x+u$ in C'' and $x+tu \notin C'$ for all $t > 0$, then for any $\mu > 0$, and $u' > 0$ which is orthogonal to u with $u'_i > 0$,

$$x - \mu u + u' \in C' \Longrightarrow (x+u) - \mu u + u' \in C''.$$

The proposition says, in a specific formal sense, that the set of substitution possibilities which favor variable i in the constraint set C'' is larger than the set of substitution possibilities which favor i in the constraint set C'. Property $(\star)$ considers two points x in C' and $x+u$ in C'', where u is positive and orthogonal to the direction i; furthermore, the point x is on the 'edge' of C' in the sense that it is not possible to add anything in the direction of u and still stay within C'. Suppose that it is possible at x to substitute μu with u' and still stay within the constraint set C' - note that this is a substitution which 'favors i' because $u_i = 0$ and $u'_i > 0$ - then property $(\star)$ requires that it is possible to make the *same substitution* at the point $x+u$ in C'' and stay within the C''.

We wish to develop some results which will allow us to generate classes of ordered sets, but before we do that we need to introduce some new functional properties. The function $f : X \to R$ is *submodular* if for any x' and y in X, we have $f(x' \vee y) - f(y) \leq f(x') - f(x' \wedge y)$. It is *$i$-convexmodular* if for any x' and y in X with $x'_i > y_i$, and for all λ in $[0,1]$,

$$f(x' \vee y - \lambda v_{x'}) - f(y) \quad \leq \quad f(x') - f(x' \wedge y + \lambda v_{x'}), \tag{4}$$

where $v_{x'} = x' \vee y - x'$. We refer to f as *convexmodular* if it is i-convexmodular for all i. Like concavemodularity, convexmodularity has a weaker, ordinal counterpart.

The function f is *i-quasiconvexmodular* if for any x' and y in X, with $x_i' > y_i$, and for any λ in $[0,1]$,

$$f(x') \leq (<) f(x' \wedge y + \lambda v_{x'}) \implies f(x' \vee y - \lambda v_{x'}) \leq (<) f(y) \tag{5}$$

We say that f is *quasiconvexmodular* if it is i-quasiconvexmodular for all i.

PROPOSITION 5: *Let $X \subset R^l$ be a convex sublattice. Then $f : X \to R$ is i-convexmodular if it is submodular and i-convex.*

Proposition 5 is analogous to Proposition 1 and has essentially the same proof. The next result is very useful and relates quasiconvexmodularity with the generalized strong set order.

PROPOSITION 6: *(i) Suppose that $C : X \to R$ is a continuous, increasing, and i-quasiconvexmodular function. Then*

$$C^{-1}((-\infty, k'']) \quad \geq_i \quad C^{-1}((-\infty, k']) \text{ whenever } k'' \geq k'. \tag{6}$$

(ii) Suppose that $C : X \to R$ is continuous, strictly increasing, and obeys property (6). Then C is i-quasiconvexmodular.[5]

Proof: We prove (i) here; the proof of (ii) is in Appendix A. Consider x' in $C^{-1}((-\infty, k'])$ and y in $C^{-1}((-\infty, k''])$ with $x_i' > y$. The problem is trivial if y is also in $C^{-1}((-\infty, k'])$, so we assume that it is not. This means that $C(y) > k'$. On the other hand, $C(x' \wedge y) \leq k'$ since C is increasing. Note that $v_{x'} = x' \vee y - x' = y - x' \wedge y$ is a positive vector, so by the fact that C is increasing and continuous, there is λ in $[0,1]$ such that $C(x' \wedge y + \lambda v_{x'}) = k$. Thus $C(x') - C(x' \wedge y + \lambda v_{x'}) \leq 0$. Since C is i-quasiconvexmodular, we must also have $C(x' \vee y - \lambda v_{x'}) \leq C(y)$, so $x' \vee y - \lambda v_{x'}$ is in $C^{-1}((-\infty, k''])$. QED

[5] Throughout this paper when we say that something 'is increasing', 'increases' or 'rises', we mean to say that it is nondecreasing. Most of the inequalities in this paper are weak, so this convention leads to less awkwardness. When we want an inequality to be strict, we will say so explicitly, as in 'strictly increasing,' etc.

Proposition 5 and 6 together means that if C is submodular and i-convex, then (6) will hold, and if C is submodular and i-convex for all i, then $C^{-1}((-\infty, k'']) \geq C^{-1}((-\infty, k'])$ whenever $k'' \geq k'$. Another useful way of generating comparable sets is given by the next result, whose proof is found in Appendix A.

Proposition 7: *Let $\widetilde{X}$ be a convex sublattice of R^{l-1}, I an interval of R, and $G : \widetilde{X} \to R$ a continuous, supermodular, concave and decreasing function. Then if $s'' \geq s' > 0$, we have $S'' \geq_l S'$ where $S'' = \{(\tilde{x}, x_l) \in R^{l-1} \times I : \tilde{x} \in \widetilde{X}, x_l \leq s'' G(\tilde{x})\}$ and S' is similarly defined.*

Comparative Statics

Let X be a convex sublattice of R^l and let F be a real valued function defined on X. We say that F has the *i-increasing property* if whenever $C'' \geq_i C'$ we also have $\arg\max_{x \in C''} F(x) \geq_i \arg\max_{x \in C'} F(x)$. We say that F has the *increasing property* if it is i-increasing for all i; in particular, this means that whenever $C'' \geq C'$, we also have $\arg\max_{x \in C''} F(x) \geq \arg\max_{x \in C'} F(x)$. If F has the i-increasing property, then Proposition 3(i) tells us the following:

(a) whenever $C'' \geq_i C'$ and x' is in $\arg\max_{x \in C'} F(x)$, and $\arg\max_{x \in C''} F(x)$ is nonempty, then there is x'' in $\arg\max_{x \in C''} F(x)$ such that $x''_i \geq x'_i$ ($x'' \geq x'$);

(b) whenever $C'' \geq_i C'$ and x'' is in $\arg\max_{x \in C''} F(x)$, and $\arg\max_{x \in C'} F(x)$ is nonempty, then there is x' in $\arg\max_{x \in C'} F(x)$ such that $x''_i \geq x'_i$ ($x'' \geq x'$).

If F has the increasing property then in addition to (a) and (b), Proposition 3(ii) tells us that the following is also true:

(c) whenever $C'' \geq C'$ and x' is in $\arg\max_{x \in C'} F(x)$, and $\arg\max_{x \in C''} F(x)$ is nonempty, then there is x'' in $\arg\max_{x \in C''} F(x)$ such that $x'' \geq x'$;

(d) whenever $C'' \geq C'$ and x'' is in $\arg\max_{x \in C''} F(x)$, and $\arg\max_{x \in C'} F(x)$ is nonempty, then there is x' in $\arg\max_{x \in C'} F(x)$ such that $x'' \geq x'$.

The main comparative statics result of this paper says that the i-increasing property is equivalent to the i-quasiconcavemodularity of the objective function. It is well known that greater sets (in the sense of the strong set order) lead to greater solution sets (with respect to the same set order) when the objective function is supermodular. (The first version of this result is due to A. Veinott; see Topkis (1978) for an early statement of this result). More precisely, the quasisupermodularity of the objective function is both sufficient and necessary for this property (see Milgrom and Shannon (1994)). The proof of our comparative statics theorem has a similar structure to those earlier proofs. Indeed, we have developed the theory in the way we did precisely so that we can now adopt the arguments they employed in their proofs, subject to certain natural modifications.

THEOREM 1: *Let X be a convex sublattice of R^l and let F be a real valued function defined on X. Then F is i-quasiconcavemodular (quasiconcavemodular) if and only if it has the i-increasing property (increasing property).*

Proof: The bracketed version of the theorem follows logically from the unbracketed version, so we shall only prove the latter. We first prove sufficiency. Assume that $C'' >_i C'$ and let x' be in $\arg\max_{x \in C'} F(x)$ and let y be in $\arg\max_{x \in C''} F(x)$. Suppose that $x'_i > y_i$; there is some $\tilde{\lambda}$ in $[0,1]$ such that $x' \wedge y + \tilde{\lambda} v_{x'}$ is in C' and $x' \vee y - \tilde{\lambda} v_{x'}$ is in C''. By revealed preference, $F(x') \geq F(x' \wedge y + \tilde{\lambda} v_{x'})$ and by i-quasiconcavemodularity, $F(x' \vee y - \tilde{\lambda} v_{x'}) \geq F(y)$, so $x' \vee y - \tilde{\lambda} v_{x'}$ is in $\arg\max_{x \in C''} F(x)$. If $F(x') > F(x' \wedge y + \tilde{\lambda} v_{x'})$, then i-quasiconcavemodularity implies that $F(x' \vee y - \tilde{\lambda} v_{x'}) > F(y)$ which contradicts the assumption that y maximizes F in C''. So we must also have $x' \wedge y + \tilde{\lambda} v_{x'}$ in $\arg\max_{x \in C'} F(x)$.

We prove the necessity part of the theorem by contradiction. Let x' and y be elements in X with $x'_i > y_i$. There are two possible violations of i-quasiconcavemodularity. One possibility is that there is λ^* in $[0,1]$ such that $F(x') \geq F(x' \wedge y + \lambda^* v_{x'})$ but $F(x' \vee y - \lambda^* v_{x'}) < F(y)$. In this case, let C' be the set with elements x' and $x' \wedge y + \lambda^* v_{x'}$ and let C'' be the set with elements $x' \vee y - \lambda^* v_{x'}$ and y. Then, clearly, $C'' >_i C'$, x' maximizes F in C' and y uniquely maximizes F in C''. This violates the monotonic property since $x'_i > y_i$.

The other possible violation of i-quasiconcavemodularity is that there is λ^* in $[0,1]$ such that $F(x') > F(x' \wedge y + \lambda^* v_{x'})$ but $F(x' \vee y - \lambda^* v_{x'}) = F(y)$. In this case, with C' and C'' defined as above, y maximizes F in C'' while x' is the unique maximizer of F in C'. Again this violates the monotonic property. QED

The next result follows immediately from Theorem 1 and Proposition 6(i). Note also that by Propositions 1 and 5 we can easily modify the assumptions in the next result: instead of i-quasiconcavemodularity, we can assume that F is supermodular and i-concave, while we can replace the i-quasiconvexmodularity of C by submodularity and i-convexity.

COROLLARY 1: *Let $F : X \to R$ be a i-quasiconcavemodular function and let $C : X \to$ be a continuous, increasing and i-quasiconvexmodular function. Then the following holds: whenever $k'' \geq k'$, we have* $\arg\max_{x \in C''} F(x) \geq_i \arg\max_{x \in C'} F(x)$. (In this case, we shall say that *the optimal value of i increases/rises with k*, but bear in mind that we are not claiming that the optimal solutions are unique.)

In certain problems, optima before and after changes to *both* the constraint set and the objective function are compared. For these problems, the next result is useful. In essence, it captures the idea that a change in the objective function which favors variable i will lead to an increase in the optimal value if i. Note that the proposition refers to the set C_i; given any set C, C_i is the set $\{r \in R : x_i = r \text{ for some } x \in X\}$.

PROPOSITION 8: *Let C be a subset of R^l and let T be a subset of R. The function F maps $C \times T$ to R, with $F(x,t) = \bar{F}(x) + f(x_i, t)$ where $f : C_i \times T \to R$ is supermodular in (x_i, t). Suppose that x' is in argmax$_{\{x \in C\}} F(x,t')$ and x'' is in argmax$_{\{x \in C\}} F(x,t'')$. If $x'_i > x''_i$, then x' is in argmax$_{\{x \in C\}} F(x,t'')$ and x'' is in argmax$_{\{x \in C\}} F(x,t')$. So, in particular, argmax$_{\{x \in C\}} F(x,t'') >_i$ argmax$_{\{x \in C\}} F(x,t')$.*[6]

[6]The parameter change to the objective function considered in this proposition is rather special, but it arises sufficiently often in applications for this simple result to be relevant. In R^2, it is known that a conclusion similar to that in Proposition 8 is true for any parameter change obeying the Spence-Mirrlees single crossing property (see Theorems 3 and 4 in Milgrom and Shannon (1994)). It

Proof: By revealed preference $\bar{F}(x')+f(x_i',t') \geq \bar{F}(x'')+f(x_i'',t')$. Since $f(x_i'',t')-f(x_i',t') \geq f(x_i'',t'')-f(x_i',t'')$ by the supermodularity of f, we have $\bar{F}(x')+f(x_i',t'') \geq \bar{F}(x'')+f(x_i'',t'')$, which means that x' is also in $\text{argmax}_{\{x\in C\}}F(x,t'')$. To see that x'' is in $\text{argmax}_{\{x\in C\}}F(x,t')$, suppose it is not, so that $\bar{F}(x')+f(x_i',t') > \bar{F}(x'')+f(x_i'',t')$. By the supermodularity of f again, we have $\bar{F}(x')+f(x_i',t'') > \bar{F}(x'')+f(x_i'',t'')$, which contradicts our assumption that x'' is in $\text{argmax}_{\{x\in C\}}F(x,t'')$. QED

To motivate the formal results we have developed so far, we will now consider their applications, beginning with their applications to demand theory.

4. Applications to Demand Theory

We have in mind a consumer who maximizes a utility function $U : X \to R$, where $X = R^l_{++}$ or R^l_+, while facing a budget constraint. At the price p in R^l_+, and income $w > 0$, we denote his budget set by $B(p,w)$, where $B(p,w) = \{x \in X : p\cdot x \leq w\}$. We refer to the set $D(p,w) = \text{argmax}_{\{x\in B(p,w)\}}U(x)$ as *the demand set at* (p,w). If the demand set is nonempty and unique at every $(p,w) \gg 0$, then the map from (p,w) in $R^l_{++} \times R_{++}$ to $D(p,w)$ will be referred to as the *demand function*.

Example 1. We say that the agent has *normal demand for good* i if $D(p,w'') \geq_i D(p,w')$ whenever $w'' \geq w'$. If the agent's utility admits a demand function then normality for good i simply means that the demand for good i is increasing with income in the usual sense.[7] Chipman (1977) has shown that if $U : R^l_{++} \to R$ is locally non-satiated, differentiably strongly concave (i.e., U has a strictly negative-definite Hessian) and obeys supermodularity, then U generates a demand *function* which has normal demand for all goods, i.e., $D(p,w'') > D(p,w')$ whenever $w'' > w'$. It is quite obvious that Chipman's result is, in essence, a special case of Corollary

may well be possible to improve on our proposition by considering some higher dimensional version of the Spence-Mirrlees property, but we will not pursue this issue here.

[7] It is well known that under standard assumptions, the normality of demand for some good also means that it obeys the law of demand, i.e., its demand falls when its price increases. For the uses of normality for comparative statics problems in general equilibrium, see Quah (2003).

1. In fact, applying Corollary 1 gives us a more nuanced version of that result and under weaker conditions.

First we note that the map $C : R^l_+ \to R$ given by $C(x) = p \cdot x$ is continuous, increasing, convex and submodular, and $B(p, w) = C^{-1}(-\infty, w]$, so by Proposition 5 and 6(i), it is convexmodular. By Corollary 1, we know that the demand for i will be normal provided U is i-concavemodular, a sufficient condition for which is that it is supermodular and i-concave. To guarantee that there is normal demand for *all* goods, it is sufficient that, in addition to supermodularity, the function U be i-concave all i, i.e., U is partially concave (which will hold if U is concave).

It is worth saying a bit about what we have *not* assumed to arrive at this conclusion. We have not made any of the assumptions needed to guarantee the existence of demand, since our result is a statement on the monotone response of demand to income change, *if* demand exists. In particular, U need not be continuous and the budget set need not be compact since we allow for some prices to be zero. (Of course, demand can still exist in a noncompact budget set provided U is not locally non-satiated.) Because we have not assumed that U is locally non-satiated, demand need not obey the budget identity, i.e., demand at (p, w) may be valued by p at strictly less than w.

Adding other assumptions usually made in demand theory will lead to slightly stronger results. We know that if U is strongly quasiconcave, demand must be *unique* if it exists. So if we add this assumption to the concavity and supermodularity of U, we obtain $D(p, w'') \geq D(p, w')$ when $w'' > w'$. If we also know that demand obeys the budget identity (for example, because U obeys local non-satiation) then we can say that $D(p, w'') > D(p, w')$.

Note also that, by Corollary 1, quasiconcavemodularity of the utility function will guarantee monotone comparative statics even in those situations where the budget set departs from the standard one. For example, the cost of bundle x can take the form $C(x) = \sum_{i=1}^{l} \phi_i(x_i)$, with ϕ_i increasing and convex; in other words we allow for the marginal cost of certain goods to increase with the amount purchased.

The flip side of this observation is that, if all we want is to guarantee monotone comparative statics for parallel shifts in *linear* budget boundaries then we could no longer use Theorem 1 to conclude that quasiconcavemodularity is *necessary* for monotonicity. This is because such constraint set changes are just some, but not all, of that required in the necessity part of Theorem 1.[8] Anticipating our discussion in Section 5 a little, we know that in the two good case, the 2-quasiconcavemodularity of the utility function U simply means that $-[\partial U/\partial x_1]/[\partial U/\partial x_2]$ decreases with x_1; in other words, the indifference curves through (x_1, x_2) becomes flatter as x_1 increases (keeping x_2 fixed). It is very obvious that under usual conditions (say, of a monotone and quasiconcave preference), flattening indifference curves is also necessary for the normality of good 2. For the general l good case, we do not know if i-quasiconcavemodularity is necessary for the normality of good i (if we only permit linear budget constraints).

That said, quasiconcavemodularity arises sufficiently often in commonly used utility functions for our results to be useful in practice. Obviously, it holds if U is additive and concave, i.e., $U(x) = \sum_{i=1}^{l} u_i(x_i)$, where $u_i : R_+ \to R$ are concave functions. Suppose we interpret the goods to be contingent commodities in l different states of the world; then the von Neumann-Morgenstern axioms guarantee that the preference over contingent consumption can be evaluated via expected utility, so that U will indeed be additive. Thus, the demand for contingent consumption will be normal if markets are complete (which guarantees that the commodity space is R^l_+ and not some lower dimensional subset).

A much studied alternative to expected utility is Choquet expected utility. In that

[8] At least in the case when we confine ourselves to linear budget constraints, a plausible alternative way of dealing with comparative statics issues in consumer theory is to abandon the product order and to use some other order in Euclidean space. This ordering must be such that it permits the comparisons we wish to make and at the same time, parallel shifts in the budget plane must be comparable in the strong set order it induces. This, in essence, distinguishes the approaches adopted by Antoniadou (1995, 2004) and Mirman and Ruble (2003).

case, the agent's utility function takes the following form $\bar{U}(x) = \min_{\mu \in C(\nu)}[\sum_{i=1}^{l} \mu_i u(x_i)]$ where $u : R_+ \to R$, ν is a convex nonadditive probability function, and $C(\nu)$ is the core of ν. It is straightforward to check that $\bar{U}$ is concave if u is concave. It is less straightforward to check, but still true, that if u is increasing, $\bar{U}$ will be supermodular (see Marinacci and Montrucchio (Theorem 35, 2004)). So by Corollary 1, $\bar{U}$ generates a normal demand for contingent consumption.

Example 2. A demand function is said to exhibit the gross substitutability property if a fall in the price of good i causes the demand for all other goods to decrease. This property is important because, amongst other things, it helps to guarantee the uniqueness and stability of the equilibrium price in general equilibrium models (see, for example, Mas-Colell et al (1995)). The most well known condition guaranteeing gross substitutability is the following. Let $U : R^l_{++} \to R$ be of the form $U(x) = \sum_{i=1}^{l} u_i(x_i)$ where each $u_i : R_+ \to R$ is C^2, with $u_i'(x_i) > 0$ and $u_i'' \leq 0$. Then the demand function $f : R^l_{++} \times R_+ \to R^l_{++}$ generated by U, f will obey gross substitutability if $-x_i u_i''(x_i)/u_i'(x_i) < 1$ for all i and $x_i > 0$.[9]

One can easily obtain this result using the techniques developed here. Assume that income is held fixed at w and consider a price change from p' to p'', where $p_i'' = p_i'$ for $i \geq 2$ and $p_1'' < p_1'$. Suppose that demand exists at both prices, with x' being a demand at p'. We wish to show that there is a demand at p'' in which the demand for good i rises and that of all other goods fall.

First, observe that x^* solves the following problem: (i) maximizing $\sum_i^l u_i(x_i)$ subject to x satisfying $p \cdot x = w$ if and only if $(s_1^*, x_2^*, ..., x_l^*)$, where $s_1^* = p_1 x_1^*$, solves the following problem: (ii) maximizing $u_1(s_1/p_1) + \sum_{i=2}^{l} u_i(x_i)$ subject to $s_1 + \sum_{i=2}^{l} p_i x_i = w$. So we can focus on problem (ii).

Since x' solves (i) at $p = p'$ we know that $(s_1', x_2', x_3', ..., x_l')$, with $s_1' = p_1' x_1'$ is a solution to (ii) at $p = p'$. Provided the map from $(s_1, 1/p_1)$ to $u_1(s_1/p_1)$ is supermodular, and since demand exists at p'' by assumption, we know from Proposition 8 that

[9]For a proof of this result see, for example, Hens and Loffler (1995).

there is a solution $(s_1'', x_2,'' , ..., x_l'')$ to (ii) at $p = p''$ such that $s_1'' \geq s_1'$. In other words, there must be a demand at $p = p''$ in which the expenditure on good 1 is higher than that at $p = p'$. In particular, $x_1'' > x_1'$.

Since U is additive, we know that $(x_2', x_3', ...x_l')$ maximizes $\bar{U}(x_2, x_3, ...x_l) = \sum_{i=2}^{l} u_i(x_i)$ subject to $\sum_{i=2}^{l} p_i x_i \leq w - s_1'$. If u_is are concave, so is $\bar{U}$; furthermore, $\bar{U}$ is additive and therefore supermodular. From Example 1, we know that $\bar{U}$ generates normal demand. When more is spent on good 1, the expenditure available for other goods is reduced from $w - s_1'$ to $w - s_1''$, and so there must be $(x_2''', x_3''', ..., x_l''')$ which maximizes $\bar{U}(x_2, x_3, ...x_l)$ subject to $\sum_{i=2}^{l} p_i x_i \leq w - s_1''$ such that $x_i''' \leq x_i'$ for $i \geq 2$. Furthermore, $(s_1'', x_2''', x_3''', ..., x_l''')$ solves (ii) at $p = p''$, which establishes gross substitutability.

It remains for us to point out what it means for the map from (s_1, a) in R^2_{++} to $u_1(as_1)$ to be supermodular. It is not hard to check that this is equivalent to the convexity of the map $\tilde{u}_1 : R \to R$ given by $\tilde{u}_1(z_1) = u_1(e^{z_1})$. In short, we have shown that the additive utility function U will generate demand satisfying gross substitutability if for all $i \geq 1$, u_i is concave and $\tilde{u}_i$ is convex. It is also not hard to check that when u_i is C^2 with $u_i' > 0$, then $\tilde{u}_i$ is convex if and only if $-x_i u_i''(x_i)/u_i'(x_i) \leq 1$ for all $x_i > 0$. In other words, we have obtained the non-differentiable version of the well known result.

Example 3. We consider the following two period saving-portfolio problem.[10] At date 1, an agent decides how much to save out of date 1 income w_1. Savings can be invested in two ways: a riskless asset A, which pays off $r > 0$ at date 2 and an asset B which has a stochastic payoff of s at date 2, with s distributed according to the density function f. In addition, the agent has a non-stochastic income of w_2 at date 2. The agent's Bernoulli utility function is $u(c_1, c_2)$ where c_1 and c_2 refer to consumption at dates 1 and 2 respectively. If he holds a portfolio with a of asset A and b of asset B, his utility given a date 1 consumption of c_1 is $U(c_1, a, b) = \int u(c_1, ar + bs + w_2) f(s) ds$. We wish to determine how his savings and investments will vary with w_1.

[10] For a discussion of other issues in the saving-portfolio problem, see Gollier (2001).

At first glance, the methods developed in this paper seem unsuited to dealing with this problem. Notice that $U_{ab}(c_1, a, b) = \int u_{22}(c_1, ar + bs + w_2) rsf(s)ds$.[11] Given risk aversion, u_{22} will always be negative; if in addition, rs is always positive, then U_{ab} will be negative so U is certainly not a supermodular function. There are various ways in which one can get around this difficulty. Most obviously, one could show that the *ordinal* conditions needed for comparative statics are satisfied, even though straightforward supermodularity is not: this is a method we will exploit in Example 8, which deals with another variation of the standard two asset portfolio problem. In this example, we will give a different, and quite intuitive, treatment of this problem.

Without loss of generality, we can assume that both assets have the same positive date 1 price $p > 0$. We can then reformulate the agent's problem by imagining him choosing between two assets: a riskless asset A with the constant payoff r and a risky free asset X which has payoff $t = s - r$. Note that X can be constructed by buying one B and selling one A. Formally, the agent solves the following problem **P**: maximize $\tilde{U}(c_1, \tilde{a}, x) = \int u(c_1, \tilde{a}r + xt + w_2) g(t)dt$ subject to $c_1 + p\tilde{a} = w_1$, where g is the density function of t. We will identify conditions under which the optimal values of c_1, $\tilde{a}$ and x are all increasing with w_1. In terms of the agent's original problem (which is the one we are interested in), this means that *optimal date 1 consumption, c_1, and also savings, i.e., $pa + pb$, which equals $p\tilde{a}$, both increase with w_1*. It also means, given the way X is constructed, that *investment in B increases with w_1*. However, since we do not know the relative magnitudes of the changes to x and $\tilde{a}$ given an increase in w_1, we cannot determine if the agent will buy more or less of A.

We shall say that **P** obeys the *regularity conditions* if the following holds: u is C^2 and concave, $u_2(c_1, c_2) > 0$ for all (c_1, c_2), the optimum is unique, obeys by the first order conditions and varies smoothly with w_1. Since $\tilde{U}$ is then concave, by Corollary 1, to guarantee normality, we require $\tilde{U}$ to be supermodular locally at the optimum;

[11] Notice that we are using subscripts to denote derivatives. We will do so whenever there is little risk of confusion.

in other words, that there is some neighborhood around the optimum in which the cross derivatives are positive.[12]

The following proposition is proved in Appendix A.

Proposition 9: *Suppose that the problem* $\mathbf{P}$ *obeys the regularity conditions and the solution* $(c_1^*, \tilde{a}^*, x^*)$ *at* w_1. *Then* $\tilde{U}$ *is locally supermodular at* $(c_1^*, \tilde{a}^*, x^*)$ *if the following conditions hold:*

(a) $-u_{22}(c_1, c_2)/u_2(c_1, c_2)$ *strictly decreases with* c_2 *for all* c_1, *i.e., there is strictly decreasing risk aversion, and*

either *(b')* $u_{12} \equiv 0$ *(in other words, u is additively time-separable)* or *(b")* $u_{12}(c_1, c_2) > 0$ *for all* (c_1, c_2) *and* $u_{12}(c_1, c_2)/u_2(c_1, c_2)$ *strictly increases with* c_2 *for all* c_1.

Note that the standard two asset portfolio problem with no date 1 consumption is covered by Proposition 9 since we can set $u(c_1, c_2) = v(c_2)$; combining Proposition 9 and Corollary 1 will then give us the familiar result that decreasing risk aversion leads to investment in the risky asset expanding with wealth. Proposition 9 covers the case when u is time separable (b'), but it also deals, more generally, with the case where it is supermodular (b"). Supermodularity may be plausible for various reasons; when interpreted as habit formation, it has been quite extensively considered as an explanation of the equity premium puzzle (see, for example, Constantinides (1990)). The supermodularity of u means that raising c_1 has a positive impact on u_2. The second condition in (b") says that this positive impact should increase in proportional terms as c_2 increases; more precisely, it says that the elasticity of u_2 with respect to c_1 increases with c_2.

4. Applications to Producer Theory

Throughout this section, we shall be considering a firm producing a single product using l inputs. We assume that when $q > 0$ is produced, the firm derives from it a

[12] More carefully: since the optimal portfolio varies smoothly with w_1, any violation of normality must mean a local violation of normality, but this is impossible since there is always an open neighborhood around each optimum in which Corollary 1 is applicable.

revenue of $R(q)$. The production function is F, so if x in R^l_+ is the input vector, the output is $q = F(x)$.

Example 4. We wish to examine the impact on optimal output of a change in input prices. To keep the discussion short we shall make the standard assumptions which guarantee the following: at the input price vector p in R^l_{++}, there is a unique cost-minimizing bundle which produces output q, which we will denote by $X(p,q)$; $X(p,q)$ coincides with the bundle maximizing output when input prices are at p and expenditure is kept at $p \cdot X(p,q)$; the envelope theorem is applicable, so that differentiating the cost function $C(p,q) = p \cdot X(p,q)$ by p_i gives us $X_i(p,q)$.

The firm chooses output q to maximize profit, which is $R(q) - C(p,q)$. By Proposition 8, the optimal q decreases with p_i if C is supermodular in (p_i, q). Since

$$\frac{\partial C}{\partial p_i}(p,q) = X_i(p,q),$$

this is in turn equivalent to the demand for i being normal.[13] Applying Corollary 1, we see that this is guaranteed if F is supermodular and i-concave.

Note that this conclusion makes no assumptions about the function R. Note also that if we want output to be decreasing with respect to *all* input prices, then a sufficient condition is that F is supermodular and partially concave. In the production context, supermodularity has the interpretation that all inputs are complements in the production process. Partial concavity means, in particular, that whenever one input is held fixed, increasing all other inputs by a multiple of k will not raise output by more than a multiple of k. However, increasing returns to scale is not excluded by the assumption, as is clear from the function $F(x_1, x_2) = x_1 x_2$.

Example 5. We wish to examine the impact of a technological change on the optimal output; specifically, if $q = AF(x)$, how would an increase in A affect the optimal level of q? It is convenient in this context to think of inputs as negative

[13] The connection between normality of input demand and the impact on marginal costs (and thus output) of an input price change is well known (see, for example, McFadden (1978) and Athey et al (1998)).

variables, so we define $\tilde{F} : R^l_- \to R$ by $\tilde{F}(\tilde{x}) = F(-\tilde{x})$. We can then formulate the firm's problem as a constrained maximization problem. Assuming that $p > 0$ is the input price vector and that $R(q)$ is the revenue from output q, the firm maximizes $\Pi(\tilde{x}, q) = R(q)+p\cdot\tilde{x}$ subject to $q \le A\tilde{F}(\tilde{x})$. Clearly Π is supermodular and q-concave. If F is continuous, supermodular, concave, and increasing, then $\tilde{F}$ is continuous, supermodular, concave and *decreasing.* Applying Proposition 7 and Theorem 1, we see that the optimal level of q rises with A.

Note that the conclusion requires no assumptions on R. Note also that the conclusion is about the optimal q and says nothing about x. In fact, it is not hard to construct examples where the optimal level of x will fall or rise.

Example 6. The classic formulation of the LeChatelier Principle in economics considers the impact of a small reduction in the price of an input (say input 1) on the demand for 1. It says that in the short run, interpreted as the time frame in which some inputs are not free to vary, the increase in the demand for 1 is smaller than in the long run, when all inputs are free to vary. Milgrom and Roberts (1996) has shown that this result also holds when the price reduction is large, provided the profit function is a supermodular function of the inputs. Our next result gives a formulation of the LeChatelier principle which extends the result of Milgrom and Roberts by enlarging on the class of permissible constraints faced by the firm in the short run.[14]

Proposition 10: *Let x^* be a solution to the problem of (i) maximizing* $\Pi(x, a')$ *subject to x in R^l_+, where* $\Pi(\cdot, a) : R^l_+ \to R$ *is given by* $\Pi(x, a) = R(F(x))-p\cdot x+ax_1$. *Suppose, also, that there are solutions to the following problems:*
(ii) maximize $\Pi(x, a'')$ *subject to $x \in C$, where $a'' > a'$ and C is a subset of R^l_+ containing x^*; and (iii) maximize* $\Pi(x, a'')$.

[14]For a recent discussion of the LeChatelier Principle in its classical form, together with an extension in a direction different from the one considered here (or in Milgrom and Roberts (1996)) see K. Roberts (1999).

Then there are x^{**} and x^{***}, solutions to problems (ii) and (iii) respectively, such that $x_1^{***} \geq x_1^{**} \geq x_1^*$, provided either *of the following conditions hold:*

(A) $R \circ F$ is supermodular and X_C is greater than C in the strong set order, where $X_C = \{x \in R_+^l : x \geq c \text{ for some } c \in C\}$.

(B) $R \circ F$ is supermodular and 1-concave, and X_C is greater than C in the generalized strong set order.

The proof of this result is in the Appendix. Note that the change in the parameter from a' to a'' can be interpreted as fall in the price of input 1 from $p_1 - a'$ to $p_1 - a''$. In the short run (case (ii)), the firm is constrained to choose inputs from the set C; in the long run (case (iii)), no constraints are imposed. The desired conclusion holds under two sets of assumptions. In both cases, $R \circ F$ is assumed to be supermodular, which guarantees the supermodularity of the profit function Π. In (B), $R \circ F$ (and thus Π) is also assumed to be 1-concave, but the class of constraint sets permitted is larger than in (A), because the set ordering requirement is weaker.

Milgrom and Roberts (1996) considers the case where certain inputs are held fixed in the short run. Formally, they are considering a constraint set of the form $C = \{x \in R_+^l : x_i = k_i \text{ for } i = m, m+1, ..., l\}$. It is quite obvious that in this case X_C is greater than C in the strong set order, so that condition (A) may be applied, but this is just one of many possible types of constraints that a firm might face in the short run. It is not hard to check that X_C dominates C in the strong set order for any set C which has the free disposal property: if x' is in C, then $x > 0$ with $x < x'$ is also in C. Short run constraints of this form are quite plausible; for example, a firm which, in the short run cannot allow its expenditure to exceed w will formally have $C = \{x \in R_+^l : p \cdot x - a'' x_1 \leq w\}$.

For an example of the type of short run constraint permitted by condition (B), let $\phi : R_+^l \to R$ be any continuous and increasing function and let r be in the range of ϕ. The set $C = \phi^{-1}(r)$ is closed and nonempty, and X_C will dominate C in the generalized strong set order. This is easy to check, and it is also easy to see that X_C

need not dominate C in the strong set order (so that condition (A) is inapplicable).[15] Short run constraints of this form are economically plausible. For example, suppose that the inputs m, $m+1$,..., l are intrinsically the same good, i.e., they have the same inherent characteristics, but are only considered as different inputs because they play different roles in the production process. Imagine that in the short run, due to contractual and technological reasons, the total amount of this good used cannot be varied, though the firm is free to employ what they already have in different ways. In that case, $C = \{x \in R^l_+ : \sum_{i=m}^l x_i = \sum_{i=m}^l x_i^*\}$, which equals $\phi^{-1}(r)$ if we define $\phi(x) = \sum_{i=m}^l x_i$ and $r = \sum_{i=m}^l x_i^*$.

5. Two-Dimensional Constrained Optimization Problems

In this section, we will focus on the comparative statics of constrained optimization problems which are set in R^2. The general theory we have developed so far in an l-dimensional context is particularly simple and intuitive in this special case. The results here are also interesting because comparative statics problems in R^2 are ubiquitous in economic theory. Throughout this section, we shall assume that $X = X_1 \times X_2$ where X_1 and X_2 are two open intervals in R^2. The next result gives sufficient conditions under which the C^1 function $f : X \to R$ is 2-quasiconcavemodular. We shall be considering functions for which either $f_1(x) > 0$ or $f_2(x) > 0$ for all x. (We shall be using subscripts to denote derivatives in this section.) We say that f has *well behaved indifference curves* if *either* of these conditions holds:

(i) $f_2(x) > 0$ for all x and at each $x^* = (x_1^*, x_2^*)$ in X, there is a differentiable curve $\psi : X_1 \to R$ such that $\psi(x_1^*) = x_2^*$ and $f(x_1, \psi(x_1)) = f(x_1^*, x_2^*)$ for all x_1 in X_1; or

(ii) $f_1(x) > 0$ for all x and at each $x^* = (x_1^*, x_2^*)$ in X, there is a differentiable curve $\phi_{x^*} : X_2 \to R$ such that $\phi(x_2^*) = x_1^*$ and $f(\phi(x_2), x_2) = f(\phi(x_2^*), x_2^*)$ for all x_2 in X_2.

[15] Let x' be in C and y be in X_C. Then $\phi(y) > r$ while $\phi(x' \wedge y) \le r$ (since ϕ is increasing). Thus there is λ in $[0,1]$ such that $\phi(x' \wedge y + \lambda v_{x'}) = r$. Clearly $x' \wedge y + \lambda v_{x'}$ is in $\phi^{-1}(r) = C$ while $x' \vee y - \lambda v_{x'}$ is in X_C.

PROPOSITION 11: *Suppose that* $f : X \to R$ *has well behaved indifference curves. Then* f *is 2-quasiconcavemodular if either of the following conditions hold:*
(i) $f_2 > 0$ *and* $f_1(x_1, x_2)/f_2(x_1, x_2)$ *is increasing in* x_1*; or*
(ii) $f_1 > 0$ *and* $f_2(x_1, x_2)/f_1(x_1, x_2)$ *is decreasing in* x_1*.*

Note that $-f_1(x_1, x_2)/f_2(x_1, x_2)$ is the slope of the indifference curve through (x_1, x_2), so all the proposition requires is that f obeys the *declining slope condition*: the slope of the indifference curve through (x_1, x_2) falls as x_1 increases. That this guarantees the 2-quasiconcavemodularity of f is quite intuitive; the formal proof is in Appendix A.[16] Note also that this condition ordinal. In particular, the function need neither be supermodular nor concave nor even quasiconcave. However, it is consistent with Proposition 1: one can check directly that if f is supermodular and 2-concave then the declining slope condition is satisfied. From our discussion in Example 1, we know that a utility function which generates indifference curves obeying this condition will have normal demand for good 2. This fact is known (see Milgrom and Shannon (1994)); we now turn to a result which generalizes this example.

Let $f : X \to R$ be the objective function and I_h and I_g be two intervals in X_2 such that I_h dominates I_g in the strong set order (essentially this means that the infimum and supremum of I_h are greater than the infimum and supremum respectively of I_g). Suppose that I_g and I_h are respectively the domains of functions g and h, which are mapped into X_1. We wish to compare the solutions of the following two problems:
P_g: maximize $f(x)$ subject to x in $G = \{x \in X : x_2 \in I_g \text{ and } x_1 \leq g(x_2)\}$ and
P_h: maximize $f(x)$ subject to x in $H = \{x \in X : x_2 \in I_h \text{ and } x_1 \leq h(x_2)\}$.

THEOREM 2: *Suppose that* $f : X \to R$ *is 2-quasiconcavemodular, and suppose that the problems* P_g *and* P_h *(as defined above) satisfy* either *of these conditions:*

[16] When the partial derivatives of f are both positive, it is fairly easy to see that this declining slope condition is also necessary for 2-quasiconcavemodularity. However, this condition is not necessary if $f_1 < 0$ and $f_2 > 0$. In this case f is (trivially) 2-quasiconcavemodular; the indifference curves must all slope upwards, but they need not obey the declining slope condition.

(i) for all x_2'' and x_2' in $I_g \cap I_h$ with $x_2'' > x_2'$, we have $g(x_2') \leq h(x_2')$ and $g(x_2'') - g(x_2') \leq h(x_2'') - h(x_2') \leq 0$; or
(ii) f is strictly increasing in x_2 and for all x_2'' and x_2' in $I_g \cap I_h$ with $x_2'' > x_2'$, we have $g(x_2') \leq h(x_2')$ and $g(x_2'') - g(x_2') \leq h(x_2'') - h(x_2')$.
Then the solution set for (P_h) is 2-greater than the solution set for (P_g).

We provide a formal proof of this result in the Appendix, but its pictorial intuition is entirely straightforward. The constraint sets G and H are depicted in Figure 2. Condition (i) guarantees that the boundary of H is to the left of G's and is also steeper at every x_2 for which comparison can be made; on the other hand, because f is 2-quasiconcavemodular, the indifference curves generated by f become flatter as x_2 increases. Together, these two properties guarantee that optimal solutions must involve an increase in the value of x_2 as one changes the constraint set from G to H. In terms of the formal language of this paper, condition (i) guarantees that $H >_2 G$, so that monotonicity follows from an application of Theorem 1. (Note that condition (ii) is similar to (i) except that it requires f to be strictly increasing in x_2, which allows for the restriction on the constraint set to be weakened.)

We now turn to some comparative statics problems in R^2 which can be solved using Theorem 2. Many of these problems can also be solved, in part or in whole, using other methods. Suppose that we are interested in how the value of x_2 which maximizes $f(x_1, x_2)$ subject to $x_1 \leq g(x_2, t)$ changes as the parameter t changes. Provided we know that f is locally non-satiated, so that the constraint is binding, this problem can be converted into the 1-dimensional problem of maximizing $f(g(x_2, t), x_2)$. This latter problem can often then be fruitfully studied using techniques already developed for studying 1-dimensional problems (see, in particular, Athey et al (1998)). However, this does not negate the value of Theorem 2 because even when other techniques can be used, this theorem provides a particularly transparent approach to many such comparative statics problems.

Example 7. Consider a profit maximizing firm producing a single product. If it charges a price $p > 0$, its demand is $D(p, \theta) > 0$ where θ is some parameter. (In a Bertrand game with differentiated products θ will represent the prices of other firms.) The cost of producing output q is $C(q)$, so that the firm's objective is to maximize $pD(p, \theta) - C(D(p, \theta))$. Suppose that, as θ increases, $\ln D(p, \theta)$ increases and the difference $\ln D(p, \theta) - \ln D(p', \theta)$, for any $p' > p$, also increases; respectively, this means that demand increases and becomes less elastic with respect to its own price as θ increases. Milgrom and Shannon (1994) has shown that with these assumptions on demand, the profit maximizing price charged by the firm increases with θ if the firm has increasing marginal costs.[17] By formulating the firm's problem as a constrained optimization problem, we can see with great clarity that their conclusion follows from an application of Theorem 2.

Let $\tilde{q}$ the be log output. Then the firm's problem is to maximize $\Pi : R_{++} \times R \to R$ given by $\Pi(p, \tilde{q}) = pe^{\tilde{q}} - C(e^{\tilde{q}})$, subject to $\tilde{q} \leq \ln D(p, \theta)$. Suppose $\theta' > \theta$; then the conditions we have imposed on demand guarantee that the maps $p \to \ln D(p, \theta)$ and $p \to \ln D(p, \theta')$ are related to each other in the way that g and h are related in Theorem 2(ii). Note also that Π is strictly increasing in p, so to apply Theorem 2(ii), we only require that Π be 2-quasiconcavemodular. To keep things simple, assume that C is differentiable; then Proposition 11(i) says that Π is 2-quasiconcavemodular if the ratio of the partial derivatives with respect to $\tilde{q}$ and to p, which is $p - C'(e^{\tilde{q}})$, is decreasing with $\tilde{q}$. This holds if $C'' \geq 0$.

It is also clear, from our formulation of the problem, that the comparative statics will hold whenever the objective function is 2-quasiconcavemodular, which potentially can accommodate other interesting objective functions besides the standard one considered so far. A particularly simple case is the following. Suppose that marginal cost is constant and that, at price p, the log-demand is stochastic,

[17] Strictly speaking, our assumptions are a bit weaker than in Milgrom and Shannon (1994). We do note assume the differentiability of demand, nor do we assume that the demand for the good is a decreasing function of its own price.

taking the value $\ln D(p,\theta)+s$, with the distribution of s governed by the density function f. The firm has the Bernoulli utility function u, so that we may consider the firm's problem as that of a constrained maximization problem: maximize $U(\tilde{q},p)=\int_R u((p-c)\exp(\tilde{q}+s))f(s)ds$ subject to $\tilde{q}\leq \ln D(p,\theta)$. U is strictly increasing in p if u is strictly increasing. The slope of the indifference curve at $(\tilde{q},p)$ is simply $c-p$, which is independent of $\tilde{q}$. So, by Proposition 11(i), U is 2-quasiconcavemodular and we conclude that the optimal price rises with θ (under the maintained assumptions on $\ln D(p,\theta)$).

Example 8. Consider the standard portfolio problem of an agent who has to choose between two assets, a safe asset with constant and positive payoff r and a risky asset with payoff s, governed by the density function f. The agent has the Bernoulli utility function $u: R\to R$, so that it's objective function is $U(a,b)=\int u(bs+ar)f(s)ds$. It is well known that the agent's investment in the risky asset will increase with wealth if his coefficient of risk aversion decreases with wealth. The standard proof of this result converts the agent's problem into a single variable (the level of risky investment) problem by making a substitution using the budget identity and then establishing that some version of the single crossing property holds (see, for example, Gollier (2001) or Athey (2002)).

Another natural way of obtaining this result is simply to prove that U is 2-concavemodular. The function U is strictly increasing in a if u is strictly increasing, so by Proposition 11(ii), we need only show

$$\frac{\int u'(bs+ar)sf(s)ds}{\int u'(bs+ar)rf(s)ds}$$

is increasing with a. For this to hold, it is sufficient that $u'(as+br)$ be log supermodular in (s,b).[18] The cross derivative of $\ln u'(as+br)$ is $(\ln u')''(as+br)ar$; it is not hard to check that u has decreasing risk aversion if and only if $(\ln u')''\geq 0$ (in other

[18]The ratio $\int g(s)\phi(s,\theta)ds/\int h(s)\phi(s,\theta)ds$ increases with θ if $g(s)/h(s)$ increases with s and ϕ is a log-supermodular function of (s,θ) (see Athey (2002)). In our case, $\theta=a$, $\phi=u'$, $g(s)=sf(s)$, and $h(s)=rf(s)$.

words, $\ln u'$ is convex), so $u'(as+br)$ is log supermodular if we restrict the domain of a to $a > 0$. As is well known (see Gollier (2001)), we can, if we prefer, make this last restriction non-binding by assuming that the risky payoff has a mean return greater than r and that u is concave (until this point, the concavity of u has not been used).

We can use our approach to generalize this standard result to the case when both assets are risky.[19] Suppose that asset A has the payoff rt, where r is a positive constant and $t > 0$ is stochastic; asset B has a payoff st, where s is also stochastic. We assume that s and t are independent and are distributed according to density functions f and h respectively. If we wish, we can interpret this as a situation in which both assets have nominal payoffs and the price level is stochastic, so that rt and st measure the real returns of the two assets. The agent's utility when he holds a of asset A and b of asset B is then given by

$$U(a,b) = \int u(bst + art) f(s) h(t) ds\, dt. \tag{7}$$

The next proposition guarantees the 2-quasiconcavemodularity of U and thus the normality of demand for asset B. Its proof is in Appendix A.

Proposition 12: *The function U as defined by (7) is 2-quasiconcavemodular if u is C^3, $u' > 0$, $u'' \leq 0$ and the coefficient of risk aversion of u is decreasing.*

6. Changing Variables in Comparative Statics Problems

Let $X = \Pi_{i=1}^{l} X_i$, where each X_i is an interval in R and let f be some real-valued function defined on X. Let $\phi : \tilde{X}_l \to X_l$ be a strictly increasing function from the interval $\tilde{X}_l$ into X_l. Writing $\tilde{X} = \Pi_{i=1}^{l-1} X_i \times \tilde{X}_l$, we define $\tilde{f} : \tilde{X} \to R$ by $\tilde{f}(x) = f(x_1, x_2, ..., \phi(x_l))$; thus, $\tilde{f}$ is related to f by a change of variables. It is straightforward to check that the following properties, if it holds for f, will be inherited by $\tilde{f}$: supermodularity, quasisupermodularity, l-concavity, l-concavemodularity, and l-quasiconcavemodularity.

[19] For other comparative statics results with two risky assets see Jewitt (2000).

To appreciate the import of these simple claims imagine a situation where we wish to examine the comparative statics of variable l and we have a result which relies on the objective function being l-quasiconcavemodular. Suppose also that we have somehow managed to establish l-quasiconcavemodularity for a particular objective function f. In that case, l-quasi-concavemodularity, and thus the comparative statics result, will also hold for $\tilde{f}$. In other words, the scope of applicability of our result has been enlarged from a particular objective function to a larger family of functions. Our observations here echo those of Milgrom (1994) who shows that in certain comparative statics problems, simplifying assumptions are not restrictive since any conclusion under those assumptions must also hold when they are replaced by less stringent and more realistic assumptions.

Similarly, for any set K in X we can define $\tilde{K}$ in $\tilde{X}$ by $\tilde{K} = \{x \in \tilde{X} : (x_1, x_2, ..., \phi(x_l)) \in K\}$. It is very easy to see that if K'' is greater than K' in the strong set order then $\tilde{K}''$ is greater than $\tilde{K}'$ in the strong set order and if K'' is l-greater than K' in the generalized strong set order than $\tilde{K}''$ is l-greater than $\tilde{K}'$ in the generalized strong set order. Our observations in the previous paragraph are also relevant in this case: such variable changes can significantly expand the scope of a comparative statics result. We illustrate this in the next example.

Example 9. Consider an open economy producing goods 1 and 2 using capital and labor. There is no joint production; good i ($i = A, B$) has the production function $f_i : R^2_+ \to R_+$, with f_i quasiconcave and homogeneous of degree one (in other words, has constant returns to scale). These two assumptions also guarantee that f_i is concave (see Champsaur and Milleron (1983)). The economy has an endowment $\bar{K}$ of capital and $\bar{L}$ of labor. Production decisions in this economy are made by two representative firms; firm i chooses (k_i, l_i) in R^2_+ to maximize $\Pi_i(l_i, k_i) = R_i(f_i(k_i, l_i)) - w_K k_i - w_L l_i$, where $R_i(x_i)$ is the revenue earned from selling x_i units of good i and w_K and w_L are prices of capital and labor respectively. An equilibrium in this economy is reached when w_K and w_L are such that the firms' demand for capital and labor equal the economy's endowments. It is well known that under standard assumptions, the equilibrium output of goods 1 and 2 can also be obtained via an optimization procedure. Let $S(\bar{K}, \bar{L})$ in R_+^2 be the production possibility set of this economy when the aggregate endowment is $(\bar{K}, \bar{L})$. Then the equilibrium output of the two goods coincides with that obtained from the following optimization problem: maximize $U(a, b) = R_A(a) + R_B(b)$ subject to (a, b) in $S(\bar{K}, \bar{L})$.

Suppose that the markets for the two goods are perfectly competitive, so $R_i(x_i) = p_i x_i$. Rybcsynski's Theorem considers an increase in the endowment of capital, say from $\bar{K}'$ to $\bar{K}''$, and identifies conditions on the relationship between f_A and f_B which guarantee that *more of good B and less of good A* is produced at equilibrium. (See Mas-Colell et al (1995); for reasons of brevity, we will not examine those conditions.) This is illustrated in Figure 3, where the optimal output moves from (a^*, b^*) to (a^{**}, b^{**}); because output prices are held fixed, the slope of the production possibility frontiers at these points are the same. Note that because the production functions, and hence the possibility sets are concave, the slope of the new production possibility set at (a^{***}, b^*) must be steeper than the slope at (a^*, b^*). In other words, following from the increase in capital, we *must* have $S(\bar{K}'', \bar{L}) >_B S(\bar{K}', \bar{L})$.[20] So we have a situation where Theorem 2 is applicable. By Theorem 2, the optimal level of B will rise - this is true whenever R_A is concave (so that U is 2-concavemodular) and does not require either R_A or R_B to be linear. However, with this departure, we will not be able to say that the output of good A falls.

Now consider an increasing transformation of the production function of good B from f_B to $\hat{f}_B = H \circ f_B$, where $H : R_+ \to R_+$ is a strictly increasing function. Like f_B, $\hat{f}_B$ will be quasiconcave, but it will not in general be concave or have constant returns to scale. We denote the production possibility sets generated by f_A and $\hat{f}_B$ by $\hat{S}$; because $\hat{f}_B$ need not be concave, neither must $\hat{S}$. One can easily check that

[20]It is not hard to prove this claim directly from the assumptions in Rybcsynski's Theorem, if one so wishes.

$\hat{S}(\bar{K}', \bar{L}) = \{(a,b) \in R^2_+ : (a, H^{-1}(b)) \in S(\bar{K}', \bar{L})\}$; obviously, $\hat{S}(\bar{K}'', \bar{L})$ is similarly related to $S(\bar{K}'', \bar{L})$. Since $S(\bar{K}'', \bar{L}) >_B S(\bar{K}', \bar{L})$ we must also have $\hat{S}(\bar{K}'', \bar{L}) >_B \hat{S}(\bar{K}', \bar{L})$. It follows from Theorem 2 that the output of B will rise with the endowment of capital: the comparative statics result is robust to increasing transformations of the production function of good B.

7. Conclusion

This paper has shown that the comparative statics of constrained optimization problems can be studied in a unified way, with a simple and geometrically intuitive method. In keeping with much of the recent literature, we have focussed on the precise conditions needed for comparative statics, distinguishing them from those conditions which may be needed for other properties like the existence or uniqueness of optimal points. We hope that we have provided a set of tools that are easy to understand and to use, and which can find application in many areas of economic theory.

APPENDIX A

Proof of Proposition 2: Suppose that there is x' and y, with $x'_i > y_i$ and λ such that (2) is violated, so

$$f(x' \vee y - \lambda v_{x'}) - f(y) \quad < \quad f(x') - f(x' \wedge y + \lambda v_{x'}). \tag{8}$$

Note that $(x' \wedge y + \lambda v_{x'})_i = y_i < x'_i$ so there is $\bar{w}_i$ such that $\bar{w}_i[x'_i - (x' \wedge y + \lambda v_{x'})_i] = f(x') - f(x' \wedge y + \lambda v_{x'})$. Furthermore, since $x' - (x' \wedge y + \lambda v_{x'}) = (x' \vee y - \lambda v_{x'}) - y$, we have $\bar{w}_i[x'_i - (x' \wedge y + \lambda v_{x'})_i] = \bar{w}_i[(x' \vee y - \lambda v_{x'})_i - y_i]$. Deducting this term from both sides of (8), we obtain

$$g_{\bar{w}_i}(x' \vee y - \lambda v_{x'}) - g_{\bar{w}_i}(y) < g_{\bar{w}_i}(x') - g_{\bar{w}_i}(x' \wedge y + \lambda v_{x'}) = 0.$$

So $g_{\bar{w}_i}$ violates i-quasiconcavemodularity and we have a contradiction.

To prove (ii), note that if (8) is true for $\lambda = 0$, then the right hand side of (8) is nonnegative (since f is increasing), while $(x' - x' \wedge y)_i = x'_i - y_i > 0$. We could then use the proof given for (i), choosing $\bar{w}_i \geq 0$. (Set the other entries of the vector $\bar{w}$ at zero.) So we consider the case when (8) is true for $\lambda > 0$. (8) can be true only if x' and y are unordered, and with $\lambda > 0$, x' and $(x' \wedge y + \lambda v_{x'})$ must also be unordered. Thus, $x' - (x' \wedge y + \lambda v_{x'})$ has both positive and negative entries, and there is $\bar{w}$ in R^l_+ such that $\bar{w} \cdot [x' - (x' \wedge y + \lambda v_{x'})] = f(x') - f(x' \wedge y + \lambda v_{x'})$. Now repeating the steps in our proof of (i), we see that $g_{\bar{w}}$ must violate i-quasiconcavemodularity. QED

Proof of Proposition 4: We first prove that $(\star)$ implies that $C''_i >_i C'$. Let x' be in C' and y be in C'' with $x'_i > y_i$. If $x' > y$, the condition for $C'' >_i C'$ requires x' to be in C'' and y to be in C': the first is true since $C' \subset C''$, while the second follows from free disposal. So we assume that x' and y are unordered. If y is in C', the condition for $C'' >_i C'$ holds with $\lambda = 1$. This leaves us with the case of x and y are unordered, with y not in C'. Since $x' \wedge y$ is in X and less than x', we know that it is in C'. By the closedness of C' and free disposal, there is λ^* in $[0, 1)$ such that $x' \wedge y + \lambda^* v_{x'}$ is in C' and $x' \wedge y + \lambda v_{x'}$ is not in C' for λ in $(\lambda^*, 1]$. Define $u = (1 - \lambda^*) v_{x'}$. Choose $\mu = \lambda^*/(1 - \lambda^*)$ and $u' = x' - x' \wedge y$. We have $u_i = 0$, $u'_i > 0$. We then have $x' \wedge y + \lambda^* v_{x'}$ in C', $(x' \wedge y + \lambda^* v_{x'}) - \mu u + u' = x$ in C', and $x + u = y$ in C''. So by $(\star)$, $(x + u) - \mu u + u' = x' \vee y - \lambda^* v_{x'}$ must be C''. Thus $C'' >_i C'$.

For the other direction, let $x' = x - \mu u + u'$. By assumption, this is in C'; also by assumption, $x + u$ is in C'' and $x'_i > x_i + u_i$. Note that $x' \wedge (x + u) = x - \mu u$. Since $C'' >_i C'$, there must be a positive t smaller than μ such that $x - tu$ is in C' and $x + u + u' - tu$ is in C''. Note that t cannot be negative because it is assumed that x is at the 'edge' of C'. Since C'' obeys free disposal, the fact that $x + u + u' - tu$ is in C'' implies that $x + u + u' - \mu u$ is also in C'', which establishes $(\star)$. QED

Proof of Proposition 6(ii): Consider x' and y, unordered, with $x'_i > y_i$ and suppose that $C(x') = k'$ and $C(y) = k''$. If $k'' < k'$, then by the fact that C is increasing, $C(x' \wedge y + \lambda v_{x'}) \leq C(y) = k'' < k' = C(x')$ for all λ in $[0, 1]$, which means that (3) is vacuously true for all λ in $[0,1]$. If $k'' = k'$, then because C is strictly increasing, (3) is vacuously true for λ in $[0,1)$ and trivially true at $\lambda = 1$. So we assume that $k'' > k'$; since $C^{-1}((-\infty, k'']) >_i C^{-1}((-\infty, k'])$ we know that there is $\bar{\lambda}$ such that $x' \wedge y + \bar{\lambda} v_{x'}$ is in $C^{-1}((-\infty, k'])$ and $x' \vee y - \bar{\lambda} v_{x'}$ is in $C^{-1}((-\infty, k''])$. Since C is continuous and increasing, there is $\lambda^* \geq \bar{\lambda}$ such that $C(x' \wedge y + \lambda^* v_{x'}) = k'$ and $C(x' \vee y - \lambda^* v_{x'}) \leq k''$. Furthermore, since C is strictly increasing, for $\lambda < \lambda^*$, we have $C(x' \wedge y + \lambda v_{x'}) < k'$ and for $\lambda > \lambda^*$, we have $C(x' \wedge y + \lambda^* v_{x'}) > k'$ and $C(x' \vee y - \lambda^* v_{x'}) < k''$. Together, this means that (3) holds. QED

Proof of Proposition 7: Define the function C acting on $X = \widetilde{X} \times I$ by $C(\tilde{x}, x_l) = x_l/s' - G(\tilde{x})$ and consider x' in $S' = C^{-1}(-\infty, 0]$ and y in S'', with $x'_l > y_l$. Assume also that y is not in S', so $C(y) > 0$, the other case being trivial. Note that C is continuous, increasing and l-convexmodular. By Proposition 6, $C^{-1}(-\infty, C(y)] \geq_l C^{-1}(-\infty, 0] = S'$, so there is λ in $[0,1]$ such that $x' \wedge y + \lambda v_{x'}$ is in S' and $z = x' \vee y - \lambda v_{x'}$ with $C(z) \leq C(y)$. Since y is S'', $C(y) + (y_l/s'') - (y_l/s') \leq 0$, so $C(z) + (y_l/s'') - (y_l/s') \leq 0$. Note that $z_l = x'_l > y_l$, and since the map from (t, s) to t/s is submodular, we have $(z_l/s'') - (z_l/s') < (y_l/s'') - (y_l/s')$. Thus $C(z) + (z_l/s'') - (z_l/s') \leq 0$, which implies that z is in S''. QED

Proof of Proposition 9: Differentiating $\tilde{U}$ by $\tilde{a}$ and x, we obtain

$$\begin{aligned} \tilde{U}_{\tilde{a},x}(c_1^*, \tilde{a}^*, x^*) &= \int u_{22}(c_1^*, \tilde{a}^* r + x^* t + w_2) r t g(t) dt \\ &= r \int \left[-\frac{u_{22}}{u_2} \right] \left(-u_2(c_1^*, \tilde{a}^* r + x^* t + w_2) \right) t g(t) dt. \end{aligned}$$

Let k_0 be the coefficient of risk aversion at $(c_1^*, \tilde{a}^* r + w_2)$ (when $t = 0$). Since the coefficient is strictly decreasing,

$$\tilde{U}_{\tilde{a},x}(c_1^*, \tilde{a}^*, x^*) > r k_0 \int \left(-u_2(c_1^*, \tilde{a}^* r + x^* t + w_2) \right) t g(t) dt = 0,$$

where the last equality follows from the first order condition $\tilde{U}_x(c_1^*, \tilde{a}^*, x^*) = 0$ (recall that X is free). So we are left with showing that

$$\tilde{U}_{c_1,\tilde{a}}(c_1^*, \tilde{a}^*, x^*) = \int u_{12}(c_1^*, \tilde{a}^* r + x^* t + w_2) r g(t) dt \text{ and}$$

$$\tilde{U}_{c_1,x}(c_1^*, \tilde{a}^*, x^*) = \int u_{12}(c_1^*, \tilde{a}^* r + x^* t + w_2) t g(t) dt$$

are both nonnegative in some neighborhood of the optimum. This is obvious under (b') since both terms will be identically zero. With the conditions in (b"), the first term is obviously positive since u is supermodular. Clearly, those conditions also guarantee that

$$\tilde{U}_{c_1,x}(c_1^*, \tilde{a}^*, x^*) = \int \left[\frac{u_{12}}{u_2}\right] \left(u_2(c_1^*, \tilde{a}^* r + x^* t + w_2)\right) t g(t) dt$$

is positive, once we apply an argument similar to the one we used for $\tilde{U}_{\tilde{a},x}$. QED

Proof of Proposition 10: Since x^* is in C and the map $(x_1, a) \to a x_1$ is supermodular, Proposition 8 guarantees that there is x^{**} solving problem (ii) such that $x_1^{**} \geq x_1^*$. (Note that this parameter change is more specific than the general parameter change considered by Milgrom and Roberts (1996).)

Since in both (A) and (B) Π is supermodular in x and has increasing differences in (x, a), and since $\mathrm{argmax}_{x \in R_+^l} \Pi(x, a'')$ is nonempty by assumption, there is some $\bar{x}$ in $\mathrm{argmax}_{x \in R_+^l} \Pi(x, a'')$ such that $\bar{x} \geq x^*$ (see Topkis (1998) or Milgrom and Shannon (1994)). Since x^* is in C, $\bar{x}$ also solves $\max_{x \in X_C} \Pi(x, a'')$; in particular $\mathrm{argmax}_{x \in X_C} \Pi(x, a'')$ is nonempty. We also know that $\mathrm{argmax}_{x \in C} \Pi(x, a'')$ contains x^{**}. Thus there is x^{***} in $\mathrm{argmax}_{x \in X_C} \Pi(x, a'')$ such that $x_1^{***} \geq x_1^{**}$; in case (A) this follows from the standard monotone comparative statics results (see Topkis (1998) or Milgrom and Shannon (1994)) while in case (B) it follows from Theorem 1. Finally note that x^{***} is in $\mathrm{argmax}_{x \in R_+^l} \Pi(x, a'')$ since $\bar{x}$ is in X_C and also in $\mathrm{argmax}_{x \in R_+^l} \Pi(x, a'')$. QED

Proof of Proposition 11: This proof is a straightforward variation of the one given by Milgrom and Shannon (1994, Theorem 3) for the single crossing property. Suppose that $x_1' < x_1''$ and $x_2' > x_2''$ with $f(x_1', x_2') \geq f(x_1'', x_2'')$. We wish to show that $f(x_1' + k, x_2') \geq f(x_1'' + k, x_2'')$ for any positive k. Define I to be the interval $[x_1', x_1'']$ and consider the indifference curve through the point (x_1'', x_2''). On the interval I, this curve can be represented by the function $\tau : I \to R$, where $\tau(x_1'') = x_2''$. Then

$f(x_1''+k,\tau(x_1''))-f(x_1'+k,\tau(x_1'))$ equals

$$\begin{aligned}\int_{x_1'+k}^{x_2''+k}\frac{df}{dt}(t,\tau(t-k))\,dt &= \int_{x_1'+k}^{x_2''+k} f_1(t,\tau(t-k))+f_2(t,\tau(t-k))\tau'(t-k)\,dt\\ &= \int_{x_1'+k}^{x_2''+k}\left[\frac{f_1(t,\tau(t-k))}{f_2(t,\tau(t-k))}+\tau'(t-k)\right]f_2(t,\tau(t-k))\,dt\end{aligned}$$

By the declining slope property in (i), this expression is less

$$\int_{x_1'+k}^{x_2''+k}\left[\frac{f_1(t-k,\tau(t-k))}{f_2(t-k,\tau(t-k))}+\tau'(t-k)\right]f_2(t,\tau(t-k))\,dt,$$

which in turn equals zero because $f(t,\tau(t))$ is identically constant and has a zero derivative. So we conclude that $f(x_1'+k,\tau(x_1'))\geq f(x_1''+k,\tau(x_1''))=f(x_1''+k,x_2'')$. Since $f(x_1',x_2')\geq(>)f(x_1'',x_2'')$ by assumption, and f is strictly increasing in variable 2, $x_2'\geq(>)\tau(x_1')$. Given that f is strictly increasing in variable 2, we have $f(x_1'+k,x_2')\geq(>)f(x_1''+k,x_2'')$.

The proof in the case of condition (ii) is similar. QED

Proof of Theorem 2: We claim that (i) implies that $H>_2 G$; the result then follows from Theorem 1. Let (x_1',x_2') be in G and (y_1,y_2) be in H such that $x_2'>y_2$ and $y_1>g(y_2)$. (All the other cases are trivial.) Consider the four points (x_1',x_2'), $(g(y_2),y_2)$, (y_1,y_2), and $(x_1'+[y_1-g(y_2)],x_2')$; they form a parallelogram, indeed a backward leaning parallelogram since $x_1'\leq g(x_2')\leq g(y_2)$. Note that the second point, $(g(y_2),y_2)$, is in G, so we need only show that the last point is in H. This is true since

$$\begin{aligned}x_1'+[y_1-g(y_2)] &\leq h(x_2')+[x_1'-h(x_2')]+[y_1-g(y_2)]\\ &\leq h(x_2')+[g(x_2')-h(x_2')]+[h(y_2)-g(y_2)]\\ &\leq h(x_2').\end{aligned}$$

Case (ii) is just a slight modification of (i). In this case, let (x_1',x_2') be a solution to P_g and let (y_1,y_2) be a solution to P_h. As in the previous case, we assume that $x_2'>y_2$ and $y_1>g(y_2)$ (the other cases being trivial) and consider the four points (x_1',x_2'),

$(g(y_2), y_2)$, (y_1, y_2), and $(x_1'+[y_1-g(y_2)], x_2')$. The same argument as before guarantees that the second point is in G and the fourth point is in H. We claim that they form a backward leaning parallelogram. If not, $g(y_2) \leq x_1'$ and so $(x_1' + [y_1 - g(y_2)] \geq y_1$. Since $(x_1' + [y_1 - g(y_2)], x_2')$ is in H so must (y_1, x_2'). By the strict monotonicity of f in variable 2, $f(y_1, x_2') > f(y_1 y_2)$, which contradicts the assumption that (y_1, y_2) solves P_h.

Thus $\{(y_1, y_2), (x_1'+[y_1-g(y_2)], x_2')\} >_2 \{(x_1', x_2'), (g(y_2), y_2)\}$. Applying Theorem 1 again, we see that $f(x_1' + [y_1 - g(y_2)], x_2') \geq f(y_1, y_2)$ and $f(g(y_2), y_2) \geq f(x_1', x_2')$. So $(x_1' + [y_1 - g(y_2)], x_2')$ solves P_h and $(g(y_2), y_2)$ solves P_g. QED

Proof of Proposition 12: By Proposition 11, we need to show that the ratio

$$R(a,b) = \frac{\int u'(bst + art)stf(s)h(t)dtds}{\int u'(bst + art)rtf(s)h(t)dtds}$$

is increasing with a. Define $v(z) = \int u(tz)h(t)dt$; then $v'(z) = \int u'(tz)th(t)dt$, so that $R(a,b) = \int v'(bs + ar)sf(s)ds / \int v'(bs + ar)rf(s)ds$. From the argument in the main body of the paper, we know that this is increasing in a provided v exhibits decreasing risk aversion. It is not hard to check (alternatively, see Gollier (2001)) that, when v is concave (which it is since u is concave), this property holds if and only if

$$-\frac{v'''(z)}{v''(z)} \geq -\frac{v''(z)}{v'(z)} \text{ for all } z.$$

To show this, we set $-v''(z)/v'(z) = \lambda$ and claim that

$$v''' + \lambda v'' = \int \left[t^3 u'''(tz) + \lambda t^2 u''(tz)\right] h(t)dt \geq 0.$$

Clearly, this is true if there is some number m such that

$$t^3 u'''(tz) + \lambda t^2 u''(tz) \quad \geq \quad m\left[\lambda u'(tz)t + u''(tz)t^2\right] \tag{9}$$

since the integral of the right hand side gives $\lambda v'(z) + v''(z) = 0$.[21] Denoting $a = -u''(tz)/u'(tz)$, and recalling that $-u'''(tz)/u''(tz) \geq a$ since u has diminishing risk

[21] In fact the existence of m is also *necessary* (see Gollier (2001) who refers to this equivalence as *the diffidence theorem*).

aversion, we can check that a sufficient condition for (9) to be true (after dividing by $t > 0$) is that $a^2t^2 - \lambda at \geq m[\lambda - at]$. This is true if we set $m = -\lambda$. QED

APPENDIX B

In this appendix, we examine the implications that concavemodularity and quasiconcavemodularity have on the concavity of a function. We call a function $g : I \to R$, where I is an interval in R, *concave* if $g(t) - g(t') \geq g(t + c) - g(t' + c)$ whenever $t < t'$, $c > 0$, and the the four points concerned lie in I. Note that the definition we have adopted is not the standard one, which says that g is concave if $g(\alpha t + (1 - \alpha)t') \geq \alpha g(t) + (1 - \alpha)g(t')$ for all t and t' in I and α in $[0, 1]$. It is not hard to show that the standard concavity property implies the one we have adopted; if g is continuous, it is also not hard to show that our concavity property implies the standard one.[22]

Similarly, we will adopt a slightly different definition of quasiconcavity. We say that g is *quasiconcave* if for all $t < t'$ and $c > 0$, $g(t) > g(t')$ implies that $g(t + c) \geq g(t' + c)$. The standard definition says that g is quasiconcave if, for any scalar M, the set $\{x \in I : g(x) \geq M\}$ is convex. It is not hard to check that this standard property implies the one we have adopted; when g is continuous, our property also implies the standard one.[23]

[22]Without continuity, our property *does not* imply the. standard concavity property. It is well known that there is a function $H : R \to R$ obeying $H(t + t') = H(t) + H(t')$ which is discontinuous at all points on R (see Hardy, Littlewood, and Polya (1952)). Since any concave function must be continuous on the interior of its domain, H is not concave. On the other hand H clearly satisfies our weaker definition of concavity. We should point out that the concavity property we *used* in Proposition 1 (analogously, the convexity property we used Proposition 5) is, in fact, the weaker property we have adopted in this appendix, rather than the standard one.

[23]The function H in the previous footnote obeys our definition of quasiconcavity, but it will violate the standard definition. This is because a quasiconcave function can only have a countable number of discontinuities, whereas H is discontinuous everywhere. There is an easy way of seeing that a quasiconcave function (in the standard sense) can only be countably discontinuous. First, note that

Let X be a convex sublattice of R^l. The function $f : X \to R$ is said to be concave (quasiconcave) in direction v if for all x in X, the map from the scalar t to $f(x + tv)$ is concave (quasiconcave). The domain of this map is taken to be the largest possible interval so that $x + tv$ lies in X. The function f is i-concave (i-quasiconcave) if it is concave (quasiconcave) in all directions v such that $v > 0$ and $v_i = 0$.

Our first result is an increasing difference formulation of concavemodularity which will be convenient for expositional purposes. We will skip the obvious proof.

LEMMA B1: *The function $f : X \to R$ is i-concavemodular if and only if for all vectors v such that $v_i < 0$ and $v_j > 0$ for some j, we have*

$$f(x + \lambda v_+) - f(x + v + \lambda v_+) \geq f(x) - f(x + v)$$

where λ is a positive scalar and $v_+ = v \vee 0$.

The next proposition says that, with some mild additional assumptions, there is a converse to Proposition 1; i.e., a function which is i-concavemodular must be i-concave.

PROPOSITION B1: *Let $X \subset R^l$ be a convex and open sublattice and suppose that $f : X \to R$ is an i-concavemodular function which is also continuous in x_i. Then f is i-concave.*

Proof: Suppose, by way of contradiction, that there is $\bar{v} > 0$ with $\bar{v}_i = 0$, and $\lambda > 0$ such that $f(x) - f(x + \bar{v}) > f(x + \lambda\bar{v}) - f(x + \bar{v} + \lambda\bar{v})$. Since f is continuous in x_i, and X is open, there is $\delta > 0$ and sufficiently close to zero such that $f(x) - f(x + \bar{v} - \delta e_i) > f(x + \lambda\bar{v}) - f(x + \bar{v} - \delta e_i + \lambda\bar{v})$, where e_i is the unit vector pointing in direction i. By Lemma B1, this is a violation of i-concavemodularity since $(\bar{v} - \delta e_i) \vee 0 = \bar{v}$. QED

The next result concerns functions which are concavemodular (in all directions). It turns out that these functions must be concave in all directions except the strictly positive and the strictly negative.

there must be some t^* in the domain of the function such that the function is increasing for $t < t^*$ and is decreasing for $t > t^*$. Second, increasing and decreasing functions can only have countably many discontinuities.

Proposition B2: *Let* $X \subset R^l$ *be a convex and open sublattice and suppose that* $f : X \to R$ *is a concavemodular function which is also continuous in each of its arguments (but not necessarily jointly continuous). Then* f *has the following concavity property: for all* x *in* X, f *is concave at* x *in all directions* v *satisfying* $v \not\gg 0$ *and* $v \not\ll 0$. *In particular,* f *must be partially concave.*

Proof: For $v > 0$ (or $v < 0$) such that $v_i = 0$ for some i, we can appeal to Proposition B1. (Note that this is the only place where the continuity property imposed on f and the openness of X is used.) We now turn to the case where v is such that $v_i < 0$ for some i and $v_j > 0$ for some j. Let t be a positive scalar. Denote $v_+ = v \vee 0$ and $v_- = v \wedge 0$. We have

$$\begin{aligned} f(x) - f(x+v) &\leq f(x+tv_+) - f(x+v+tv_+) \\ &\leq f(x+tv_+ + tv_-) - f(x+v+tv_+ + tv_-) \\ &= f(x+tv) - f(x+v+tv); \end{aligned}$$

the first inequality arises from i-concavemodularity and the second from j-concavemodularity. (Note that all the elements referred to in the inequalities are in X because it is a convex lattice.) QED

As a simple illustration, consider the function $f : R^2_{++} \to R$ given by $f(x_1, x_2) = x_1 x_2$. Clearly this function is partially concave and supermodular. By Proposition 1, it is concavemodular, which means by Proposition B2 that it is concave in all directions except possibly those which are strictly positive or strictly negative. To check this, consider the behavior of the function along the ray emanating from the point $(\bar{x}_1, \bar{x}_2)$ and in the direction (a, b): $f(\bar{x}_1 + at, \bar{x}_2 + bt) = \bar{x}_1\bar{x}_2 + (b\bar{x}_1 + a\bar{x}_2)t + abt^2$, which is a concave function of t whenever a and b are of different signs, but convex whenever a and b are *both* strictly positive or strictly negative.

The results we have reported so far have analogs for quasiconcavemodular functions. The next result is obviously analogous to Proposition B1.

Proposition B3: *Let* $X \subset R^l$ *be a convex and open sublattice and suppose that*

$f : X \to R$ is an i-quasiconcavemodular function which is also continuous in x_i. Then f is i-quasiconcave.

Proof: Suppose, by way of contradiction, that there is $\bar{v} > 0$ with $\bar{v}_i = 0$, and a positive scalar λ such that $f(x) - f(x + \bar{v}) > 0$ and $f(x + \lambda\bar{v}) - f(x + \bar{v} + \lambda\bar{v}) < 0$. Since f is continuous in x_i, and X is open, there is $\delta > 0$ and sufficiently close to zero such that $f(x) - f(x + \bar{v} - \delta e_i) > 0$ and $f(x + \lambda\bar{v}) - f(x + \bar{v} - \delta e_i + \lambda\bar{v}) < 0$, where e_i is the unit vector pointing in direction i. By Lemma B1, this is a violation of i-quasiconcavemodularity since $(\bar{v} - \delta e_i) \wedge 0 = \bar{v}$. QED

Our final result shows that any continuous and increasing quasiconcavemodular function must have convex upper contour sets (in other words, that it is quasiconcave in the standard sense). Note, once again, that this is borne out by the example $f(x_1, x_2) = x_1 x_2$.

Proposition B4: *Let $X \subset R^l$ be a convex sublattice and suppose that $f : X \to R$ is a continuous, increasing and quasiconcavemodular function. Then for any M, the set $S_M = \{x \in X : f(x) \geq M\}$ is convex.*

Proof: By adapting the proof of Proposition B2, we can easily establish that f is quasiconcave in direction v for all v with $v_i > 0$ for some i and $v_j < 0$ for some j. Suppose that $f(x) - f(x + v) > 0$. By the i-quasiconcavemodularity of f, we have $f(x + tv_+) - f(x + v + tv_+) > 0$ and by j-quasiconcavemodularity, we have

$$f(x + tv_+ + tv_-) - f(x + v + tv_+ + tv_-) = f(x + tv) - f(x + v + tv) > 0,$$

which shows that f is quasiconcave in direction v. Now consider two distinct points x' and x'' in S_M. If $x'' > x'$ or $x' > x''$, it is clear that $f(tx' + (1 - t)x'') \geq M$ for t in $[0, 1]$ since f is increasing. So we assume that x' and x'' are not ordered, in which case we know that f is quasiconcave in the direction $\bar{v} = x'' - x'$. Recall, however, that our definition of quasiconcavity is nonstandard (see the second paragraph of this appendix); nonetheless, when f is continuous this coincides with the standard definition, and the latter says that $f(tx' + (1 - t)x'') \geq M$ for all t in $[0, 1]$. QED

REFERENCES

ANTONIADOU, E. (1995): "Lattice Programming and Economic Optimization Part II: Applications to Consumer Theory," *Stanford University mimeo.*

ANTONIADOU, E. (2004): "Lattice Programming and Economic Optimization: Applications to Consumer Theory," *mimeo.*

ATHEY, S., P. MILGROM, AND J. ROBERTS (1998): *Robust Comparative Statics.* (Draft Chapters.)

ATHEY, S. (2002): "Monotone Comparative Statics Under Uncertainty," *The Quarterly Journal of Economics,* 117(1), 187-223.

CHAMPSAUR, P. AND J.-C. MILLERON (1983): *Advance Exercises in Microeconomics.* Cambridge: Harvard University Press.

CHIPMAN, J. (1977): "An Empirical Implication of Auspitz-Lieben-Edgeworth-Pareto Complementarity," *Journal of Economic Theory,* 14, 228-231.

CONSTANTINIDES, G. M. (1990): "Habit Formation: A Resolution of the Equity Premium Puzzle," *Journal of Political Economy,* 98, 519-43.

GOLLIER, C. (2001): *The Economics of Risk and Time.* Cambridge: MIT Press.

HARDY, G., E. LITTLEWOOD, AND G. POLYA (1952): *Inequalities, 2nd Edition.* Cambridge: Cambridge University Press.

HENS, T. AND A. LOFFLER (1995): "Gross Substitution in Financial Markets," *Economic Letters*, 49, 39-43.

JEWITT, I. (2000): "Risk and Risk Aversion in the Two Risky Asset Problem," *mimeo.*

MARINACCI, M. AND L. MONTRUCCHIO (2003): "Introduction to the mathematics of uncertainty," in GILBOA, I (ed.), *Uncertainty in economic theory: essays in honor of David Schmeidler's 65th birthday.* London: Routledge.

MAS-COLELL, A. (1985): *The Theory of General Economic Equilibrium: A differentiable approach.* Cambridge: Cambridge University Press.

MAS-COLELL, A., M. D. WHINSTON, AND J. R. GREEN (1995): *Microeconomic Theory.* Oxford: Oxford University Press.

MCFADDEN, D. L. (1978); "Cost, Revenue, and Profit Functions," in FUSS, M. AND D. L. MCFADDEN (eds.), *Production Economics: A dual approach to theory and applications. Vol. I: A Theory of Production.* Amsterdam: North-Holland.

MILGROM, P. (1994): "Comparing Optima: Do Simplifying Assumptions Affect Conclusions?" *Journal of Political Economy*, 102(3), 607-615.

MILGROM, P. AND C. SHANNON (1994): "Monotone Comparative Statics," *Econometrica*, 62(1), 157-180.

MILGROM, P. AND J. ROBERTS (1996): "The LeChatelier Principle," *American Economic Review,* 86(1), 173-179.

MIRMAN, L. J. AND R. RUBLE (2003): "Lattice Programming and the Consumer's Problem" (mimeo).

QUAH, J. K.-H. (2003): "Market Demand and Comparative Statics When Goods are Normal," *Journal of Mathematical Economics*, 39, 317-333.

ROBERTS, K. (1999): "Rationality and the LeChatelier Principle," *Journal of Economic Theory*, 87, 416-428.

TOPKIS, D. M. (1978): "Minimizing a Submodular Function on a lattice," *Operations Research*, 26, 305-321.

TOPKIS, D. M. (1998): *Supermodularity and Complementarity.* Princeton: Princeton University Press.

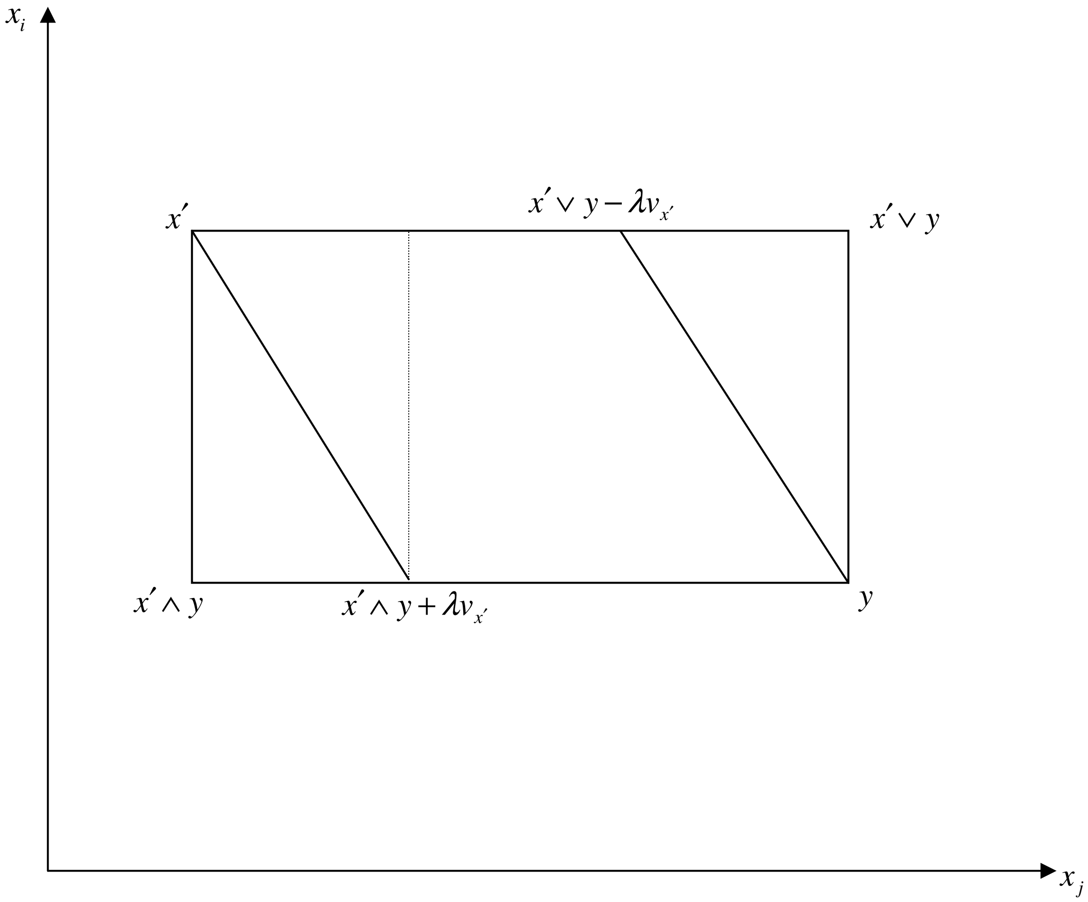

FIGURE 1

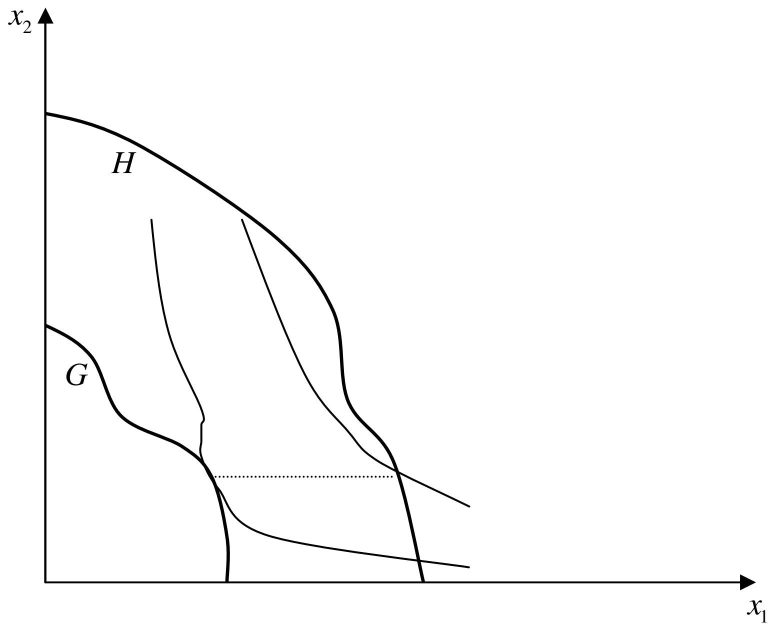

FIGURE 2

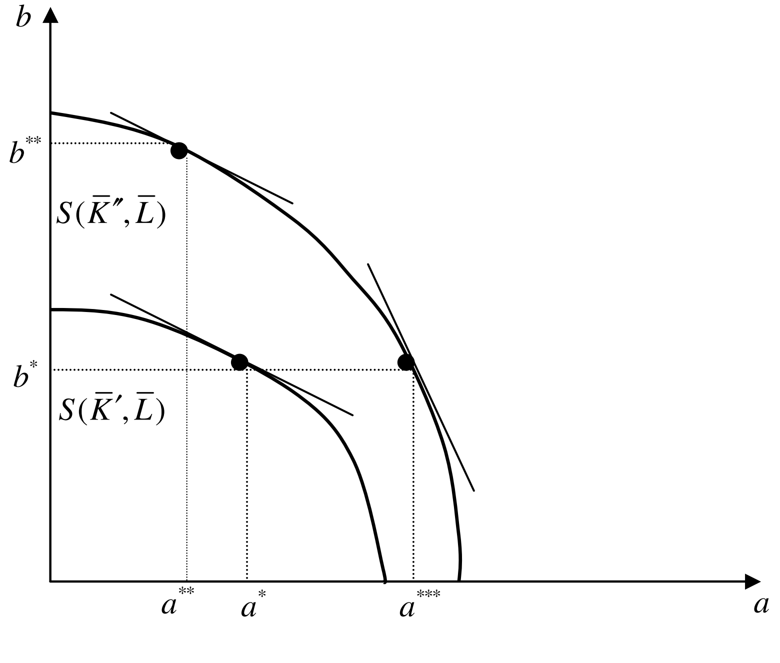

FIGURE 3