Event Exchangeability:

Probabilistic Sophistication without Continuity or Monotonicity*

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Abstract

Building on the Ramsey-de Finetti idea of event exchangeability, we derive a parsimonious and novel characterization of probabilistic sophistication on an algebra of events without requiring any of the various versions of monotonicity, continuity, or comparative likelihood assumptions imposed by Savage (1954), Machina and Schmeidler (1992) and Grant (1995). Our characterization identifies a unique and finitely-additive subjective probability measure over an algebra of events for a decision maker under weaker conditions than have hitherto been identified in the literature.

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1 Introduction

In their pioneering studies, Ramsey (1926) and de Finetti (1937) originated the idea of distinguishing events according to whether they are 'exchangeable' or 'ethically neutral', providing the basis for their construction of a decision maker's subjective probability over events. Savage's (1954) subsequent formulation departs from this direction and nevertheless yields an overall subjective probability on a sigma algebra of events. Building on Savage's approach, Machina and Schmeidler (1992) and subsequently Grant (1995) provide more parsimonious characterizations of what is termed probabilistic sophistication, in which the choice behavior of a decision maker reflects her probabilistic belief in the sense that events are distinguished only by their subjective probabilities.

Their contributions notwithstanding, some of the axioms employed by Machina and Schmeidler (1992) and Grant (1995) are arguably too strong for the notion of probabilistic sophistication. Consider, for instance, a decision maker with preferences over mappings from finite partitions of the state space $[0,1]$ to an outcome set X (i.e., simple acts). Suppose the decision maker translates each act into a lottery by associating with the i^{th} partition element its measure, $p_i \in [0,1]$, and its assigned outcome, $x_i \in X$. Denote such a lottery as, say, $L = (x_1, p_1; ...; x_n, p_n)$. *As long as the decision maker is indifferent between two acts that induce the same lottery, it seems reasonable to conclude that she is probabilistically sophisticated.* For instance, let X be the real line and suppose that the decision maker ranks any simple act according to the expected value of the lottery it induces, and if two lotteries have the same mean, the one with a smaller variance is preferred. According to the preceding notion of probabilistic sophistication, the decision maker is probabilistically sophisticated. However, this lexicographic preference satisfies all the axioms of Savage (1954), Machina and Schmeidler (1992) and Grant (1995), except for P6 ('continuity').

As another example, suppose X is the two dimensional positive orthant, $\mathbb{R}^2_+$. Let $E[L] = \sum_{i=1}^n p_i \mathbf{x}_i$ and $V(L) = \sum_{i=1}^n p_i ||\mathbf{x}_i - E[L]||$ with $|| \cdot ||$ the Euclidean metric. Suppose the decision maker's preferences can be represented by the utility function $U(L) = \sum_{i=1}^n p_i (1 + x_i^1)(1 + x_i^2) - \frac{1}{2} V(L)$. Here too, the decision maker is probabilistically sophisticated (according to the preceding criteria). Moreover, if $x^1 > y^1$ and $x^2 > y^2$, then this decision maker strictly prefers mixing any lottery L and (x^1, x^2) with probability $p \in [0, 1)$ over mixing L and (y^1, y^2) with probability p. Since these preferences are continuous and strictly increasing in payoffs, they are arguably unobjectionable on normative grounds. Yet, it is straight forward to show that they violate axioms P3 and P4 ('monotonicity' and 'comparative likelihood') in Machina and Schmeidler (1992), as well as their analogues (P3^{CU}, P3^{CL} and P4^{CE}) in Grant (1995).[1]

Say that two events are *exchangeable* if the decision maker is always indifferent to permuting their payoffs. Building on exchangeability as the primitive, we develop a notion of comparability to capture the intuition behind a likelihood relation among events. Specifically, two disjoint events are *comparable* when one contains a subevent that is exchangeable with the other. Informally, one is motivated to view one event as 'larger' or 'more likely' than the other. When all disjoint events are comparable in this way, we show that very weak conditions - far weaker than Savage's assumptions of monotonicity (P3), comparative likelihood (P4), and continuity (P6) - suffice to deliver probabilistic sophistication on the part of the decision maker. Indeed, the example of the lexicographic ranking and the ranking implied by $U(\cdot)$ satisfy our axioms.

The next section introduces preliminary notions including formal definitions of event exchangeability and comparability, presents our main result concerning probabilistic sophistication, and relates our result to the existing literature.

2 A Parsimonious Axiomatization for Subjective Probabilities

2.1 Exchangeability and Comparability

Let Ω be a space whose elements correspond to all states of the world. Let X be a set of payoffs and Σ an algebra on Ω. Elements of Σ are events. If $e, E \in \Sigma$ and $e \subseteq E$, then we say that e is a *subevent* of E. The set of simple acts, $\mathcal{F}$, comprises all Σ-adapted and X-valued functions over Ω that have a finite range. As is customary, $x \in X$ is identified

[1] It is important to note that P3 ('monotonicity') is synonymous with monotonicity in payoffs *only* if X is one-dimensional (e.g., monetary payoffs). We thank Uzi Segal for suggesting this example.

with the constant act that pays x in every state. Throughout the paper we assume that the decision maker has a non-degenerate binary preference relation, $\succeq$, on $\mathcal{F}$ as in Savage's P1 and P5.[2]

For any collection of pairwise disjoint events, $E_1, E_2, ..., E_n \subset \Omega$, and $f_1, f_2, ..., f_n, g \in \mathcal{F}$, let $f_1E_1f_2E_2...f_nE_ng$ denote the act that pays $f_i(\omega)$ if the true state, $\omega \in \Omega$, is in E_i, and pays $g(\omega)$ otherwise. We say that $E \in \Sigma$ is *null* if $fEh \sim gEh\ \forall f, g, h \in \mathcal{F}$.

To capture the sense in which events are similar, we introduce a binary relation over events via $\succeq$:

Definition 1 (Event Exchangeability)**.** *For any pair of disjoint events $E, E' \in \Sigma$, $E \approx E'$ if for any $x, x' \in X$ and $f \in \mathcal{F}$, $xEx'E'f \sim x'ExE'f$.*

Whenever $E \approx E'$ we will say that E and E' are *exchangeable.* Note that all null events are exchangeable. Exchangeability may be viewed as a pre-notion of 'equally likely': two events are 'equally likely' if the decision maker is indifferent to a permutation of their payoffs. Without further structure this interpretation is not formally justified since, as the next example demonstrates, $\approx$ is not necessarily transitive, and therefore not an equivalence relation.

Example 1. Consider the partition $\{A, B_1, B_2, C\}$ of Ω. Let $X \equiv [0, 1]$ and the utility representation over acts $xAy_1B_1y_2B_2z$ be given by

$$V(x, y_1, y_2, z) = x + z + \frac{y_1 + y_2}{2} + \frac{y_1 - y_2}{4}x$$

It is straight forward to check that the representation satisfies first order dominance. It should also be clear that $A \approx B_1 \cup B_2$ and $C \approx B_1 \cup B_2$. On the other hand, it is certainly not the case that $A \approx C$ due to the asymmetry between x and z arising in the last term of the utility function. $\square$

Intuitively, an event is 'at-least-as-likely' as any of its subevents. Exchangeability supplies the motivation underlying a similar comparison across disjoint events, $E, E' \in \Sigma$: if a

[2] As usual, $\succ$ (resp. $\sim$) is the asymmetric (resp. symmetric) part of $\succeq$. Under Savage's P1, $\succeq$ is a weak order on $\mathcal{F}$, while P5 asserts that there exists $f, g \in \mathcal{F}$ such that $f \succ g$.

subevent of E is exchangeable with E', then it is also natural to view E as 'at-least-as-likely' as E'. Building on this, we define the following exchangeability based relation between any two events.

Definition 2 (Event Comparability)**.** *For any events, $E, E' \in \Sigma$, $E \succeq^C E'$ whenever $E \setminus E'$ contains a subevent, e, that is exchangeable with $E' \setminus E$. Moreover, e is referred to as a* ***comparison event****.*

Just as $\approx$ gives a pre-notion of 'equal likelihood' among events, $\succeq^C$ provides a pre-notion of an 'at-least-as-likely' relation. The event E is 'at least as likely' as E' if outside their intersection the 'more likely' event (i.e., $E \setminus E'$) contains a 'copy' (i.e., the comparison event) of the 'less likely' event (i.e., $E' \setminus E$). Since $\emptyset$ is a subevent of any event and $\emptyset$ is exchangeable with itself, $E' \subseteq E$ implies $E \succeq^C E'$.

For any $E, E' \in \Sigma$, we say that E and E' are *comparable* whenever $E \succeq^C E'$ or $E' \succeq^C E$. Finally, define $E \succ^C E'$ whenever $E \succeq^C E'$ and it is not the case that $E' \succeq^C E$. Likewise, define $\sim^C$ as the symmetric part of $\succeq^C$.

We also need the following definitions:

Definition 3. *$\succeq^\circ$ is a* ***likelihood relation*** *over Σ if the following conditions hold:*

i) $\succeq^\circ$ is a weak order over Σ

ii) $\Omega \succ^\circ \emptyset$ and for every $A \in \Sigma$, $A \succeq^\circ \emptyset$ and $\Omega \succeq^\circ A$

iii) for every $A, B, C \in \Sigma$ such that $C \cap (A \cup B) = \emptyset$, $A \succeq^\circ B \;\Leftrightarrow\; A \cup C \succeq^\circ B \cup C$

Note that the second requirement is satisfied by $\succeq^C$ by virtue of the non-triviality of $\succeq$, while the last requirement is satisfied by the definition of $\succeq^C$. Thus establishing that $\succeq^C$ is a likelihood relation reduces to demonstrating that condition (i) holds.

Definition 4. *μ is an* ***agreeing probability measure*** *for $\succeq^\circ$ over Σ, if it is a probability measure over Σ and for every $A, B \in \Sigma$, $A \succeq^\circ (\succ^\circ) B \;\Leftrightarrow\; \mu(A) \geq (>) \mu(B)$.*

For any probability measure, μ on Σ and act $f \in \mathcal{F}$, refer to $\{(\mu\big(f^{-1}(x)\big), x) \mid x \in X\}$ as the lottery induced by the act, $f \in \mathcal{F}$ with respect to μ. We say that μ is *purely and uniformly atomic* whenever the union of all atoms has unit measure and all atoms have equal measure. Finally, we say that μ is *solvable* if for every $A, B \in \Sigma$, $\mu(A) \geq \mu(B)$ implies the existence of a subevent $a \subseteq A$ with $\mu(a) = \mu(B)$. Note that requiring μ to be solvable is weaker than requiring it to be convex-ranged.[3]

2.2 Axioms and Main Result

Given a non-null event, e_0, consider asking a decision maker to identify a disjoint event (say e_1) that is exchangeable with e_0, then find another event (say e_2) disjoint from $e_0 \cup e_1$ and exchangeable with e_1, then find another event (say e_3) disjoint from $e_0 \cup e_1 \cup e_2$ and exchangeable with e_2, and so on; then the following 'Archimedean' condition asserts that this procedure must end after a finite number of steps:

Axiom A (Event Archimedean Property)**.** *Any sequence of pairwise disjoint and non-null events, $\{e_i\}_{i=0}^{n} \subseteq \Sigma$, such that $e_i \approx e_{i+1}$ for every $i = 0, ...,$ is necessarily finite.*

Axiom A can also be restated to say that if $\{e_i\}_{i=0}^{\infty} \subseteq \Sigma$ is a sequence of pairwise disjoint events with $e_i \approx e_{i+1}$ for every $i = 0, ...,$ then e_0 is null.

Suppose that the decision maker behaves as if she assigns a unique probability measure to each event, and the measure of events along with their assigned payoffs are the only relevant characteristics for the purpose of her decision making. Clearly, if two events are equally likely then their set differences are also equally likely and thus exchangeable. Thus, if Σ is sufficiently 'fine' any event will contain a subevent with arbitrary yet smaller likelihood, and therefore any two events in the decision maker's world are comparable. When Σ is free of atoms, the latter appears to be a fundamental attribute of probabilistic sophistication in the absence of state dependence.[4] The next assumption asserts this by requiring completeness of $\succeq^C$.

[3] μ is convex-ranged if for every $\alpha \in [0, 1]$ and $A \in \Sigma$ there is a subevent $a \subseteq A$ with $\mu(a) = \alpha\mu(A)$.
[4] An atom is an event that cannot be partitioned into two or more non-null subevents.

Axiom C (Completeness of $\succeq^C$)**.** *Given any disjoint pair of events, one of the two must contain a subevent exchangeable with the other.*

While completeness of $\succeq^C$ may be appealing, added to Axiom A it is not sufficient for the existence of a likelihood relation, let alone a unique agreeing probability measure representing $\succeq^C$. Consider the following condition, which appears much weaker than Savage's P3 and P4:[5]

Axiom N (Event Non-satiation)**.** *For any pairwise disjoint $E, A, E' \in \Sigma$, if $E \approx E'$ and A is non-null, then no subevent of E' is exchangeable with $E \cup A$.*

Axiom N is equivalent to requiring that whenever two events are exchangeable, adding a disjoint non-null event to one of them makes the combined event strictly more 'likely' (i.e., $E \cup A \succ^C E'$). How 'minimal' is Axiom N? The next result establishes that it is necessary for any exchangeability based likelihood relation in which non-null sets are strictly more likely than the empty set. Thus to the extent that the latter is desirable, Axiom N is a minimal requirement for any theory of probabilistic sophistication in which exchangeable events are equally likely.

Lemma 1. *Assume that $\succeq^\circ$ is a likelihood relation over Σ with (i) a symmetric part that agrees with $\approx$ on disjoint sets, and (ii) $A \succ^\circ \emptyset$ for all non-null $A \in \Sigma$. Then for any pairwise disjoint $E, E', A \in \Sigma$ such that A is not null, $E \approx E'$ implies that $E \cup A \succ^\circ E'$.*

Proof: Assume that $E, E', A \in \Sigma$ are pairwise disjoint, A is not null, and $E \approx E'$ (meaning that $E \sim^\circ E'$). Note that $A \succeq^\circ \emptyset \;\Leftrightarrow\; E \cup A \succeq^\circ E$. Transitivity of $\succeq^\circ$ implies that $E \cup A \succeq^\circ E'$. If $E \cup A \sim^\circ E'$ then $E \cup A \sim^\circ E$. In particular, the cancellation property (iii) of a likelihood relation means that $A \sim^\circ \emptyset$ – a contradiction. Thus $E \cup A \succ^\circ E'$. □

We now state several simple yet key implications of Axioms C and N that are useful in proving our main theorem as well as relating the intuition behind its proof.

[5] Savage's P3 states that for any non-null event, $E \subseteq \Omega$, act $f \in \mathcal{F}$ and any $x, y \in X$, $x \succeq y \;\Leftrightarrow\; xEf \succeq yEf$. Savage's P4 states that for any events $E, E' \in \Sigma$ and $x^*, x_*, y^*, y_* \in X$ with $x^* \succ x_*$, $y^* \succ y_*$, $x^*Ex_* \succeq x^*E'x_*$ implies $y^*Ey_* \succeq y^*E'y_*$. Machina and Schmeidler's (1992) more restrictive P4* requires that for any $f, g \in \mathcal{F}$ and whenever $E \cap E' = \emptyset$, $x^*Ex_*E'f \succeq x^*E'x_*Ef$ implies $y^*Ey_*E'g \succeq y^*E'y_*Eg$.

Lemma 2. *Axioms C and N imply for any $E, E', E'' \in \Sigma$, E and E' disjoint: $E \succeq^C E'$ and $E'' \subseteq E' \Rightarrow \exists \hat{e} \subseteq E$ with $\hat{e} \approx E''$. Moreover, $E \setminus \hat{e}$ is not null whenever $E' \setminus E''$ is not null.*

Proof: Let $e \subseteq E$ be the comparison set for $E \succeq^C E'$. If E'' contains a subevent $e'' \approx e$ with $E'' \setminus e''$ not null, then $e'' \cup (E' \setminus e'') \approx e$, in violation of N. Thus, by Axiom C, $e \succeq^C E''$ and $\exists \hat{e} \subseteq e \subseteq E$ with $\hat{e} \approx E''$. If $E' \setminus E''$ is not null, then $e \setminus \hat{e}$ cannot be null (and thus $E \setminus \hat{e}$ is not null), otherwise $e \approx E''$ and $e \approx (E' \setminus E'') \cup E''$, in violation of N. □

Lemma 3. *Axiom N implies for any disjoint $E, E' \in \Sigma$: $E \sim^C E' \Leftrightarrow E \approx E'$.*

Proof: $E \sim^C E' \Rightarrow \exists e \subseteq E$ with $e \approx E'$. $E \setminus e$ must be null (in which case $E \approx E'$). Otherwise, $E' \sim^C e \cup (E \setminus e)$ implies that E' contains a subevent exchangeable with $e \cup (E \setminus e)$ in violation of N. Now, $E \approx E'$ implies $E \succeq^C E'$ and $E' \succeq^C E$, thus implying $E \sim^C E'$. □

Lemma 4. *For any pairwise disjoint $a, b, c, d \in \Sigma$: $a \approx b$ and $c \approx d$ imply $a \cup c \approx b \cup d$.*

Proof: This is a direct consequence of Definition 1. □

Lemma 5. *Given Axioms C and N, and any pairwise disjoint $a, b, c, d \in \Sigma$: $a \cup b \approx c \cup d$ and $a \approx c$ imply $b \approx d$.*

Proof: If $b \not\approx d$ then Axiom C implies, without loss of generality, there is some $b' \subset b$ such that $b' \approx d$ and $b \setminus b'$ is not null. By Lemma 4, $a \cup b' \approx c \cup d$ which violates Axiom N since $a \cup b \setminus a \cup b'$ is not null. □

Our main result delivers exchangeability-based probabilistic sophistication as necessary and sufficient for Axioms A, C and N.

Theorem 1. *Axioms A, C and N are satisfied if and only if there exists a unique, solvable, and finitely additive agreeing probability measure, μ, for $\succeq^C$ over Σ. Moreover, μ is either atomless or purely and uniformly atomic, any two events have the same measure iff they are exchangeable, and the decision maker is indifferent between any two acts that induce the same lottery with respect to μ.*

We emphasize that for probabilistic sophistication we do not require Σ to be a σ-algebra, thus μ need not be convex-ranged in the atomless case. In particular, when applied to cases in which μ is solvable, our approach is more parsimonious than that of Kopylov (2004) who derives probabilistic sophistication on event domains requiring less structure than σ-algebras.[6]

2.3 Discussion

We now turn to a discussion of Theorem 1. We begin by examining the intuition behind the derivation. We then compare our axioms with their counterparts in the literature.

2.3.1 Intuition

To derive probabilistic sophistication it is sufficient to prove that $\succeq^C$ is a likelihood relation that can be represented by a unique finitely additive measure, μ. If two acts, f and g, induce the same lottery with respect to μ, the equivalence between $\sim^C$ and $\approx$ for two disjoint events (Lemma 3) can be used to permute the payoffs of f so as to show that the decision maker is indifferent between f and g. The non-trivial steps involve demonstrating that $\succeq^C$ is transitive and that Σ either consists of finitely many equal mass atoms or $\succeq^C$ is fine and tight – both cases known to be associated with a unique representing measure.[7]

To get a better sense of how Axioms A, C and N imply transitivity of $\succeq^C$, consider $E \succeq^C E' \succeq^C E''$, and assume for simplicity that $E, E', E'' \in \Sigma$ are pairwise disjoint. The general idea is to establish that if $E'' \succ^C E$, then one can construct an infinite sequence of non-null pairwise disjoint events in violation of Axiom A. To see how this is done, we first note a simple implication of Lemmas 2 and 5: $E'' \succeq^C E$ implies that for any subevent $e \subseteq E$ there exists $e'' \subseteq E''$ such that $e'' \approx e$ and $E'' \setminus e'' \succeq^C E \setminus e$. Essentially, comparability and event non-satiation enable one to 'cleave' equally sized pieces from E'' and E, while maintaining the ordering between the residual events.

[6] Kopylov's results extend to non-solvable as well as non-algebraic structures.

[7] A relation on Σ is fine if it contains no atoms and for any event, E, there exists a partition of Σ where no partition element is strictly more likely than E. The relation is tight whenever $E \succ^C E'$, there are $A, B \in \Sigma$ where $A \cap E' = \emptyset$ and $B \subset E$ such that $E \succ^C A \cup E'$ and $E \setminus B \succ^C E'$.

Now, consider that $E'' \succ^C E$ implies one can find a non-null subevent of E'', say e_1, such that $E'' \setminus e_1 \approx E$. Thus after 'cutting' e_1 from E'' one arrives at $E'' \setminus e_1 \succeq^C E \succeq^C E'$. Next, since $E' \succeq E''$, one can cleave a piece, say e_2, from E' such that $e_2 \approx e_1$, giving $E' \setminus e_2 \succeq^C E'' \setminus e_1 \succeq^C E$. This can be continued (e.g., cleave $e_3 \subset E$ such that $e_3 \approx e_2$, etc.) and yields the infinite sequence of non-null events. The contradiction with Axiom A forces $E \succeq^C E''$ and the desired transitivity of $\succeq^C$. The actual proof, found in the Appendix, makes use of such a construction, albeit in the more involved case where E, E' and E'' are not pairwise disjoint.

If Σ is atomless, tightness follows easily from Axiom N; fine-ness relative to a non-null event, E, can be established by 'cleaving' pairwise disjoint and equally sized pieces from $\Omega \setminus E$, which by Axiom A can only be done a finite number of times before one ends up with a 'remainder' event that is smaller than E. Clearly, this construction leads to a finite partition whose elements are no more likely than E. The fact that $\succeq^C$ is a fine and tight likelihood relation can then be used to deduce the unique existence of an agreeing probability measure.[8]

If Σ contains an atom, then completeness requires that every other event contain a subevent that is exchangeable with the atom. Thus, in particular, completeness of $\succeq^C$ implies that atoms must come in only one 'size', and that one can partition the state space with a set of such atoms.[9] In turn, Axiom A implies that any such partition is finite. While the result in the atomic case is 'trivial' and of limited interest, it does shed some light on a limitation of our approach: while Axiom C may be sensible when Σ is atomless, it is far from innocuous otherwise. Interesting cases involving atoms require a relaxation or at least a re-examination of the structure imposed. Moreover, additional assumptions will be required to pin down a unique representing measure for $\succeq^C$ when it is atomic.[10]

[8]See Wakker (1981).

[9]Details can be found in the proof of Theorem 1.

[10]This issue is not unique to our work – the majority of papers in this literature tend to focus on atomless state spaces and those that do not require considerably more structure than we do; see Wakker (1984), Chateauneuf (1985), Nakamura (1990), Gul (1992), Chew and Karni (1994), and Kobberling and Wakker (2003).

2.3.2 Relation to the Literature

In relating our approach to prior literature on probabilistic sophistication we focus primarily on Machina and Schmeidler (1992) and Grant (1995).[11]

Comparison with Machina and Schmeidler (1992)

Machina and Schmeidler (1992) show that the existence of a continuous probabilistically sophisticated utility representation of $\succeq$ agreeing with first degree stochastic dominance is equivalent to P1, P3, P4*, P5 and P6.[12] This result delivers a unique convex-ranged probability measure where the measures of two events coincide if and only if the events are exchangeable. Thus their axioms imply that $\succeq^C$ is complete (i.e., all events are mutually comparable) as well as Axiom A. It is also easy to show that monotonicity (i.e., P3) implies event non-satiation. One can therefore interpret that, in establishing probabilistic sophistication in Theorem 1, we weaken the Machina and Schmeidler axioms as follows: P3 $\rightarrow$ Axiom N, P4 and P6 $\rightarrow$ Axioms A and C. Indeed, in light of the preceding discussion, it seems that completeness, together with Axiom A, endows the state space with a uniform character reminiscent of the role typically played by P6. The latter, however, is much stronger given that it leads to a continuous representation, whereas continuity is not required in our case. Thus aside from monotonicity considerations, our assumptions substantively weaken those of Savage (1954) or Machina and Schmeidler (1992). Moreover, consider the following result:

Proposition 1. *Assume Savage's P3, and Axiom C. Then for any $x^*, x_* \in X$ with $x^* \succ x_*$, disjoint $E, E' \in \Sigma$, and $f \in \mathcal{F}$, $x^*Ex_*E'f \succeq x^*E'x_*Ef \ \Leftrightarrow \ E \succeq^C E'$.*

The proposition establishes two things: given a weak ordering satisfying P3, Machina and Schmeidler's P4* is implied by completeness of $\succeq^C$; moreover, $\succeq^C$ *is*, in this case, the comparative likelihood relation represented in their probabilistically sophisticated setting.

[11]Other related works include Sarin and Wakker (2000).

[12]See Footnote 2 for definitions of P1 and P5, and Footnote 5 for definitions of P3, P4 and P4*. First degree stochastic dominance, as used by Machina and Schmeidler, is essentially an expression of P3 in terms of lotteries and, in the case of multi-dimensional outcomes, is more restrictive than monotonicity in outcomes (see Footnote 1). Savage's P6 requires that whenever $f \succ g$, then for any $x \in X$ there is a sufficiently fine finite partition of Ω, say $\{E_i\}_{i=1}^n \subset \Sigma$, such that $xE_if \succ g$ and $f \succ xE_ig$ for every $i = 1...n$.

In other words, to arrive at their representation theorem one need only replace N with P3 and add a stronger form of continuity to our list of conditions.

Comparison with Grant (1995)

The following highlights the limitations of an exchangeability based approach to probabilistic sophistication.

Example 2. Consider the 'mother' example supplied by Grant (1995). If there are only two outcomes in the world of the decision maker - namely, receipt of an indivisible good by Child 1 or by Child 2 - then a plausible representation for the mother's preferences is the utility function $U(p) = p(1-p)$, where p is the probability that Child 1 receives the indivisible good and is subjectively generated by some device deemed by the mother to be uniform. According to the definition of exchangeable events, any event with probability $p \in [0, 0.5]$ is exchangeable with its complement. $\square$

In the example, $\approx$ fails to deliver a notion of likelihood because given three disjoint events, E, E' and A such that $\mu(E) = \mu(E') = 0.4$ and $\mu(A) = 0.2$, the mother's preference behavior leads to the conclusion that $E \approx E'$ while $E \cup A \approx E'$, in violation of Axiom N. Failure of the latter to deliver what is clearly probabilistically sophisticated behavior can be attributed to the highly restricted nature of the outcome space. If the good is divisible, say chocolate, or there is an outcome in which nothing is given to either child, then it will likely no longer be the case that any event is exchangeable with its complement; for instance, if E is a probability 0.6 event, then it is reasonable to suppose that the mother is not indifferent between giving *each* child a piece of chocolate if E is realized and nothing otherwise, versus giving each child a piece of chocolate if the complement of E is realized and nothing otherwise.

As stated, our axioms do not encompass those of Grant (1995) whose approach, in particular, can accommodate Example 2. Grant (1995) weakens P3 to either one of two variants: conditional upper (or lower) eventwise monotonicity (P3^{CU} or P3^{CL}).[13] However, the pre-

[13]These are formally stated in the proof to the next Proposition.

ceding discussion suggests that a sufficiently enriched outcome space may not be subject to the peculiarities of the mother example.[14] Specifically, consider the following result which establishes that in the presence of a 'rich' outcome space, either one of Grant's P3^{CU} and P3^{CL} implies Axiom N.

Proposition 2. *Assume that for every non-null $A \in \Sigma$, $f \in \mathcal{F}$, there exist $x, x' \in X$ such that $xAf \succ f \succ x'Af$. Then either one of Grant's P3^{CU} or P3^{CL} implies Axiom N.*

The condition "for every non-null $A \in \Sigma$, $f \in \mathcal{F}$ there exist $x, x' \in X$ such that $xAf \succ f \succ x'Af$" is a form of non-satiation in outcomes: there is always something sufficiently good (resp. bad) that the decision maker is happy (resp. reluctant) to substitute for the payoff scheme determined by f on A. It can therefore be viewed as a 'richness' assumption on both $\succeq$ and the outcome set, X. Indeed, it is a challenge to find an intuitively behavioral example in a state independent setting where the state space cannot be so 'enriched'.

Under the conditions in Proposition 2, Grant's unique measure representing probabilistic sophistication agrees with $\succeq^C$, and his axioms (taken together) imply both completeness of $\succeq^C$ and Axiom A. In other words, probabilistically sophisticated preferences that satisfy Grant's axioms also satisfy ours *provided that the outcome space is sufficiently rich to ensure that Axiom N is also satisfied.* Thus, in practice, Grant's axioms are more demanding than ours in the sense that they require a form of continuity and monotonicity not needed in Theorem 1, and rule out many probabilistically sophisticated functional forms that are admissible under our axioms.

A Appendix

Proof of Theorem 1: We prove the Theorem in several stages:

[14]To further emphasize the importance of 'enriching' the outcome space, we note that whenever X contains only two outcomes, Σ is atomless, and $\succeq$ can be represented via a continuous and probabilistically sophisticated utility function, Axiom N is satisfied if and only if the representation is monotonic in the sense of P3. We thank I. Gilboa for pointing this out.

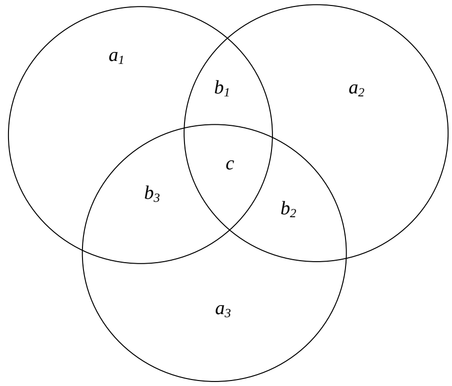

Figure 1: Venn diagram useful in proving Theorem 1.

Stage A. *If E, E' and E'' are pairwise disjoint events, then $E \approx E'$ and $E' \approx E''$ imply $E \approx E''$.*

This is trivial if any of the events are null, so assume otherwise. If $E \not\approx E''$, then without loss of generality, there is some non-null event $e_1 \subset E$ such that $E \setminus e_1 \approx E''$. Lemma 2 implies the existence of a non-null event $e_2 \subset E'$ such that $E' \setminus e_2 \approx E \setminus e_1$. The events e_1 and e_2 are disjoint, so Lemma 5 gives $e_1 \approx e_2$. The fact that $E'' \approx E'$ can be similarly used to establish the existence of a set $e_3 \subset E''$ disjoint from e_1 and e_2 such that $e_3 \approx e_2$. Similarly, $E \setminus e_1 \approx E''$ leads to $e_4 \subset E \setminus e_1$ such that $e_4 \approx e_3$, etc. Clearly this can be continued to construct an infinite sequence of non-null events that are disjoint such that $e_{i+1} \approx e_i$, in violation of Axiom A. $\square$

Stage B. *Let $a_1, a_2, a_3, b_1, b_2, b_3$ be pairwise disjoint events in Σ (see Figure 1). Then $a_1 \cup b_3 \approx a_2 \cup b_2$ and $a_2 \cup b_1 \approx a_3 \cup b_3$ imply $a_1 \cup b_1 \approx a_3 \cup b_2$.*

The idea is to demonstrate the existence of events a_1', a_3', b_1' and b_2' where $a_1' \subseteq a_1, a_3' \subseteq a_3, b_1' \subseteq b_1$, and $b_2' \subseteq b_2$, such that $a_1' \approx a_3'$, $a_1 \setminus a_1' \approx b_2'$, $a_3 \setminus a_3' \approx b_1'$ and $b_1 \setminus b_1' \approx b_2 \setminus b_2'$. This enables one to write, using Lemma 4, $a_1 \cup b_1 = a_1' \cup (a_1 \setminus a_1') \cup b_1' \cup (b_1 \setminus b_1') \approx a_3' \cup b_2' \cup (a_3 \setminus a_3') \cup (b_2 \setminus b_2') = a_3 \cup b_2$, which is the desired result.

Step 1: Lemma 2 implies the existence of $\hat{a}_1 \cup \hat{b}_3 \approx a_2$ and $\check{a}_3 \cup \check{b}_3 \approx a_2$, with $\hat{a}_1 \subseteq a_1, \check{a}_3 \subseteq a_3$, and $\hat{b}_3, \check{b}_3 \subseteq b_3$. Similarly, Lemma 2 also implies the existence of $\hat{a}_2 \approx \hat{b}_3$ and $\check{a}_2 \approx \check{b}_3$, where

$\hat{a}_2, \check{a}_2 \subseteq a_2$. Set $a_2' \equiv a_2 \setminus (\hat{a}_2 \cup \check{a}_2)$ and note that using Lemma 5, $a_2' \subseteq a_2 \setminus \hat{a}_2 \approx \hat{a}_1$ and $a_2' \subseteq a_2 \setminus \check{a}_2 \approx \check{a}_3$. Lemma 2 implies the existence of $a_1' \subseteq a_1$ and $a_3' \subseteq a_3$ such that $a_1' \approx a_2' \approx a_3'$. Stage A gives $a_1' \approx a_3'$.

Step 2: Defining $b_3' \equiv \hat{b}_3 \cup \check{b}_3$ gives $a_1' \cup b_3' \approx a_2 \approx a_3' \cup b_3'$ (using Lemma 4). From $a_1 \cup b_3 \approx a_2 \cup b_2$, $a_3 \cup b_3 \approx a_2 \cup b_1$, and the last step, Lemma 5 implies that $(a_1 \setminus a_1') \cup (b_3 \setminus b_3') \approx b_2$ and $(a_3 \setminus a_3') \cup (b_3 \setminus b_3') \approx b_1$. Lemma 2 implies there are $b_2' \subseteq b_2$ and $b_1' \subseteq b_1$ such that $b_2' \approx a_1 \setminus a_1'$ and $b_1' \approx a_3 \setminus a_3'$. By Lemma 5, $b_1 \setminus b_1' \approx b_3 \setminus b_3'$ and $b_3 \setminus b_3' \approx b_2 \setminus b_2'$, thus Stage A implies that $b_1 \setminus b_1' \approx b_2 \setminus b_2'$. □

Stage C. $\succeq^C$ *is transitive*

Now, given $E, E', E'' \in \Sigma$, suppose that $E \succeq^C E'$ and $E' \succeq^C E''$. Let $e' \subseteq E' \setminus E''$ be a comparison subevent between E' and E'' (i.e., $e' \approx E'' \setminus E'$). Lemma 2 implies there is some $\hat{e} \subseteq E \setminus \big(e' \cup (E' \cap E'')\big)$ such that $\hat{e} \approx \big(e' \cup (E' \cap E'')\big) \setminus E$. We can now apply Stage B as follows. Let the lower circle in Figure 1 correspond to E''. This can be broken up into two pieces: $E'' \setminus E' \equiv a_3 \cup b_3$ and $E'' \cap E' \equiv b_2 \cup c$. Likewise, let e' correspond to $a_2 \cup b_1$, so that $a_2 \cup b_1 \approx a_3 \cup b_3$. Finally, let $a_1 \cup b_3 \equiv \hat{e}$ and set: $\xi = (e' \cup (E' \cap E'')) \cap E$. Diagrammatically, ξ corresponds to $b_1 \cup c$. Note that we identify the left and right circles with *subevents* of E and E', respectively. It follows that: $b_1 = \xi \cap e'$, $a_2 = e' \setminus b_1$, $b_3 = \hat{e} \cap E''$, $a_1 = \hat{e} \setminus b_3$, $b_2 = \Big(\big(e' \cup (E' \cap E'')\big) \setminus E\Big) \cap E''$, and $a_3 = E'' \setminus \big(\hat{e} \cup E'\big)$. Now, $\hat{e} \approx \big(e' \cup (E' \cap E'')\big) \setminus E$ means that $a_1 \cup b_3 \approx a_2 \cup b_2$. Since $a_2 \cup b_1 \approx a_3 \cup b_3$, Stage B implies $a_1 \cup b_1 \approx a_3 \cup b_2$. Moreover, since $E'' \setminus E = a_3 \cup b_2$ and $a_1 \cup b_1 \subseteq E \setminus E''$, by definition $E \succeq^C E''$. □

Stage D. $\succeq^C$ *is a likelihood relation*

Stage C establishes that $\succeq^C$ is a weak order (transitive and complete) over Σ. Condition (ii) in the definition of a likelihood relation is satisfied by $\succeq^C$ due to the presence of non-null events (P5) and Axiom N, while condition (iii) is automatically satisfied by the definition of comparability. □

Stage E. $\succeq^C$ *is either atomless and tight or purely and uniformly atomic.*

Assume first that Σ contains an atom, a, and denote a^c as its relative complement in Ω. Note that for any $e \in \Sigma$ it cannot be that $a \succ^C e$ since a cannot be partitioned into two or more non-null events. Thus $a^c \succeq^C a$. If $a \approx a^c$ then, by Axiom N, a^c must also be an atom and Σ therefore consists of two exchangeable atoms. Suppose instead that $a^c \not\approx a$. Then there is some event $a_1 \subset a^c$ with $a_1 \approx a$ and $a^c \setminus a_1$ not null. By Axiom N, a_1 must be an atom in Σ, and by Axiom C, $a^c \setminus a_1 \succeq^C a$. In turn this implies the presence of another atom $a_2 \approx a$ in $a^c \setminus a_1$ with a, a_1 and a_2 disjoint and pairwise exchangeable. According to Axiom A, this can be continued at most a finite number of times, proving that the set of non-null events in Σ is finite. Transitivity of $\approx$ (Stage A) implies that all atoms are pairwise exchangeable.

Assume now that Σ is atomless. To demonstrate tightness (see Footnote 7), consider that $E \succ^C E'$ implies there is some $e \subset E \backslash E'$ such that $e \approx E' \backslash E$ and $E \backslash (e \cup E')$ is not null. Since Σ is atomless, $E \setminus (e \cup E')$ can be split into two disjoint non-null events, ξ_1 and ξ_2, both in Σ, subsets of E and disjoint from $e \cup E'$. By Axiom N, no subevent of $E' \backslash E$ is exchangeable with $e \cup \xi_2$. Thus Axiom C implies that $E = e \cup \xi_2 \cup (E \cap E') \cup \xi_1 \succ^C (E' \backslash E) \cup (E \cap E') \cup \xi_1 = E' \cup \xi_1$ where $\xi_1 \cap E' = \emptyset$. A similar argument implies $E \setminus \xi_1 \succ^C E'$, implying $\succeq^C$ is tight. □

Stage F. *If $\succeq^C$ is atomless, then it is fine.*

To show this, for any $E \in \Sigma$ we construct a finite partition of Ω at least as fine as E, $\{e_i\}$, starting with $e_1 \equiv E$. Next, Axiom C implies that either $E \succeq^C E^c$ or $E^c \succeq^C E$. In the former case, let $e_2 \equiv E^c$ and $\{e_1, e_2\}$ forms a partition containing events at least as fine as E. In the latter case, define e_2 as the comparison subevent of E^c that, by definition, is exchangeable with E. Once again, Axiom C implies that either $E \succeq^C (E \cup e_2)^c$ or $(E \cup e_2)^c \succeq^C E$, and we can continue constructing events exchangeable with E and disjoint from each other in the obvious way. By Axiom A this construction must be finite and constitutes a partition of Ω consisting of events at least as fine as E. Thus $\succeq^C$ is fine. □

Stage G. *Conclusion*

In either the atomic or the fine and tight case, there exists a unique finitely additive probability measure, μ, that agrees with $\succeq^C$ (see Wakker, 1981). μ is therefore solvable;

moreover, it is a countably additive convex-ranged measure if Σ is a σ-algebra (as in Savage's original treatment). Finally, whenever the measure of two events, E, E', coincides, it must be that $E \succeq^C E'$ and $E' \succeq^C E$; in turn, Lemma 2 implies that $E \approx E'$.

To prove that the decision maker is indifferent between all acts inducing the same distribution one can use the arguments in steps 4 and 5 in the proof of Theorem 1, Machina and Schmeidler (1992), or step 5 in the proof of Theorem 1, Grant (1995). Proving necessity of Axioms C and A is trivial; necessity of Axiom N follows from Lemma 1. □

Proof of Proposition 1: Assume $E \succeq^C E'$. For any $x^*, x_* \in X$ with $x^* \succ x_*$ and $f \in \mathcal{F}$, write $x^* E x_* E' f = x^* \xi \cup \xi' x_* E' f$, where $\xi \cup \xi' = E$ and $\xi' \approx E'$. By definition of $\approx$, we have that $x^* \xi \cup \xi' x_* E' f \sim x^* \xi \cup E' x_* \xi' f$. By P3, the latter dominates $x^* E' x_* \xi \cup \xi' f = x^* E' x_* E f$. Summarizing: $x^* E x_* E' f \succeq x^* E' x_* E f$.

If $E' \succ^C E$, then E' contains a non-null subevent e' such that $E' \setminus e' \approx E$. Using P3: $E' \succ^C E \;\Rightarrow\; x^* E' x_* E f \succ x^* E x_* E' f$. Axiom C and the contrapositive of the latter gives $x^* E x_* E' f \succeq x^* E' x_* E f \;\Rightarrow E \succeq^C E'$. □

Proof of Proposition 2: Grant's axioms state that for any $x, y \in X$, $h \in \mathcal{F}$ and disjoint non-null $E, E' \in \Sigma$,

P3^{CU}: $\quad x(E \cup E')f \succ y(E \cup E')f \;\Rightarrow\; xEyE'f \succ y(E \cup E')f$

P3^{CL}: $\quad x(E \cup E')f \succ y(E \cup E')f \;\Rightarrow\; x(E \cup E')f \succ xEyE'f$

We first establish, under the hypothesis, Property †: for any disjoint $E, E', A \in \Sigma$, if $x(E \cup A)x'E'f \sim xEx'(E' \cup A)f$ for every $x, x' \in X$ and $f \in \mathcal{F}$ then A is null. Specializing to $f = x'$, this becomes $x(E \cup A)x' \sim xEx'$ for every $x, x' \in X$. Note that, under the hypothesis of the Proposition, when E is null A too must be null. Assuming E is not null, for each $z \in X$ there are $y, y' \in X$ such that $z \succ y(E \cup A)z$ and $y'(E \cup A)z \succ z$. If P3^{CU} is satisfied and A is not null, it must be that $zAyEz = yEz \succ y(E \cup A)z$, a contradiction of $x(E \cup A)x' \sim xEx'$ for every $x, x' \in X$. On the other hand, if P3^{CL} is satisfied and A is not null, it must be that $y'(E \cup A)z \succ y'Ez = zAy'Ez$, also a contradiction. Thus A is null.

We now demonstrate that Property † implies Axiom N. Suppose $E, A, E' \in \Sigma$ are pairwise disjoint, such that $E \approx E'$, and A is non-null. Let ξ' be a subevent of E' that is exchangeable with $E \cup A$. By exchanging ξ' for $E \cup A$, we have for any $x, x' \in X$ and $f \in \mathcal{F}$ that, $x'(E \cup A)xE'f \sim x'\xi'x\big((E \cup A) \cup (E' \setminus \xi')\big)f = x'\xi'x\big(E \cup (A \cup E' \setminus \xi')\big)f$. Similarly, by exchanging E with E' it follows that $x'(E \cup A)xE'f \sim x'(E' \cup A)xEf = x'\big(\xi' \cup (A \cup E' \setminus \xi')\big)xEf$. Property † implies $A \cup E' \setminus \xi'$ is null, contradicting the fact that A is not null. Avoiding the contradiction requires that no subevent of E' is exchangeable with $E \cup A$. $\square$

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