Event Exchangeability:

Small Worlds Probabilistic Sophistication without Continuity or Monotonicity*

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Abstract

Building on the Ramsey-de Finetti idea of event exchangeability, we derive a parsimonious and novel characterization of global probabilistic sophistication on an algebra of events without requiring monotonicity or continuity. An exchangeability-based approach also enables us, under similarly weak behavioral conditions, to characterize probabilistic sophistication without monotonicity or continuity over smaller domains of events, or 'small worlds'. This offers a perspective to modeling attitudes towards different sources of uncertainty encompassing and extending the distinction between risk and 'ambiguity' often associated with Ellsberg-type behavior.

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1. Introduction

In their pioneering studies, Ramsey (1926) and de Finetti (1937) originated the idea of distinguishing events according to whether they are 'exchangeable' or 'ethically neutral', providing the basis for their construction of a decision maker's subjective probability over events.[1] Savage's (1954) subsequent formulation departs from this direction and nevertheless yields an overall subjective probability on the decision maker's 'big world' consisting of all exhaustive and mutually exclusive contingencies. Building on Savage's approach, Machina and Schmeidler (1992) and subsequently Grant (1995) provide more parsimonious characterizations of what is termed probabilistic sophistication, in which the choice behavior of a decision maker reflects her probabilistic belief in the sense that events are distinguished only by their subjective probabilities.

Say that two events are *exchangeable* if the decision maker is always indifferent to permuting their payoffs. Building on exchangeability as the primitive, we develop a notion of comparability to capture the intuition behind a likelihood relation among events. Specifically, two disjoint events are *comparable* when one contains a subevent that is exchangeable with the other. Informally, one is motivated to view one event as 'larger' or 'more likely' than the other. When all disjoint events are comparable in this way, we show that very weak conditions - far weaker than Savage's assumptions of eventwise monotonicity (P3), comparative likelihood (P4), and continuity (P6) - suffice to deliver probabilistic sophistication on the part of the decision maker.

Our approach to probabilistic sophistication is also suited to dealing with the case in which not all events are comparable via a likelihood relation (as exemplified in Ellsberg's (1961) influential work). In particular, we can provide answers to the following natural questions: What are appropriate subjective conditions that characterize self-contained *domains* of comparable events, such that the decision maker's preference over acts restricted to any one domain exhibits probabilistic sophistication? Second, given two comparable events, can one constructively tell whether both events are contained in a single domain? Our exchangeability-based view on these issues extends Savage's idea of 'small worlds' to

[1] de Finetti is associated with a number of results that have been described in terms of exchangeability. We refer specifically to his thinking on "Exchangeable Events" (Chapter 3) and his "Reflections on the Notion of Exchangeability" (Chapter 5) in Kyburg's translation of de Finetti (1937). Ramsey focused on the partition of the state space into two 'exchangeable' events by what he called 'ethically neutral' events.

situations where not all pairs of events are comparable, admitting the possibility of endogenously distinguishing sources of uncertainty and the events they encompass through event noncomparability.[2] The approach can also be used to shed more light on the distinction between subjectively ambiguous and unambiguous events.

We illustrate these ideas with an example. Consider a single draw from a three-color urn containing 50 red balls, numbered from 1 to 50, plus another 50 balls numbered from 51 to 100, each of which can be either blue or green. Refer to the event of drawing a ball of a certain color by the first letter of the color drawn, and let the union of color events be represented by the conjunction of the appropriate letters (e.g., GB is the event 'Green or Blue ball drawn'). Furthermore, denote by $[k; n]$ the event consisting of drawing a ball whose number is between k and n. It seems reasonable to require that any two events involving an equal number of balls are exchangeable, and that G is exchangeable with B.

We first consider the case where all events are comparable, and outline how under weak conditions this leads to global probabilistic sophistication. Assuming comparability between all events implies, in particular, that G should be exchangeable with $[1; n]$ for some $n \leq 50$; if one assumes that $n \leq 25$, this further suggests that B is exchangeable with $[n + 1; 2n]$.[3] By the definition of exchangeability, the union of G and B (i.e., GB) must be exchangeable with the union of $[1, n]$ and $[n + 1; 2n]$, so that for any pair of consequences, x and x':

$$\begin{pmatrix} x & \text{in event } R \\ x' & \text{otherwise} \end{pmatrix} = \begin{pmatrix} x & \text{in event } [1; 2n] \\ x & \text{in event } [2n+1; 50] \\ x' & \text{otherwise} \end{pmatrix} \sim \begin{pmatrix} x' & \text{in event } [1; 2n] \\ x & \text{in event } [2n+1; 50] \\ x & \text{otherwise} \end{pmatrix}$$

where $\sim$ denotes indifference and we have used the exchangeability of GB with $[1; 2n]$ to permute x and x'. Since R and its complement (each having 50 balls) are exchangeable,

[2] Savage (1954, pages 9, 27, 81) discusses how decision making tends to take place in a 'small world' - events relevant to a particular decision situation that partition the state space. Our notion of a small world admits partitions of events smaller than the whole space (i.e., 'conditional small worlds'). Moreover, whereas Savage intended events to be comparable across small worlds, we explicitly incorporate the possibility that this is not the case. In this paper, we shall be using the terms 'domain', 'small world', and 'source of uncertainty' interchangeably.

[3] The example is a bit more complicated if $n > 25$ and is treated in greater generality in Section 2.3.1.

it must also be true that

$$\begin{pmatrix} x' & \text{in event } [1;2n] \\ x' & \text{in event } [2n+1;50] \\ x & \text{otherwise} \end{pmatrix} = \begin{pmatrix} x' & \text{in event } R \\ x & \text{otherwise} \end{pmatrix} \sim \begin{pmatrix} x & \text{in event } R \\ x' & \text{otherwise} \end{pmatrix}$$

Combining the last two results, we are led to conclude that

$$\begin{pmatrix} x' & \text{in event } [1;2n] \\ x' & \text{in event } [2n+1;50] \\ x & \text{otherwise} \end{pmatrix} \sim \begin{pmatrix} x' & \text{in event } [1;2n] \\ x & \text{in event } [2n+1;50] \\ x & \text{otherwise} \end{pmatrix} \tag{1.1}$$

Since this is true for *any* two consequences, x and x', the decision maker appears to be insensitive to the type of consequence received in the event $[2n+1;50]$, suggesting that she deems the event to be null (i.e., one that is exchangeable with the empty set). The only sensible $n \leq 25$ that renders $[2n+1;50]$ null is $n = 25$, meaning that the decision maker views $\{[1;25],[26;50],G,B\}$ as a partition of mutually exchangeable events. The decision maker's behavior is consistent with a choice procedure that uniquely assigns equal probability to $[1;25],[26;50],G$ and B.

The two crucial steps in the urn example above involved (i) the assumption that the comparability relation is *complete* in that any two events are comparable, and (ii) that the relation in (1.1), when true for all pairs of consequences, implies that $[2n+1;50]$ is null. We generalize the second property, which we term *'Event Non-satiation'*, to require that if two disjoint events, E and E', are exchangeable (e.g., $[1;2n]$ and GB) then for any A that is mutually disjoint with E and E' (e.g., $[2n+1;50]$), it cannot be the case that $A \cup E$ is exchangeable with a subevent of E' unless A is null. By adding to completeness and Event Non-satiation one more condition taken for granted in the example – that any sequence of exchangeable non-null events must be finite – we are able to deliver proof of global probabilistic sophistication for any algebraic system of events.

It should be emphasized that pairwise comparability between all events is not an innocuous condition. Experimental evidence (e.g., Ellsberg, 1961) sugggests that no subset of R is exchangeable with G or B. Aside from ruling out Ellsbergian behavior, 'completeness' with respect to comparability also excludes state dependence: for instance, while one future temperature interval may be exchangeable with another when payoffs are monetary, this may not be the case if the payoff is, say, a picnic. Thus, in principle, one might wish to have a notion of exchangeability that depends on the specification of the set of payoffs. We briefly elaborate on this later in the paper.

Earlier, we posed two questions in the context of deviation from global probabilistic sophistication. Of the assumptions leading to the latter the strongest appears to be completeness of the event comparability relation. Our exchangeability-based view offers a direction to answering the questions posed. Specifically, we relax the completeness assumption and define a *domain* as a suitably maximal collection of comparable events that is self-contained in the following key senses: (i) any pair of domain events are comparable; (ii) the union of all domain elements is in the domain; (iii) if E is the 'more likely' of two comparable events in the domain, then both the 'copy' of the other event within E, as well as the remainder, are in the domain; and (iv) the collection resulting from adding any other collection of non-null event to the domain is not compatible with (i)-(iii). The first three conditions require the domain to be self contained, while the last requires it to be 'maximal'. This description of a domain renders it a λ-system whose universal set may be a proper subset of the full state space.[4] Domains of events can be viewed as preference induced small worlds, or subjectively distinct sources of uncertainty. Here too, weak behavioral assumptions lead to source specific probabilistic sophistication.[5] Moreover, the definition can be used to determine, in principle, whether two given events can belong to a single domain without prior knowledge of what that domain is.

For instance, given Ellsbergian attitudes in our three-color urn example, G and B cannot belong to the same domain as R, or alternatively, the decision maker views R as associated with a source of uncertainty distinct from what characterizes B or G. Likewise, the fact that B and G are exchangeable guarantees that there exists some domain that includes both, leading to the interpretation that the two events belong to the same subjective small world. This can also serve as a basis for identifying those events that appear to be subjectively unambiguous.

The remainder of the paper is organized as follows. Section 2 introduces preliminary

[4] A λ-system, defined in Section 3.1, is a collection of events not necessarily closed under intersections, yet suitable for a definition of a probability measure.

[5] While here we characterize the case in which a domain is an algebra, the approach lends itself nicely to a more general characterization in which domains are λ-systems. We pursue this in a separate paper.

notions including formal definitions of event exchangeability and comparability, presents our main result concerning global probabilistic sophistication, and relates our result to the existing literature. Section 3 defines domains and discusses brief applications to small worlds probabilistic sophistication, subjectively unambiguous events, and 'two-stage' utility.

2. Global Probabilistic Sophistication without Monotonicity or Continuity

2.1. Exchangeability and Comparability

Let Ω be a space whose elements correspond to all states of the world. Let X be a set of payoffs and Σ an algebra on Ω. Elements of Σ are events. If $e, E \in \Sigma$ and $e \subseteq E$, then we say that e is a *subevent* of E. The set, $\mathcal{F}$, of simple acts comprises all Σ-adapted and X-valued functions over Ω that have a finite range. As is customary, $x \in X$ is identified with the constant act that pays x in every state. Throughout the paper we assume that the decision maker has a complete, transitive and non-degenerate binary preference relation, $\succeq$, on $\mathcal{F}$ as in Savage's P1 and P5.[6]

For any collection of pairwise disjoint events, $E_1, E_2, ..., E_n \subset \Omega$, and $f_1, f_2, ..., f_n, g \in \mathcal{F}$, let $f_1E_1f_2E_2...f_nE_ng$ denote the act that pays $f_i(\omega)$ if the true state, $\omega \in \Omega$, is in E_i, and pays $g(\omega)$ otherwise. We say that $E \in \Sigma$ is *null* if $fEh \sim gEh\ \forall f, g, h \in \mathcal{F}$.[7]

To capture the sense in which events are similar, we introduce a binary relation over events via $\succeq$:

Definition 1 (Event Exchangeability). *For any pair of disjoint events $E, E' \in \Sigma$, $E \approx E'$ if for any $x, x' \in X$ and $f \in \mathcal{F}$, $xEx'E'f \sim x'ExE'f$.*

Whenever $E \approx E'$ we will say that E and E' are *exchangeable.* Note that all null events are exchangeable. Exchangeability may be viewed as a pre-notion of 'equally likely': two events are 'equally likely' if the decision maker is indifferent to a permutation of their

[6]As usual, $\succ$ (resp. $\sim$) is the asymmetric (resp. symmetric) part of $\succeq$. Under Savage's P1, $\succeq$ is a weak (complete and transitive) order on $\mathcal{F}$, while P5 asserts that there exists $f, g \in \mathcal{F}$ such that $f \succ g$.

[7] The intuition underlying this definition may not be robust relative to the presence of 'extreme' forms of state dependence. For instance, in Aumann's example of a man who will lose his taste for life should his sick wife die, the event 'sick wife dies' will also be classified here as null. For further discussion of this issue see Karni (2003) and references therein.

payoffs. Without further structure this interpretation is not formally justified since $\approx$ is not necessarily transitive, and therefore not an equivalence relation. The next example, to which we will refer again in the context of defining subjectively unambiguous events, demonstrates this.

Example 1. A randomly thrown dart can land in one of four rectangular regions: A, B_1, B_2, and C. The following information is available: the regions represented by A and C are each at least as large as either of the regions B_1 or B_2; moreover, A is larger than C if and only if B_1 is larger than B_2. Let Ω be $[0,1] \times \{A, B_1, B_2, C\}$, so that each of A, B_1, B_2, and C can be split into arbitrarily fine partitions.[8] Furthermore, let $X \equiv [-1,1]$ and let $E_A[f]$ be the expected value (under the uniform measure) of the act f conditional on the dart falling in region A (and similar for B_1, B_2, and C). Consider the utility representation that assigns the act f a utility of

$$V(f) = \Big(\frac{E_A[f] + E_C[f] + \frac{E_{B_1}[f]+E_{B_2}[f]}{2}}{3}\Big) \min\Big\{1, 1 - \frac{(E_A[f] - E_C[f])(E_{B_1}[f] - E_{B_2}[f])}{8}\Big\}$$

This can be interpreted as follows: The decision maker first calculates the expected value of an act by assigning each of A and C a probability of $\frac{1}{3}$, and a probability of $\frac{1}{6}$ to each of B_1 and B_2; she then assesses utility by adjusting this calculated expected value with a 'hedging premium' that gives preference to acts in which a higher (resp. lower) average payoff on A versus C is balanced by a higher (resp. lower) average payoff on B_2 versus B_1. It is straight forward to check that the representation is monotonic. It should also be clear that $A \approx B_1 \cup B_2$ and $C \approx B_1 \cup B_2$. On the other hand, it is certainly not the case that $A \approx C$ due to the 'preference for hedging' (i.e., the asymmetry between payoffs on A and C arising in the multiplicative term of the utility function). ∎

Intuitively, an event is 'at-least-as-likely' as any of its subevents. Exchangeability supplies the motivation underlying a similar comparison across disjoint events, $E, E' \in \Sigma$: if a subevent of E is exchangeable with E', then it is also natural to view E as 'at-least-as-likely' as E'. Building on this, we define the following exchangeability based relation between any two events.

[8]E.g., one can speak of the dart landing in the top third of region A. We identify the event $[0,1] \times A$ with A, and similar for the other regions. Σ is generated by the product of the Borel sets of $[0,1]$ with the power set of $\{A, B_1, B_2, C\}$.

Definition 2 (Event Comparability). *For any events, $E, E' \in \Sigma$, $E \succeq^C E'$ whenever $E \setminus E'$ contains a subevent, e, that is exchangeable with $E' \setminus E$. Moreover, e is referred to as a* **comparison event**.

Just as $\approx$ gives a pre-notion of 'equal likelihood' among events, $\succeq^C$ provides a pre-notion of an 'at-least-as-likely' relation. The event E is 'at least as likely' as E' if outside their intersection the 'more likely' event (i.e., $E \setminus E'$) contains a 'copy' (i.e., the comparison event) of the 'less likely' event (i.e., $E' \setminus E$). Since $\emptyset$ is a subevent of any event and $\emptyset$ is exchangeable with itself, $e \subseteq E$ implies $E \succeq^C e$.

For any $E, E' \in \Sigma$, we say that E and E' are *comparable* whenever $E \succeq^C E'$ or $E' \succeq^C E$. Finally, define $E \succ^C E'$ whenever $E \succeq^C E'$ and it is not the case that $E' \succeq^C E$. Likewise, define $\sim^C$ as the symmetric part of $\succeq^C$.

We also need the following definitions:

Definition 3. *$\succeq^C$ is a* ***likelihood relation*** *over Σ if the following conditions hold:*

i) $\succeq^C$ is a weak order over Σ

ii) $\Omega \succ^C \emptyset$ and for every $A \in \Sigma$, $A \succeq^C \emptyset$ and $\Omega \succeq^C A$

iii) for every $A, B, C \in \Sigma$ such that $C \cap (A \cup B) = \emptyset$, $A \succeq^C B \;\Leftrightarrow\; A \cup C \succeq^C B \cup C$

Note that the second requirement is satisfied by $\succeq^C$ by virtue of the non-triviality of $\succeq$, while the last requirement is satisfied by the definition of $\succeq^C$. Thus establishing that $\succeq^C$ is a likelihood relation reduces to demonstrating that condition (i) holds.

Definition 4. *μ is an* ***agreeing probability measure*** *for $\succeq^C$ over Σ, if it is a probability measure over Σ and for every $A, B \in \mathcal{A}$, $A \succeq^C (\succ^C) B \;\Leftrightarrow\; \mu(A) \geq (>) \mu(B)$.*

For any probability measure, μ on Σ and act $f \in \mathcal{F}$, refer to $\{\Big(x, \mu\big(f^{-1}(x)\big)\Big) \mid x \in X\}$ as the lottery induced by the act, $f \in \mathcal{F}$ with respect to μ.

2.2. Axioms and Main Result

Given a non-null event, e_0, consider asking a decision maker to identify a disjoint event (say e_1) that is exchangeable with e_0, then find another event (say e_2) disjoint from $e_0 \cup e_1$ and exchangeable with e_1, then find another event (say e_3) disjoint from $e_0 \cup e_1 \cup e_2$ and exchangeable with e_2, and so on; then the following 'Archimedean' condition asserts that this procedure must end after a finite number of steps:

Axiom A (Event Archimedean Property). *Any sequence of pairwise disjoint and non-null events, $\{e_0, e_1, e_2, \ldots\} \subset \Sigma$, such that $e_i \approx e_{i+1}$ for every $i = 0, ...,$ is finite.*

Suppose that the decision maker behaves as if she assigns a unique probability measure to each event and the measure of an event is its only relevant characteristic for the purpose of her decision making. Clearly, if two events are equally likely then their set differences are also equally likely and thus exchangeable. If Σ is sufficiently 'fine' any event will contain a subevent with arbitrary yet smaller likelihood, and therefore any two events in the decision maker's world are comparable. When Σ is free of atoms, the latter appears to be a fundamental attribute of global probabilistic sophistication in the absence of state dependence.[9] The next assumption asserts this by requiring completeness of $\succeq^C$.

Axiom C (Completeness of $\succeq^C$). *Given any disjoint pair of events, one of the two contains a subevent exchangeable with the other.*

On its own, completeness of $\succeq^C$ is not sufficient for the existence of a likelihood relation, let alone a unique agreeing probability measure. As we discuss later, Savage's Eventwise Monotonicity (P3) in conjunction with Axiom A is sufficient to ensure the existence of a unique agreeing probability measure that coincides with $\succeq^C$ when it is complete over Σ.[10] Grant (1995), however, convincingly argues that an insistence on Savage's P3 may descriptively exclude some decision makers that rely on subjective probabilities for decision making. In particular, he cites an example of a mother that strictly prefers tossing a coin to determine how an indivisible treat is to be distributed among her two children. Another example noted by Grant (1995) is that of induced preferences, which are quasi-convex. Examples of probabilistically sophisticated preferences abound that violate both P3 and P4 (as well as both of Grant's (1995) weakening of P3 and Machina and Schmeidler's (1992) P4*).[11] In addition to questioning the *universal* appeal

[9] An atom is an event that cannot be partitioned into two or more non-null subevents.

[10] Savage's P3 states that for any non-null event, $E \subseteq \Omega$, act $f \in \mathcal{F}$ and any $x, y \in X$, $x \succeq y \Leftrightarrow xEf \succeq yEf$.

[11] Savage's P4 states that for any events $E, E' \in \Sigma$ and $x^*, x_*, y^*, y_* \in X$ with $x^* \succ x_*$, $y^* \succ y_*$, $x^*Ex_* \succeq x^*E'x_*$ implies $y^*Ey_* \succeq y^*E'y_*$. Machina and Schmeidler's (1992) more restrictive P4* requires that for any $f, g \in \mathcal{F}$ and whenever $E \cap E' = \emptyset$, $x^*Ex_*E'f \succeq x^*E'x_*Ef$ implies $y^*Ey_*E'g \succeq y^*E'y_*Eg$.

of eventwise monotonicity, it seems desirable that the basic set of assumptions be as parsimonious as possible if one is to develop a better understanding of deviations from global probabilistic sophistication.

The example in the Introduction motivates the following condition, which appears much weaker than P3:

Axiom N (Event Non-satiation). *For any pairwise disjoint $E, A, E' \in \Sigma$, if $E \approx E'$ and A is non-null, then no subevent of E' is exchangeable with $E \cup A$.*

Axiom N is equivalent to requiring that whenever two events are exchangeable, adding a disjoint non-null event to one of them makes the combined event strictly more 'likely' (i.e., $E \cup A \succ^C E'$). Axiom N is also sufficient (though not necessary) for another intuitively sensible implication:

$$E' \sim^C E \;\Leftrightarrow\; E \approx E'.$$

How 'minimal' is Axiom N? The next result establishes that it is necessary for any exchangeability based likelihood relation in which non-null sets are strictly more likely than the empty set. Thus to the extent that the latter is desirable, Axiom N is a minimal requirement for any theory of probabilistic sophistication in which exchangeable events are equally likely.

Lemma 1. *Assume that $\succeq^\circ$ is a likelihood relation over Σ with (i) a symmetric part that agrees with $\approx$ on disjoint sets, and (ii) $A \succ^\circ \emptyset$ for all non-null $A \in \Sigma$. Then for any pairwise disjoint $E, E', A \in \Sigma$ such that A is not null, $E \approx E'$ implies that $E \cup A \succ^\circ E'$.*

Proof: Assume that $E, E', A \in \Sigma$ are pairwise disjoint, A is not null, and $E \approx E'$ (meaning that $E \sim^\circ E'$). Note that $A \succeq^\circ \emptyset \;\Leftrightarrow\; E \cup A \succeq^\circ E$. Transitivity of $\succeq^\circ$ implies that $E \cup A \succeq^\circ E'$. If $E \cup A \sim^\circ E'$ then $E \cup A \sim^\circ E$. In particular, the cancellation property (iii) of a likelihood relation means that $A \sim^\circ \emptyset$ – a contradiction. Thus $E \cup A \succ^\circ E'$. ∎

Our main result delivers exchangeability-based global probabilistic sophistication as necessary and sufficient for Axioms A, C and N.

Theorem 1. *Axioms A, C and N are satisfied if and only if there exists a unique finitely additive probability measure, μ, that is either atomless or purely and uniformly atomic, where μ represents $\succeq^C$ on Σ such that any two events with the same measure are exchangeable, and the decision maker is indifferent between any two acts that induce the same lottery with respect to μ.*

We emphasize that for probabilistic sophistication we do not require Σ to be a σ-algebra, thus μ need not be convex-valued in the atomless case. In particular, when applied to algebraic structures our approach is more parsimonious than that of Kopylov (2004) who derives probabilistic sophistication on event domains requiring less structure than σ-algebras.[12]

2.3. Discussion

We now turn to a comprehensive discussion of Theorem 1. We begin by examining the intuition behind the derivation. We then compare our axioms with their counterparts in the literature. Finally, we discuss how the choice of an outcome space can impact the results in the sense that 'state independence' is implicitly present in our framework.

2.3.1. Intuition

To highlight the important steps in the proof, we return momentarily to the three color urn example of the Introduction. If all events are comparable (Axiom C), then G is exchangeable with some subevent of R, say R_G. Given that $R \approx GB$, one would like to conclude that subtracting the 'equally sized' pieces, R_G and G, from R and GB, respectively, results in $R \setminus R_G \approx B$. Lemma A.1 in the Appendix demonstrates that this is indeed a simple consequence of Axiom N. We therefore have that $R_G \approx G \approx B \approx R \setminus R_G$. If $\approx$ is transitive, then one may be justified in identifying R_G with $[1; 25]$, so that $\{[1; 25], [26; 50], G, B\}$ can be viewed as a uniform partition of Ω associated with a unique probability measure. It is apparent from this thinking that transitivity of $\approx$, or alternatively $\succeq^C$, is key to establishing global probabilistic sophistication.

More generally, to derive probabilistic sophistication it is sufficient to prove that $\succeq^C$ is a likelihood relation that can be represented by a unique finitely additive measure, μ.

[12] An important distinction is that Kopylov's results extend to non-algebraic structures.

The equivalence between $\sim^C$ and $\approx$ for two disjoint events can then be used to show that the decision maker is indifferent between any two acts that induce the same lottery via μ. The non-trivial steps involve demonstrating that $\succeq^C$ is transitive and that Σ either consists of finitely many equal mass atoms or is fine and tight – both cases known to be associated with a unique representing measure.[13]

To get a better sense of how Axioms A, C and N imply transitivity of $\succeq^C$, consider $E \succeq^C E' \succeq^C E''$, and assume for simplicity that $E, E', E'' \in \Sigma$ are pairwise disjoint. The general idea is to establish that if $E'' \succ^C E$, then one can construct an infinite sequence of non-null pairwise disjoint events in violation of Axiom A. To see how this is done, we first note that Axiom N and the assumed completeness of $\succeq^C$ imply that whenever $E'' \succeq^C E$, then for any subevent $e \subset E$ there exists $e'' \subset E''$ such that $e'' \approx e$ and $E'' \setminus e'' \succeq^C E \setminus e$. Essentially, comparability and event non-satiation enable one to 'cleave' equally sized pieces from E'' and E, while maintaining the ordering between the residual events.

Now, consider that $E'' \succ^C E$ implies one can find a non-null subevent of E'', say e_1, such that $E'' \setminus e_1 \approx E$. Thus after 'cutting' e_1 from E'' one arrives at $E'' \setminus e_1 \succeq^C E \succeq^C E'$. Next, since $E' \succeq E''$, one can cleave a piece, say e_2, from E' such that $e_2 \approx e_1$, and giving $E' \setminus e_2 \succeq^C E'' \setminus e_1 \succeq^C E$. This can be continued (e.g., cleave $e_3 \subset E$ such that $e_3 \approx e_2$, etc.) and yields the infinite sequence of non-null events. The contradiction with Axiom A forces $E \succeq^C E''$ and the desired transitivity of $\succeq^C$. The actual proof, found in the Appendix, makes use of such a construction, albeit in the more involved case where E, E' and E'' are not pairwise disjoint.

If Σ is atomless, tightness follows easily from Axiom N; fine-ness relative to a non-null event, E, can be established by making use of Axiom N to 'cleave' pairwise disjoint and equally sized pieces from $\Omega \setminus E$, which by Axiom A can only be done a finite number of times before one ends up with a 'remainder' event that is smaller than E. Clearly, this construction leads to a finite partition whose elements are no more likely than E. The fact that $\succeq^C$ is a fine and tight likelihood relation can then be used to deduce the unique existence of an agreeing probability measure.[14]

[13] A likelihood relation is fine if it contains no atoms and it is tight if for any event, E, there exists a partition of Σ where no parition element is strictly more likely than E. The relation is tight whenever $E \succ^C E'$, there are $A, B \in \Sigma$ where $A \cap E' = \emptyset$ and $B \subset E$ such that $E \succ^C A \cup E'$ and $E \setminus B \succ^C E'$.

[14] See Wakker (1981).

If Σ contains an atom, then completeness requires that every other event contain a subevent that is exchangeable with the atom. Thus, in particular, completeness of $\succeq^C$ implies that atoms must come in only one 'size', and that one can partition the state space with a set of such atoms.[15] In turn, Axiom A implies that any such partition is finite. While the result in the atomic case is 'trivial' and of limited interest, it does shed some light on a limitation of our approach: while Axiom C may be sensible when Σ is atomless, it is far from inoccuous otherwise. Interesting cases involving atoms require a relaxation or at least a re-examination of the structure imposed. One possible direction is to supplement Definition 2 with the statement that E is also deemed comparable to E' whenever $E \setminus E'$ is exchangeable with an event that contains $E' \setminus E$. While this may be useful in exploring cases of unequal atoms, changing Definition 2 comes at a non-trivial cost in additional structure if one is to obtain probabilistic sophistication. Moreover, additional assumptions will be required to pin down a unique representing measure for $\succeq^C$ when it is atomic.[16]

2.3.2. Relation to the Literature

In relating our approach to prior literature on global probabilistic sophistication we focus primarily on Machina and Schmeidler (1992) and Grant (1995).[17]

Comparison with Machina and Schmeidler (1992)

Machina and Schmeidler (1992) show that the existence of a continuous probabilistically sophisticated utility representation of $\succeq$ agreeing with first degree stochastic dominance is equivalent to P1, P3, P4*, P5 and P6.[18] This result delivers a unique convex valued probability measure where the measures of two events coincide if and only if the events are exchangeable. Thus their axioms imply that $\succeq^C$ is complete (i.e., all events are mutually comparable) as well as Axiom A. It is also easy to show that monotonicity (i.e.,

[15] Details can be found in the Proof of Theorem 1.

[16] This issue is not unique to our work – the majority of papers in this literature tend to focus on atomless state spaces and those that do not require considerably more structure than we do; see Wakker (1984), Chateauneuf (1985), Nakamura (1990), Gul (1992), Chew and Karni (1994), and Kobberling and Wakker (2003).

[17] Other related works include Sarin and Wakker (2000).

[18] See Footnotes 10 for a definition of P3, and Footnote 11 for definitions of P4 and its strengthened form, P4*. Savage's P6 requires that whenever $f \succ g$, then for any $x \in X$ there is a sufficiently fine finite partition of Ω, say $\{E_i\}_{i=1}^n \subset \Sigma$, such that $xE_if \succ g$ and $f \succ xE_ig$ for every $i = 1...n$.

P3) implies weak event non-satiation. One can therefore interpret that, in establishing global probabilistic sophistication in Theorem 1, we weaken the Machina and Schmeidler axioms as follows: P3 $\rightarrow$ Axiom N, P4* and P6 $\rightarrow$ Axioms A and C. Indeed, in light of the preceding discussion, it seems that completeness, together with Axiom A, endows the state space with a uniform character reminiscent of the role typically played by P6. The latter, however, is much stronger given that it leads to a continuous representation, whereas continuity is not required in our case. As an example, consider $\Omega = [0, 1]$ with Σ the usual Borel σ-algebra. Let $X = [0, 1]$ and assume that the decision maker ranks any simple act according to the expected value of the lottery it induces (with respect to the uniform measure on $[0, 1]$), and if two lotteries have the same mean the decision maker prefers the one with smaller variance. It should be clear that the decision maker's preferences are lexicographic and therefore not continuous. Moreover these preferences are compatible with first degree stochastic dominance. Since acts are evaluated based on the probability measure they induce, the decision maker is probabilistically sophisticated and her preferences satisfy the hypothesis of Theorem 1.

Thus aside from monotonicity considerations, our assumptions substantively weaken those of Savage (1954) as well as Machina and Schmeidler (1992). Moreover, consider the following result:

Proposition 1. *Assume Savage's P3 and Axiom C. Then for any x^*, $x_* \in X$ with $x^* \succ x_*$, disjoint $E, E' \in \Sigma$, and $f \in \mathcal{F}$, $x^* E x_* E' f \succeq x^* E' x_* E f \;\Leftrightarrow\; E \succeq^C E'$.*

The proposition establishes two things: given a weak ordering satisfying P3, Machina and Schmeidler's P4* is implied by completeness of $\succeq^C$; moreover, $\succeq^C$ *is*, in this case, the comparative likelihood relation represented in their probabilistically sophisticated setting. In other words, to arrive at their representation theorem one need only add P3 and a form of continuity to our list of conditions.

Comparison with Grant (1995)

The following highlights the limitations of an exchangeability based approach to probabilistic sophistication.

Example 2. Consider the 'mother' example supplied by Grant (1995) and mentioned earlier. If there are only two outcomes in the world of the decision maker - namely, receipt of an indivisible good by Child 1 or by Child 2 - then a plausible representation for the mother's preferences is the utility function $U(p) = p(1-p)$, where p is the probability that Child 1 receives the indivisible good and is subjectively generated by some device deemed by the mother to be uniform. According to the definition of exchangeable events, any event with probability $p \in [0, 0.5]$ is exchangeable with its complement. ∎

In the example, $\approx$ fails to deliver a notion of likelihood because given three disjoint events, E, E' and A such that $\mu(E) = \mu(E') = 0.4$ and $\mu(A) = 0.2$, the mother's preference behavior leads to the conclusion that $E \approx E'$ while $E \cup A \approx E'$, in violation of Axiom N. Failure of the latter to deliver what is clearly probabilistically sophisticated behavior can be attributed to the highly restricted nature of the outcome space. If the good is divisible, say chocolate, or there is an outcome in which nothing is given to either child, then it will likely no longer be the case that any event is exchangeable with its complement; for instance, if E is a probability 0.6 event, then it is reasonable to suppose that the mother is not indifferent between giving *each* child a piece of chocolate if E is realized and nothing otherwise, versus giving each child a piece of chocolate if the complement of E is realized and nothing otherwise.

As stated, our axioms do not encompass those of Grant (1995) whose approach, in particular, can accommodate Example 2. Grant (1995) weakens P3 to either one of two variants: conditional upper (or lower) eventwise monotonicity (P3^{CU} or P3^{CL}).[19] On the other hand, Grant's axioms are more demanding in the sense that taken as a whole they imply completeness of $\succeq^C$, require a form of continuity not needed in Theorem 1, and rule out many probabilistically sophisticated functional forms that are admissible under our axioms. Moreover, the preceding discussion suggests that a sufficiently enriched outcome space may not be subject to peculiarities of the mother example. Specifically, consider the following result which establishes that in the presence of a 'rich' outcome space, either one of Grant's P3^{CU} and P3^{CL} implies Axiom N.

Proposition 2. *Assume that for every non-null $A \in \Sigma$, $f \in \mathcal{F}$, there exist $x, x' \in X$ such that $xAf \succ f \succ x'Af$. Then either one of Grant's P3^{CU} or P3^{CL} implies Axiom N.*

The premise of Proposition 2 is a form of non-satiation in outcomes: there is always something sufficiently good (resp. bad) that the decision maker is happy (resp. reluctant)

[19]These are formally stated in the proof to the next Proposition.

to substitute for the payoff scheme determined by f on A. It can therefore be viewed as a 'richness' assumption on both $\succeq$ and the outcome set, X. Indeed, it is a challenge to find an intuitively behavioral example in a state independent setting where the state space cannot be so 'enriched'. To further emphasize the importance of 'enriching' the outcome space, we note that whenever X contains only two outcomes, Σ is atomless, and $\succeq$ can be represented via a continuous and probabilistically sophisticated utility function, Axiom N is satisfied if and only if the representation is monotonic in the sense of P3.[20]

Under the conditions in Proposition 2, Grant's unique measure representing probabilistic sophistication agrees with $\succeq^C$, and his axioms (taken together) imply both completeness of $\succeq^C$ and Axiom A. In other words, probabilistically sophisticated preferences that satisfy Grant's axioms also satisfy ours *provided that the outcome space is sufficiently rich to ensure that Axiom N is also satisfied.* In practice, Theorem 1 applies to many instances in which Grant's axioms are violated, including but not limited to cases where upper contour sets of preferences over distributions are not always quasi-concave or quasi-convex.[21] Moreover, there is another advantage to an exchangeability based approach to probabilistic sophistication beyond the fact that it encompasses other known models (given a 'richness' assumption): as argued later, our exchangeability based approach lends itself particularly well to studying deviations from global probabilistic sophistication, and does not confound such deviations with other behavioral conditions such as monotonicity or continuity.

2.3.3. State Independence

One often interprets Savage's P3 and P4 to jointly assert a separation between tastes and beliefs, or 'state independence' (see, for example, Sarin and Wakker, 2000). Theorem 1 implies state independence in the sense that the decision maker to which this theorem applies only cares about the likelihood of an event and not its identity. The axioms leading to this result are weaker than P3 and P4, suggesting that the latter axioms do not fully capture the notion of 'state independence' and thus prompting the obvious question:

[20] We thank I. Gilboa for pointing this out.

[21] Our approach generally accommodates probabilistically sophisticated preferences violating first degree stochastic dominance including mean-variance, trimmed mean, Winsorized mean, and certain lexicographic preferences. We note that in addition to violating monotonicity assumptions, some of these (e.g., mean-variance preferences) also violate P4 and its variants.

which of our assumptions embodies the sense of 'state independence'? A possible answer is the same one suggested in the Introduction for ruling out Ellsbergian or source dependent behavior; i.e., $\succeq^C$ is complete, and thus all events are comparable. For instance, in the Aumann-Savage correspondence documented in Dreze (1987), two equally likely events – 'rain in Chicago' and 'no-rain in Chicago' – are not exchangeable with respect to the outcomes 'trip to Chicago with umbrella' and 'trip to Chicago without an umbrella'. While they may well be exchangeable with respect to other outcomes (e.g., trip to San Francisco with, versus without, an umbrella), the two events are not exchangeable with respect to the 'Chicago plus/minus umbrella' payoffs even though they are associated with unique probabilities. In particular, the two events are not comparable via $\succeq^C$, which is therefore incomplete.

Informally, one is able to distinguish between incompleteness of $\succeq^C$ due entirely to Ellsbergian or similar behavior, versus the type of state dependence outlined above. To be able to do so formally, let $\succeq_Y$ correspond to the preference relation obtained by restricting the set of outcomes to $Y \subseteq X$. Defining $\approx_Y$ and $\succeq_Y^C$ similarly, we offer a definition of state independent preferences that captures the idea that essentially any outcome set can be used to elicit the same subjective probabilities over Σ. Specifically, we say that $\succeq$ is **state independent** whenever $\succeq_Y^C = \succeq_{Y'}^C$ for every pair of outcome sets, $Y, Y' \subseteq X$, such that $\succeq_Y$ and $\succeq_{Y'}$ satisfies Axiom N. We require Axiom N so as to exclude scenarios such as Example 2 which seems intuitively state independent.

We caution the reader that our definition does not fully capture the intuition behind 'state independence'. For instance, in the Aumann-Savage example one might consider a very simple outcome space, spanning 'trip to Chicago with x', where $x \in X = \{$expensive umbrella, average umbrella, cheap umbrella$\}$, in which one suspects there to be state dependence relative to X. However, since the 'connection' between the outcomes and the states is uniform across outcomes, $\succeq$ is 'state independent' according to our definition.[22] Notice that this mis-labeling is shared by other approaches to state independence, such as the requirement that $\succeq$ satisfies P3 and P4. In applications, however, our notion of state independence seems to accord with intuition. Moreover, we take as uncontentious the idea that if $\succeq$ is *not* 'state independent' according to our definition, then it is *state*

[22]See Footnote 7 for another example.

dependent. Regardless, if $\succeq$ is state independent, then any structure characterizing $\succeq^C$ – in particular, deviations from global probabilistic sophistication – can be traced to 'small worlds' behavior. Thus, at least in this case, we can identify the source of incompleteness of $\succeq^C$.

We end by observing that in order to accommodate the coin-tossing mother from Example 2 we require the outcome space to contain more than two outcomes. However, if the outcome space is too rich, $\succeq$ may not satisfy our definition of state independence. This emphasizes that, as with other approaches, ours relies on the specification of X.[23]

3. Small Worlds Probabilistic Sophistication

In what has come to be known as Ellsberg's (1961) two-urn problem, originally proposed in Keynes (1921), one urn contains 50 red and 50 black balls while the second urn contains an unspecified combination of the two. It is commonly observed that a bet on the event of drawing a red ball is interchangeable with a bet on the event of drawing a black ball from the same urn, but not across urns. In particular, people seem to prefer betting using the first ('known') rather than the second ('unknown') urn. By contrast, all the above events are deemed to have the same subjective probability in the Savage setting, and bets on them are therefore interchangeable regardless of their sources. This suggests that each urn can be associated with a restricted domain of events – i.e., small world, or source of uncertainty – such that the decision maker is probabilistically sophisticated within each small world, but "equally likely" events in one small world may not be comparable with "equally likely" complementary events in another small world.

The observed choice behavior over Ellsbergian urns has inspired a substantial literature in axiomatic models of decision making.[24] Similar non-comparability of events is not specific to situations where information about objective probabilities is imprecise. Patterns of behavior involving multiple sources of uncertainty and seemingly 'equally likely'

[23] The literature on state dependence, whose primary focus concerns global probabilistic sophistication based on the expected utility specification (see Karni, Schmeidler and Vind, 1983; Dreze, 1987; Karni, 1993; and Karni, 2003), often makes use of rich outcome spaces (e.g., mixture spaces) to derive results.

[24] Some key references include Schmeidler (1989), Gilboa and Schmeidler (1989), and Nakamura (1990). A number of works posit the Knightian (1921) distinction between risk and uncertainty and classify events as being either subjectively ambiguous or subjectively unambiguous (see Epstein and Zhang, 2001; Ghirardato and Marinacci, 2002; Klibanoff, Marinacci and Mukerji, 2003; Nehring, 2001, 2002; Ghirardato, Marinacci and Maccheroni, 2003).

yet non-comparable events appear pervasive. For instance, betting on the rainfall in one's city of birth versus the rainfall in one's city of residence (if they are different). One can come up with different ways to differentiate among urns, say a favorite celebrity determines the color mixture in the unknown urn. When betting on whether the 16th digit in the decimal expansion of $\sqrt{13}$ and $\sqrt{14}$ is even or odd, a Hong Kong resident may prefer to bet on $\sqrt{13}$ while a Bay area resident may prefer to bet on $\sqrt{14}$ (in Cantonese, 14 and 13 sound respectively like "die for sure" and "live for sure".) In playing Lotto, customers may prefer selecting their own numbers rather than having them picked by a computer.[25] In all of these choice problems one might find probabilistic sophistication restricted to specific sources of uncertainty but not across them.

These examples provide vivid instances of deviation from global probabilistic sophistication as well as the seeming presence of restricted collections of events over which the decision maker may exhibit probabilistic sophistication. This leads naturally to the questions: "What is an appropriate preference-based characterization of such restricted collections?" and "Is there a constructive approach to assess whether two given events belong to a collection on which the decision maker is probabilistically sophisticated?" The Machina-Schmeidler axioms can be used to check whether a restricted collection of events satisfies a limited form of probabilistic sophistication. However, given two events, it is not obvious how these axioms might themselves be used to constructively determine if any collection of events, on which the decision maker is probabilistically sophisticated, contains both events. In Subsection 3.1 below we outline how our exchangeability based approach can address this issue.

The above discussion is related to a strand in the decision theory literature having to do with the distinction between ambiguous and unambiguous events as well as attitudes towards ambiguity. We revisit this literature in Subsection 3.2 and relate it to our setting. Finally, we illustrate in Subsection 3.3 the potential usefulness of an exchangeability-based approach with a brief application that efficiently models small worlds behavior in a variant of the Ellsberg 3-color urn.

[25]There is a growing literature recognizing the presence of source preference in decision making. See, for example, Heath and Tversky (1991), Fox and Tversky (1995), Keppe and Weber (1995), Tversky and Wakker (1995), as well as an early reference in Fellner (1961).

3.1. Domains

Given the weak setup of the preceding section, departures from probabilistic thinking can be traced to violations of Axiom A, Axiom N, or the assumption that $\succeq^C$ is complete. Despite the example of the coin-tossing mother (in a highly restricted outcome space), it seems more fruitful to explore deviations from completeness. As motivated above, there are decision making situations where completeness of $\succeq^C$ over Σ in a state independent setting is not a compelling assumption. In such cases $\succeq^C$ is not guaranteed to be transitive even if Axioms A and N hold (see Example 1). One may hope, however, that when restricted to certain collections of events $\succeq^C$ can be shown to be a likelihood relation if $\succeq$ satisfies Axioms analogous to A and N.[26]

In pursuit of this idea, consider a collection of events, $\mathcal{A} \subseteq \Sigma$, and define $\widehat{\mathcal{A}} \equiv \bigcup_{E \in \mathcal{A}} E$ as the *envelope* of $\mathcal{A}$. We emphasize that $\widehat{\mathcal{A}}$ may be *strictly* contained in Ω. We say that $f \in \mathcal{F}$ *is adapted to a collection of events*, $\mathcal{A} \subseteq \Sigma$, whenever $f^{-1}(x) \cap \widehat{\mathcal{A}} \in \mathcal{A}$ for every $x \in X$. We are led to identify collections of events suitable for a restricted notion of probabilistic sophistication according to the following intuition: first, as in the case of global probabilistic sophistication discussed earlier, every event in the collection should be comparable with every other event; second, likelihood is generally defined relative to some benchmark event - in the case of global probabilistic sophistication the benchmark event is Ω. Thus the collection, say $\mathcal{A}$, should contain a 'universal' event which we take to be its envelope, $\widehat{\mathcal{A}}$. Finally, consider two events, E and E', in the collection, $\mathcal{A}$, that can potentially be described via a probability measure relative to $\widehat{\mathcal{A}}$. If $E \succeq^C E'$ then $E \setminus E'$ contains a subevent, say e, that is 'as likely as' $E' \setminus E$. Thus if the likelihood (relative to $\widehat{\mathcal{A}}$) of $E' = (E \cap E') \cup (E' \setminus E)$ is known, it should be equal to that of $\xi \equiv (E \cap E') \cup e$, and it seems sensible to require $\xi \in \mathcal{A}$. If, in addition, the likelihood (relative to $\widehat{\mathcal{A}}$) of E is known, then since $\xi \subseteq E$ one can readily calculate the likelihood of $E \setminus \xi = E \setminus (E' \cup e)$, which should therefore also be in $\mathcal{A}$. Given these considerations, we characterize collections over which the decision maker may be probabilistically sophisticated as follows:

[26] Siniscalchi (2003) considers a property akin to probabilistic sophistication over domains of *acts*, this approach does not share our aim of characterizing probabilistic attitudes towards collections of *events*. Vind (2003) is closer to us in illustrating how probabilistic sophistication can simultaneously exist on multiple subalgebras of Σ, yet not across them.

Definition 5. *A collection of events, $\mathcal{A} \subseteq \Sigma$ is* ***homogeneous*** *if it satisfies the following:*

i) If $E, E' \in \mathcal{A}$, then E and E' are comparable.

ii) $\widehat{\mathcal{A}} \in \mathcal{A}$.

iii) For any $E, E' \in \mathcal{A}$ such that $E \succeq^C E'$, if $e \subseteq E \setminus E'$ is a comparison event, then $(E \cap E') \cup e \in \mathcal{A}$ and $E \setminus (E' \cup e) \in \mathcal{A}$.

Observe that if every event is comparable to every other event, then Σ itself is homogeneous. For proper homogeneous subsets of Σ, the definition implies that if E and $E' \subseteq E$ are in $\mathcal{A}$, then $E \setminus E'$ is in $\mathcal{A}$ ($\emptyset$ plays the role of the comparison event). In particular, this is true when $E = \widehat{\mathcal{A}}$ (i.e., if $E' \in \mathcal{A}$ then $\widehat{\mathcal{A}} \setminus E' \in \mathcal{A}$). The logic behind the definition of a homogeneous collection resembles that which leads to the construction of λ-systems of events: a λ-system in Σ is a collection of events, $\mathcal{A} \subseteq \Sigma$ that contains $\widehat{\mathcal{A}}$, $\emptyset$, the relative complement of any member, and the union of any two disjoint members. Indeed, if $\mathcal{A} \subseteq \Sigma$ is homogeneous, then $\mathcal{A}$ is a λ-system. To see this, note first that by parts (ii) and (iii) of Definition 5, $\emptyset$ is in a homogeneous collection as well as any relative complement, $\widehat{\mathcal{A}} \setminus E$, of a member, $E \in \mathcal{A}$. Finally, to see that the union of any two disjoint members of a homogeneous collection is in the collection, consider $E, E' \in \mathcal{A}$ that are pairwise disjoint. Then since $\widehat{\mathcal{A}} \setminus E \in \mathcal{A}$ and $E' \subseteq \widehat{\mathcal{A}} \setminus E$, it must be that $\big(\widehat{\mathcal{A}} \setminus E\big) \setminus E'$ is in $\mathcal{A}$, or alternatively, $\widehat{\mathcal{A}} \setminus (E \cup E') \in \mathcal{A}$. The latter, in turn, implies that $E \cup E'$, the relative complement of $\widehat{\mathcal{A}} \setminus (E \cup E')$, is also in $\mathcal{A}$.

Homogeneous collections are our candidates for the kind of restricted domains on which the decision maker may be probabilistically sophisticated. In checking whether two events might belong to the same homogeneous collection of events, Definition 5 gives an immediate answer when the events are not comparable. Given two exchangeable and disjoint events, the answer is also clear.[27] In all other cases, Definition 5 can be applied in principle to systematically arrive at either a homogeneous collection housing the two events, or the conclusion that they do not share a common homogeneous collection. This is illustrated in the following variant of Ellsberg's 3-color urn.

Example 3. Consider a single draw from a four-color urn containing 25 red balls, 25 yellow balls, plus a distribution of blue and green balls that sum to 50. Refer to the event

[27] If $E, E' \in \Sigma$ are disjoint and exchangeable, then $\{\emptyset, E, E', E \cup E'\}$ is homogeneous.

of drawing a ball of a certain color by the first letter of the color drawn, and let the union of events be represented by the conjunction of the appropriate letters (e.g., GB is the event 'Green or Blue ball drawn'). Here, Σ is generated by unions of $\{Y, R, G, B\}$.

Assuming G and B are exchangeable, as are Y and R, and GB is exchangeable with YR, three obvious homogeneous collections are $\mathcal{D}_0 = \{\emptyset, YR, GB, \Omega\}$, $\mathcal{D}_1 = \{\emptyset, G, B, GB\}$, and $\mathcal{D}_2 = \{\emptyset, Y, R, YR\}$.

Note that YRB is comparable with YRG. Can these two events belong to a single homogeneous collection? If so, under what conditions? To answer these questions, note first that by part (i) of Definition 5, such a system must necessarily include $\Omega = YRGB$. Since the homogeneous collection is a λ-system, it *must* also include both G, B and $G \cup B$, implying that YR is there as well. Note that part (i) of Definition 5 requires YR to be comparable with each of B and G; recaling that YR and BG are exchangeable, the problem now resembles the 3-color urn discussed at the Introduction and at the beginning of Section 2.3.1. If Axiom N holds, the inherent symmetry in the problem in conjunction with an argument similar to that used at the beginning of Section 2.3.1 leads to the conclusion that $Y \approx G \approx R$, and similar for B. In particular, the candidate homogeneous collection must coincide with Σ.

On the other hand, Ellsbergian behavior suggests that Y or R are not exchangeable with G or B, and therefore Σ is not homogeneous. In summary, under reasonable preference assumptions, while YRG and YRB are comparable, they can only belong to the same 'small world' if the decision maker is globally probabilistically sophisticated. ∎

From this example, it is apparent that homogeneity is a demanding requirement and that comparability between two events is not sufficient to conclude that they are members of the same small world. Moreover, if two comparable events can be contained in the same homogeneous collection, then such a collection can be constructed in principle as we did in the example.

It is easy to find examples of strict subsets of a homogeneous collection that are themselves homogeneous. For instance, any event along with the empty set forms a homogeneous collection. Given that investigating a homogeneous subset of a homogeneous collection appears redundant, we focus on the following 'maximal' construction:

Definition 6 (Domains). *$\mathcal{D} \subseteq \Sigma$ is a* ***small world event domain*** *if it is a homogeneous subset of Σ with the following properties:*

i) $\mathcal{D}$ contains more than one non-null event.

ii) If $\mathcal{A} \subseteq \Sigma$ is a non-empty collection of events and $\mathcal{A} \cup \mathcal{D}$ is homogeneous, then $\mathcal{A} \subseteq \mathcal{D}$.

For brevity, we will henceforth refer to any small world event domain simply as a 'domain'. The first condition is akin to 'non-triviality'. The second condition guarantees that a domain is a 'maximal' homogeneous collection. Note the following properties: (i) If Σ is a domain then it is the only domain and under the hypothesis of Theorem 1 the decision maker exhibits global probabilistic sophistication, (ii) if Σ does not contain domains, then no two non-null and disjoint events are exchangeable.

Probabilistic sophistication on a domain with an algebraic structure is an immediate corollary of Theorem 1:

Corollary to Theorem 1 *Assume Axiom A, Axiom N and that the domain, $\mathcal{D}$ is an algebra. Then there exists a unique finitely additive probability measure, μ, that is either atomless or purely and uniformly atomic, where μ represents $\succeq^C$ on $\mathcal{D}$ such that any two events in $\mathcal{D}$ with the same measure are exchangeable. Moreover, the decision maker is indifferent between any two acts $f, g \in \mathcal{F}$ that are adapted to $\mathcal{D}$ such that $f(\omega) = g(\omega)$ $\forall \omega \in \Omega \backslash \widehat{\mathcal{D}}$ whenever f and g induce the same lottery with respect to μ on $\mathcal{D}$.*

Are Axioms A and N sufficient for probabilistic sophistication more generally within a domain, $\mathcal{D}$? Absent an algebraic structure we are unable to show that Axioms A and N are sufficient for transitivity of $\succeq^C$. In particular, unless $\mathcal{D}$ is an algebra, the existence of a unique measure on $\mathcal{D}$ representing $\succeq^C$ does not guarantee that the decision maker is indifferent between any two acts, say f and f', that are identical outside a domain, while inducing the same lottery within the domain. This situation is not specific to our approach; in both Epstein and Zhang (2001) and Kopylov (2004) monotonicity is crucial in demonstrating indifference between such a pair of acts. Elsewhere, we show that one can achieve the desired goal in an exchangeability based approach without monotonicity by requiring slight strengthenings of Axioms A and N, adding a continuity axiom, and assuming that the decision maker is indifferent between two acts defined over distinct and 'equally sized' partitions of a domain whenever the payoff schemes of the acts coincide.[28]

To summarize, our exchangeability-based approach enables a straight-forward characterization of self-contained domains of events, such that the decision maker exhibits probabilistic sophistication when acts are adapted to these domains. Moreover, it is possible to construct a homogeneous collection from two comparable events, if a homogeneous collection containing both exists; this enables one to determine whether any two comparable events share the same endogenously characterized source of uncertainty.

3.2. Application: Subjectively Unambiguous Events

One strand of the current literature on decision making under uncertainty is to distinguish events on which a decision maker is globally probabilistically sophisticated, from those on which the decision maker's likelihood relation does not admit a probabilistic representation. The former are often called 'unambiguous' while the latter are termed 'ambiguous' events. For instance, Epstein (1999) asserts that there are exogenously given unambiguous events on which the decision maker is probabilistically sophisticated in the Machina-Schmeidler sense and defines the decision maker's risk attitude there as being ambiguity neutral, thus providing a benchmark for ambiguity aversion.

In their 2001 paper, Epstein and Zhang posited four desiderata for a definition of subjectively unambiguous events: (D1) behavioral; (D2) model free; (D3) explicit and constructive; and (D4) consistent with probabilistic sophistication on unambiguous acts. An event, $E \in \Sigma$, is subjectively unambiguous in their sense (henceforth **EZ-unambiguous**) if the following two conditions are satisfied (i) for any $x, x', z, z' \in X$, $f \in \mathcal{F}$ and $A, B \in \Sigma$ such that $E', E'' \subseteq E^c$ and $E' \cap E'' = \emptyset$,

$$zExE'x'E''f \succeq zEx'E'xE''f \ \Rightarrow \ z'ExE'x'E''f \succeq z'Ex'E'xE''f.$$

and, (ii) condition (i) holds if E is everywhere replaced by E^c. Epstein and Zhang (2001) show that the set of subjectively unambiguous events is a λ-system under P5 and some technical conditions, and further that desideratum D4 is satisfied under P3 and modified forms of P4 and P6.

[28] Proof of this was in an earlier draft of this paper (Chew and Sagi, 2003).

In our setting, an attempt to distinguish between ambiguous and unambiguous events can also be made in relation to homogeneous collections of events. Specifically, define $E \in \Sigma$ to be **EB-unambiguous** (for 'exchangeability-based unambiguous') if it belongs to *some* homogeneous collection, $\mathcal{A} \subseteq \Sigma$, such that $\widehat{\mathcal{A}} = \Omega$. Under the conditions in Theorem 1, if a domain exists with envelope equal to Ω, then each of its elements is EB-unambiguous. Notice that our definition satisfies the Epstein-Zhang desiderata in that it is (i) behavioral by virtue of being exchangeability-based, (ii) 'model free' in that it does not refer to the decision maker's ambiguity attitudes or exogenous structure about the state space, (iii) constructive in the sense that one can in principle systematically investigate whether a homogeneous collection can constructed by starting with the event under scrutiny and its complement, and (iv) consistent with probabilistic sophistication.

According to our definition, if only one domain has an envelope equal to Ω, then that domain uniquely characterizes the set of unambiguous events. If there are multiple domains whose envelope is Ω, then no single domain spans the set of 'unambiguous events'. It is in this latter case that what intuitively appears to be an ambiguous event may be classified as unambiguous according to our definition. For instance, in the case of a decision maker facing Ellsberg's two urns, an outside observer who does not know which urn contains an equal number of balls would not be able to discern the 'known' versus 'unknown' urn based on our definition alone. The source of this difficulty is the presence of more than one homogeneous collection whose envelope is Ω.[29] Specifically, what is needed is additional information concerning the decision maker's preference for betting on events with known probabilities versus those without.

The difficulty posed by the case of multiple unambiguous event domains may also account for the diversity of approaches in the literature addressing the distinction between ambiguous and unambiguous events. As with our case, the Epstein-Zhang definition may give a confounded analysis of ambiguous events in the two urn example. The definition of subjective ambiguity in Nehring (2001, 2002), Ghirardato and Marinacci (2002) and Ghirardato, Marinacci and Maccheroni (2003), on the other hand, rely on prior knowledge of ambiguity attitudes, while Epstein (1999), Nau (2003) and Klibanoff, Marinacci and Mukerji (2003) appeal to the existence of an exogenously specified set of unambiguous

[29]The event of drawing a red ball is exchangeable with the event of drawing a black ball from the same urn, thus events restricted to a single urn comprise a homogeneous collection.

events on which the decision maker is probabilistically sophisticated. One advantage of the definition we offer is that it is clear when it should be supplemented with additional information in order to conform with intuition.

Given that our definition appears to satisfy the Epstein-Zhang desiderata, it is appropriate to make a more careful comparison between the two. We first note that the two definitions are not nested in that events classified as unambiguous by Epstein and Zhang (2001) may not be so classified by us, and vice versa. The latter is easy to see from the two urn example where our definition regards any single-urn event as unambiguous since the decision maker is probabilistically sophisticated within each urn domain. To see the former, return to Example 1. It is straight forward to show that the set of EZ-unambiguous events contains $B_1 \cup B_2$ and $A \cup C$. On the other hand, $B_1 \cup B_2$ cannot be part of a homogeneous collection with envelope Ω, since that would imply that A and C are comparable.

Our definition of unambiguity is 'small world' compatible in the sense that one does not require knowledge about the decision maker's attitudes towards events *outside* of the homogeneous collection used to establish unambiguity. This is not the case in the Epstein-Zhang definition, in which one must consider *all* pairs of events E' and E'' in the complement of E. This can pose difficulties if 'big world' events (especially those not modeled) might spoil the unambiguity of what might otherwise appear to be clearly unambiguous. To see that this concern is legitimate, return once more to Example 1.

The unique domain with envelope Ω – and therefore the unique domain of EB-unambiguous events – consists of all events containing the *same proportion* of each region (e.g., 'the dart falls in the top half of one of the regions'); we refer to this domain as $\mathcal{D}_{[0,1]}$, since it is essentially equivalent to the algebra of subsets of the unit interval. If $E \in \mathcal{D}_{[0,1]}$ then the event generated by permuting the identity of the four regions is also in $\mathcal{D}_{[0,1]}$. The decision maker is probabilistically sophisticated, indeed risk-neutral, with respect to all acts adapted to $\mathcal{D}_{[0,1]}$. On the other hand, no non-null event in $\mathcal{D}_{[0,1]}$ is EZ-unambiguous. To see this, consider without loss of generality the event $E = I \times \{A, B_1, B_2, C\}$ where $I \subset [0,1]$ has Lebesgue measure of $q \in (0,1)$. Let I^c be the complement of I in $[0,1]$, $E' = I^c \times B_1$, and $E'' = I^c \times B_2$. Let $f = xAx'$ and observe that $V(zExE'x'E''f) - V(zEx'E'xE''f)$ is given by

$$-\Big(qz + \frac{(1-q)(x+x')}{2}\Big)\frac{(1-q)^2(x-x')^2}{8}$$

A necessary condition for E to be EZ-unambiguous is that for any $x, x' \in [-1,1]$ the above expression does not change sign as z varies in $[-1,1]$. To see that this cannot be, set $x = q$ and $x' = 0$. In summary, whereas the unambiguity of E in our setting depends only on events in the 'small world', $\mathcal{D}_{[0,1]}$, the Epstein-Zhang definition explicitly depends on events where permuting the identities of A, B_1, B_2 and C matters to the decision maker (e.g., E', E''). This example suggests that given a seemingly unambiguous collections of events, such as an algebra on the unit interval with the uniform measure, it might be possible to render certain events in the collection EZ-*ambiguous* by taking account of further choice comparisons in an enriched state space. In this sense, the Epstein-Zhang definition is not 'small world' compatible.

When Σ is an atomless σ-algebra, we may also alternatively identify unambiguous events as follows. First, let $\{a_{i,m}\}_{i=1}^{m}$ be a *uniform m-partition of* Ω whenever the $a_{i,m}$'s are pairwise disjoint, $\bigcup_{i=1}^{m} a_{i,m} = \Omega$, and $a_{i,m} \approx a_{j,m}$ for any $i, j = 1, ..., m$. Without loss of generality set $a_{0,m} \equiv \emptyset$. Define $E \in \Sigma$ to be subjectively EB-unambiguous if it is equal to Ω, is null, or whenever for every $m \geq 1$ there exists a uniform m-partitions of Ω and $0 \leq j \leq m$ such that

$$\bigcup_{i=0}^{j} a_{i,m} \subset E \subseteq \bigcup_{i=0}^{j+1} a_{i,m}$$

Here, an event is said to be EB-unambiguous if it can be approximated arbitrarily well by a union of events from a partition of mutually exchangeable events. This alternative definition also captures the intuition underlying the frequentist view of objective probability and likewise satisfies the Epstein-Zhang desiderata.

3.3. Application: Domain Independence

The discussion in the Introduction argues that non-exchangeability may arise from distinct sources of uncertainty leading to deviation from global probabilistically sophisticated behavior. The Corollary to Theorem 1 establishes that in situations involving multiple (algebraic) domains, choice behavior is probabilistically sophisticated when restricted to bets purely within any single domain. For instance, betting on rainfall in one of several distinct cities may lead to the impression that the decision maker is probabilistically sophisticated with well defined risk preferences over each 'city domain'. This suggests domains may be distinguished by differences in 'risk attitudes' of the decision maker across them. In particular, if the decision maker consistently prefers a lottery (with respect to one domain-specific subjective probability) in one city over an 'identical' lottery in another city, then it seems reasonable to say that she is more risk averse in the domain associated with the latter city. Similarly, in the 2-urn Ellsberg example, the observed preference for betting on the known urn is compatible with the decision maker being more risk averse when facing even-chance bets from the unknown urn. Note that this can allow for additional discrimination between distinct domains that are each classified as subjectively EB-unambiguous.

To identify possible representations for mixed acts involving payoffs across multiple domains, we need more structure. One can propose a 'domain independence property' requiring that a certainty equivalent for a conditional act that is adapted to a domain be *independent* of payoffs outside the domain.

To illustrate this and the modeling potential afforded by our approach, consider a single draw from the four-color urn of Example 3. Ellsbergian behavior corresponds to the case where neither of Y or R are exchangeable with B or G. In this case, the obvious domains correspond to: $\mathcal{D}_0 = \{\emptyset, Y, R, YR, GB, \Omega\}$ and $\mathcal{D}_1 = \{\emptyset, G, B, GB\}$. Since these are algebras, the Corollary to Theorem 1 applies and the decision maker treats acts adapted to a domain as lotteries within the domain. Consistent with the separability of decisions across domains proposed above, let $V(x_Y, x_R, x_G, x_B)$ be the utility of the act $x_Y Y x_R R x_G G x_B$. Then domain separability associates a certainty equivalent, c_{GB} with the payoffs x_G, x_B adapted to $\mathcal{D}_1$ *independently of* x_R and x_Y. Let $W_1(x_G, x_B)$ represent the risk preferences in $\mathcal{D}_1$, such that $W_1(x, x) = x$ (i.e., W_1 is a certainty equivalent function). Similarly, let $W_0(x_Y, x_R, x_{GB})$ represent the risk preferences in $\mathcal{D}_0$. Then according to the domain independence property, one can write (up to a monotonic transformation),

$$V(x_Y, x_R, x_G, x_B) = W_0(x_Y, x_R, c_{GB}) = W_0(x_Y, x_R, W_1(x_G, x_B))$$

The domain recursive preference structure in this example is reminiscent of a growing literature that rationalizes Ellsbergian attitudes by means of a two stage valuation approach (see Segal 1987, 1990; Nau, 2003; Klibanoff, Marinacci and Mukerji, 2003; and Ergin and Gul, 2004). Whereas the recursive stages arise endogenously in our setting via domain independence, it is posited or exogenously imposed in the references. Our approach appears to be complementary to this literature in that it can provide a subjective basis for the state space structure required in a recursive representation.

A. Appendix

Proof of Theorem 1: We first prove $\succeq^C$ is a likelihood relation with a unique agreeing measure for any homogeneous algebra (see Definition 5). The result for Σ follows from the fact that, by Axiom C, it is homogeneous. Note that if E, E' are members of a homogeneous algebra, then any comparison event between E and E' is also in the algebra. The latter property is used extensively in the proof. First consider the following proposition:

Proposition A.1. *Assume Axiom N and let $\mathcal{A} \subseteq \Sigma$ be a homogeneous algebra of events. Then for any $E, E', E'' \in \mathcal{A}$, $E \succeq^C E'$ and $E' \succeq^C E''$ imply $E \succeq^C E''$.*

Proof: We first prove related results.

Lemma A.1. *Assume Axiom N and consider any pairwise disjoint $a, b, c, d \in \Sigma$. Then $a \cup b \approx c \cup d$, b and d are comparable, and $a \approx c$ implies $b \approx d$*

Proof: Assume that it is not the case that $b \approx d$. Then without loss of generality, let $b \succ^C d$, which implies, by definition, that there is some $b' \subset b$ such that $b' \approx d$ and $b \setminus b'$ is not null. By the definition of exchangeability, $a \cup b' \approx c \cup d$ which violates Axiom N since $a \cup b \setminus a \cup b'$ is not null. ∎

Lemma A.2. *Assume Axiom A and Axiom N. Then if E, E' and E'' are pairwise disjoint members of a homogeneous algebra, $\mathcal{A}$, and $E \approx E'$ and $E' \approx E''$ then $E \approx E''$.*

Proof: This is trivial if any of the events are null, so assume otherwise. Suppose that $E \succ^C E''$. Then by Axiom N there is some non-null $e_1 \subset E$ such that

$E \setminus e_1 \approx E''$, and since $\mathcal{A}$ is a homogeneous algebra, $e_1 \in \mathcal{A}$. By homogeneity of $\mathcal{A}$ and Axiom N, $E' \succ^C E \setminus e_1$; thus there is some $e_2 \subset E'$ such that $e_2 \in \mathcal{A}$ and $E' \setminus e_2 \approx E \setminus e_1$. The events e_1 and e_2 are disjoint, so Lemma A.1 implies that $e_1 \approx e_2$. The fact that $E'' \approx E'$ can be similarly used to establish the existence of a set $e_3 \subset E''$ disjoint from e_1 and e_2 such that $e_3 \approx e_2$. Similarly, $E \setminus e_1 \approx E''$ leads to $e_4 \subset E \setminus e_1$ such that $e_4 \approx e_3$, etc. Clearly this can be continued to construct an infinite sequence of non-null events that are disjoint such that $e_{i+1} \approx e_i$, in violation of Axiom A. ∎

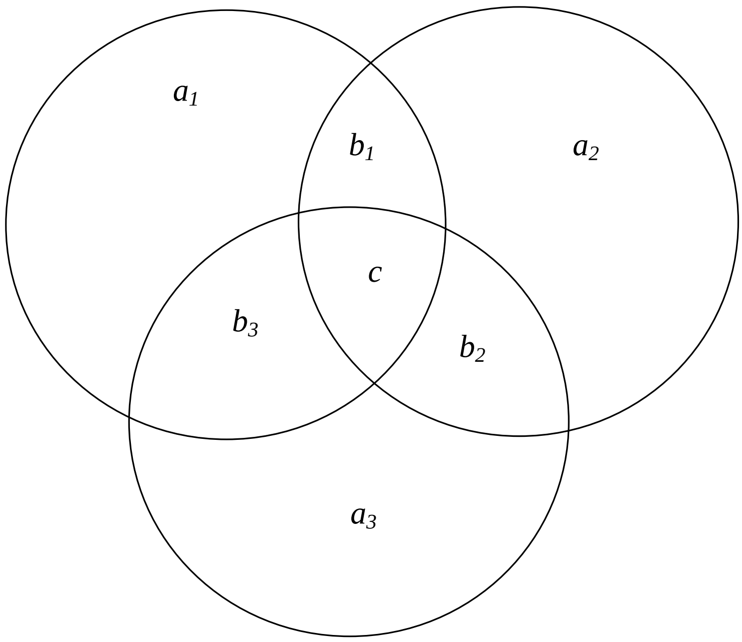

Figure 1: Venn diagram useful in proving Theorem 1.

Lemma A.3. *Assume Axiom N. Let a_1, a_2, a_3, b_1, b_2 and b_3 be pairwise disjoint events in a homogeneous algebra $\mathcal{A}$, and consider the configuration of events as illustrated in Figure 1. Suppose $a_1 \cup b_3 \approx a_2 \cup b_2$, $a_2 \cup b_1 \approx a_3 \cup b_3$, then $a_1 \cup b_1 \approx a_3 \cup b_2$.*

Proof: The proof is accomplished in 3 steps. Step 1 consists of establishing that there exist events a_1', a_3' and b_3' such that $a_1' \approx a_3'$ and $a_1' \cup b_3' \approx a_2 \approx a_3' \cup b_3'$ (all events are in $\mathcal{A}$ unless otherwise indicated). Step 2 consists of proving that there are events $b_1' \subseteq b_1, b_2' \subseteq b_2$ such that $b_1' \approx a_3 \setminus a_3', b_2' \approx a_1 \setminus a_1'$, and $b_1 \setminus b_1' \approx b_2 \setminus b_2'$. The final step concludes from this that $a_1 \cup b_1 \approx a_3 \cup b_2$.

Step 1: Since $a_1 \cup b_3 \succeq^C a_2$ and $a_3 \cup b_3 \succeq^C a_2$ there is some $\hat{a}_1 \cup \hat{b}_3 \approx a_2$ and $\check{a}_3 \cup \check{b}_3 \approx a_2$, with $\hat{a}_1 \subseteq a_1, \check{a}_3 \subseteq a_3$, and $\hat{b}_3, \check{b}_3 \subseteq b_3$. Let $\hat{a}_2 \approx \hat{b}_3$ and $\check{a}_2 \approx \check{b}_3$, where $\hat{a}_2, \check{a}_2 \subseteq a_2$. Set $a_2' \equiv a_2 \setminus (\hat{a}_2 \cup \check{a}_2)$. By Axiom N and Lemma A.1 there must be $a_1' \subseteq a_1$ and $a_3' \subseteq a_3$ such that $a_1' \approx a_2' \approx a_3'$. Lemma A.2 proves that $a_1' \approx a_3'$. Defining $b_3' \equiv \hat{b}_3 \cup \check{b}_3$ gives $a_1' \cup b_3' \approx a_2$ and $a_2 \approx a_3' \cup b_3'$.

Step 2: From $a_1 \cup b_3 \approx a_2 \cup b_2$, $a_3 \cup b_3 \approx a_2 \cup b_1$, and the last step, Lemma A.1 implies that $(a_3 \setminus a_3') \cup (b_3 \setminus b_3') \approx b_1$ and $(a_1 \setminus a_1') \cup (b_3 \setminus b_3') \approx b_2$. Thus there are $b_1' \subseteq b_1$ and $b_2' \subseteq b_2$ such that $b_1' \approx a_3 \setminus a_3'$ and $b_2' \approx a_1 \setminus a_1'$. By Lemma A.1, $b_1 \setminus b_1' \approx b_3 \setminus b_3'$ and $b_3 \setminus b_3' \approx b_2 \setminus b_2'$, thus Lemma A.2 implies that $b_1 \setminus b_1' \approx b_2 \setminus b_2'$.

Step 3: Write $a_1 \cup b_1 = a_1' \cup (a_1 \setminus a_1') \cup b_1' \cup (b_1 \setminus b_1') \approx a_3' \cup b_2' \cup (a_3 \setminus a_3') \cup (b_2 \setminus b_2') = a_3 \cup b_2$. ∎

Now, given $E, E', E'' \in \mathcal{A}$, suppose that $E \succeq^C E'$ and $E' \succeq^C E''$. Let $e' \in \mathcal{A}$ be a comparison subset between E' and E''. Note that $e' \subseteq E' \setminus E''$; moreover $e' \approx E'' \setminus E'$. Axiom N and the assumption that $\mathcal{A}$ is an algebra lead to $E \succeq^C e' \cup (E' \cap E'')$. Thus there is some $\hat{e} \subseteq E \setminus \big(e' \cup (E' \cap E'')\big)$ such that $\hat{e} \approx \big(e' \cup (E' \cap E'')\big) \setminus E$. We can now apply Lemma A.3 as follows. Let the lower circle in Figure 1 correspond to E''. This can be broken up into two pieces: $E'' \setminus E' \equiv a_3 \cup b_3$ and $E'' \cap E' \equiv b_2 \cup c$. Likewise, let e' correspond to $a_2 \cup b_1$, so that $a_2 \cup b_1 \approx a_3 \cup b_3$. Finally, let $a_1 \cup b_3 \equiv \hat{e}$ and set:

$$\xi = (e' \cup (E' \cap E'')) \cap E$$

Diagrammatically, ξ corresponds to $b_1 \cup c$. Note that we identify the left and right circles with *subsets* of E and E', respectively. It follows that: $b_1 = \xi \cap e'$, $a_2 = e' \setminus b_1$, $b_3 = \hat{e} \cap E''$, $a_1 = \hat{e} \setminus b_3$, $b_2 = \Big(\big(e' \cup (E' \cap E'')\big) \setminus E\Big) \cap E''$ and $a_3 = E'' \setminus \big(\hat{e} \cup E'\big)$. Now, $\hat{e} \approx \big(e' \cup (E' \cap E'')\big) \setminus E$ means that $a_1 \cup b_3 \approx a_2 \cup b_2$. Since $a_2 \cup b_1 \approx a_3 \cup b_3$, Lemma A.3 implies $a_1 \cup b_1 \approx a_3 \cup b_2$. Moreover, since $E'' \setminus E = a_3 \cup b_2$ and $a_1 \cup b_1 \subseteq E \setminus E''$, by definition $E \succeq^C E''$. ∎

Proposition A.1 establishes that $\succeq^C$ is a weak order (transitive and complete) over $\mathcal{A}$. Condition (ii) in the definition of a likelihood relation is satisfied by $\succeq^C$ due to the presence of non-null events (P5) and Axiom N. The last condition is automatically satisfied by the definition of comparability. The following result helps to complete the proof.

Lemma A.4. $\succeq^C$ *is either atomless and tight or* $\mathcal{A}$ *consists of a finite number of exchangeable atoms.*

Proof: Assume first that $\mathcal{A}$ contains an atom, a, and let a^c be its relative complement in $\widehat{\mathcal{A}}$. Note that for any $e \in \mathcal{A}$ it cannot be that $a \succ^C e$ since a cannot be partitioned into two or more non-null events. Thus $a^c \succeq^C a$. If $a \succeq^C a^c$ then $a \sim^C a^c$ meaning, by Axiom N, that $a \approx a^c$ and $\mathcal{A}$ therefore consists of two atoms. Suppose instead that $a^c \succ^C a$. Then there is some $a_1 \subseteq a^c$ with $a_1 \approx a$; note that a_1 must be an atom in $\mathcal{A}$. Moreover, since $a^c \setminus a_1 \in \mathcal{A}$ it must be that $a^c \setminus a_1 \succeq^C a$. In turn this implies the presence of another atom $a_2 \approx a$ in $a^c \setminus a_1$ with a, a_1 and a_2 disjoint. According to Axiom A, this can be continued at most a finite number of times, proving that the set of non-null events in $\mathcal{A}$ is finite. Transitivity of $\approx$ (Lemma A.2) implies that each atom is exchangeable.

Assume now that $\mathcal{A}$ is atomless. To demonstrate tightness, we must show that whenever $E \succ^C E'$, there are $A, B \in \mathcal{A}$ such that $A \cap E' = \emptyset$ and $B \subset E$ such that $E \succ^C A \cup E'$ and $E \setminus B \succ^C E'$. By definition, $E \succ^C E'$ implies that there is some $e \subset E \setminus E'$ such that $e \approx E' \setminus E$ and $E \setminus (e \cup E')$ is not null. Since $\mathcal{A}$ is atomless, $E \setminus (e \cup E')$ can be split into two disjoint non-null events, ξ_1 and ξ_2, both in $\mathcal{A}$, subsets of E and disjoint from $e \cup E'$. Axiom N implies that $E \succ^C E' \cup \xi_1$ where $\xi_1 \cap E' = \emptyset$, as well as $E \setminus \xi_1 \succ^C E'$. Thus $\succeq^C$ is tight. ∎

Lemma A.4 establishes that $\succeq^C$ is a likelihood relation that is either atomic with equal likelihood over each atom, or atomless and tight. In the latter case, we can show that $\succeq^C$ is fine. To do this, for any $E \in \Sigma$ we construct a finite partition of $\widehat{\mathcal{A}}$ at least as fine as E, $\{e_i\}$, starting with $e_1 \equiv E$. Next, homogeneity implies that either $E \succeq^C \widehat{\mathcal{A}} \setminus E$ or $\widehat{\mathcal{A}} \setminus E \succeq^C E$. In the former case, let $e_2 \equiv \widehat{\mathcal{A}} \setminus E$ and $\{e_1, e_2\}$ forms a partition containing events at least as fine as E. In the latter case, define e_2 as the comparison subset of $\widehat{\mathcal{A}} \setminus E$ that, by definition, is exchangeable with E. Once again, homogeneity implies that either $E \succeq^C \widehat{\mathcal{A}} \setminus (E \cup e_2)$ or $\widehat{\mathcal{A}} \setminus (E \cup e_2) \succeq^C E$, and we can continue constructing events exchangeable with E and disjoint from each other in the obvious way. By Axiom A this construction must be finite and therefore constitutes a partition of $\widehat{\mathcal{A}}$ consisting of events at least as fine as E. Thus $\succeq^C$ is fine.

In either the atomic or the fine and tight case, there exists a unique finitely additive probability measure that agrees with $\succeq^C$ (see Wakker, 1981). This can be extended to a countably additive convex-valued measure setting if $\mathcal{A}$ is a fine and tight σ-algebra, as in Savage's original treatment. Finally, whenever the measure of two events, $E, E' \in \mathcal{A}$, coincides, it must be that $E \succeq^C E'$ and $E' \succeq^C E$; in turn, Axiom N implies that $E \approx E'$.

We now prove that the decision maker is indifferent between all acts inducing the same distribution. If Σ is atomic, the outcomes can be permuted to generate one act from the other, proving the result. Consider, therefore, the case where Σ has no atoms. Consider any two acts that induce the same lottery with respect to μ: $f = x_1 A_1 ... x_{n-1} A_{n-1} x_n$ and $f' = x_1 A'_1 ... x_{n-1} A'_{n-1} x_n$, where n is finite and where for every $i = 1, ..., n$ $\mu(A_i) = \mu(A'_i)$ (where A_n and A'_n are the preimages of x_n under f and f', respectively). Let $A'_{1i} \equiv A'_1 \cap A_i$ for $i = 1, ..., n$. Now,

$$f \sim x_1 A'_{11} x_1 (A_1 \setminus A'_{11}) x_2 A'_{12} x_2 (A_2 \setminus A'_{12}) f$$

Since $A_1 \setminus A'_{11}$ is comparable to A'_{12}, and $\mu(A_1 \setminus A'_{11}) \geq \mu(A'_{12})$, there is some $\xi \subset A_1 \setminus A'_{11}$ such that $\mu(\xi) = \mu(A'_{12})$; consequently, $\xi \approx A'_{12}$, and we can write

$$\begin{aligned} f &\sim x_1 A'_{11} x_1 A'_{12} x_1 \big(A_1 \setminus (A'_{11} \cup \xi)\big) x_2 \xi x_2 (A_2 \setminus A'_{12}) f \\ &\sim x_1 A'_{11} x_1 A'_{12} x_1 \big(A_1 \setminus (A'_{11} \cup \xi)\big) x_2 \xi x_2 (A_2 \setminus A'_{12}) x_3 A'_{13} x_3 (A_3 \setminus A'_{13}) f \end{aligned}$$

This time, $A_1 \setminus (A'_{11} \cup \xi)$ is comparable to A'_{13}, and $\mu\big(A_1 \setminus (A'_{11} \cup \xi)\big) \geq \mu(A'_{13})$. Thus one can exchange A'_{13} with a subset of $A_1 \setminus (A'_{11} \cup \xi)$. Since $\mu(A_1) = \sum_{i=1}^n \mu(A'_{1i})$, this can be continued until one arrives at:

$$f \sim x_1 A'_1 x_2 A^1_2 ... x_{n-1} A^1_{n-1} x_n$$

for some collection of events, A_j^1, $j = 2, ..., n$, where $\mu(A_j^1) = \mu(A_j) = \mu(A_j')$ for $j = 2, ..., n$. This procedure can clearly be continued to write

$$\begin{aligned} f &\sim x_1 A_1' x_2 A_2^1 ... x_{n-1} A_{n-1}^1 x_n \\ &\sim x_1 A_1' x_2 A_2' x_3 A_3^2 ... x_{n-1} A_{n-1}^2 x_n \\ &\vdots \\ &\sim x_1 A_1' x_2 A_2' x_3 A_3' ... x_{n-1} A_{n-1}' x_n \end{aligned}$$

Thus $f \sim f'$, and the decision maker is indifferent between any two acts that induce the same lottery with respect to μ.

Proving necessity of the axioms is trivial. ∎

Proof of Proposition 1: Since there exists $x^*, x_* \in X$ with $x^* \succ x_*$, Axiom N holds. Assume $E \succeq^C E'$. For any $x^*, x_* \in X$ with $x^* \succ x_*$ and $f \in \mathcal{F}$, write

$$\begin{aligned} x^* E x_* E' f = x^* \, \xi \cup \xi' \, x_* \, E' \, f & \\ & \text{where } \xi \cup \xi' = E \text{ and } \xi' \approx E' \\ & \sim \quad \underbrace{x^* \, \xi \cup E'}_{\text{By definition of } \approx} x_* \, \xi' \, f \\ & \succeq x^* E' \underbrace{x_* \, \xi \cup \xi' \, f}_{\text{By P3}} = x^* E' x_* E f \end{aligned}$$

Note that by Axiom N and P3 $E \succ^C E' \;\Rightarrow\; x^* E x_* E' f \succ x^* E' x_* E f$. The latter along with completeness of $\succeq^C$ implies that $x^* E x_* E' f \succeq x^* E' x_* E f \;\Rightarrow E \succeq^C E'$. ∎

Proof of Proposition 2: Grant's axioms state that for any $x, y \in X$, $h \in \mathcal{F}$ and disjoint non-null $E, E' \in \Sigma$,

P3^{CU}: $\quad x(E \cup E')f \succ y(E \cup E')f \;\Rightarrow\; xEyE'f \succ y(E \cup E')f$

P3^{CL}: $\quad x(E \cup E')f \succ y(E \cup E')f \;\Rightarrow\; x(E \cup E')f \succ xEyE'f$

Suppose there are pairwise disjoint $E, E', A \in \Sigma$, $x(E \cup A)x'E'f \sim xEx'(E' \cup A)f$ for every $x, x' \in X$ and $f \in \mathcal{F}$. Specializing to $f = x'$, it must be that $x(E \cup A)x' \sim xEx'$

for every $x, x' \in X$. Note that this is not consistent with the hypothesis when E is null unless A is null. Assuming E is not null, there are $y, y' \in X$ such that $x \succ y(E \cup A)x$ and $y'(E \cup A)x \succ x$. If P3^{CU} is satisfied and A is not null, it must be that $xAyEx = yEx \succ y(E \cup A)x$, a contradiction. On the other hand, if P3^{CL} is satisfied and A is not null, it must be that $y'(E \cup A)x \succ y'Ex = xAy'Ex$, also a contradiction. Thus A is null. Thus, under the hypothesis, for any disjoint $E, E', A \in \Sigma$, if $x(E \cup A)x'E'f \sim xEx'(E' \cup A)f$ for every $x, x' \in X$ and $f \in \mathcal{F}$ then A is null. We now demonstrate that the latter property (Property †) implies Axiom N.

Suppose $E, A, E' \in \Sigma$ are disjoint, such that $E \approx E'$, and A is non-null. Assume it is not the case that $E \cup A \succ^C E'$. Since $E \cup A$ and E' are comparable, $E' \succeq^C E \cup A$. Thus E' contains a subset, ξ', that is exchangeable with $E \cup A$. In particular, by exchanging ξ' for $E \cup A$, we have for any $x, x' \in X$ and $f \in \mathcal{F}$ that

$$x'(E \cup A)xE'f \sim x'\xi'x\big((E \cup A) \cup (E' \setminus \xi')\big)f = x'\xi'x\big(E \cup (A \cup E' \setminus \xi')\big)f$$

Similarly, by exchanging E with E' it follows that

$$x'(E \cup A)xE'f \sim x'(E' \cup A)xEf = x'\big(\xi' \cup (A \cup E' \setminus \xi')\big)xEf$$

Note that the set $A \cup E' \setminus \xi'$ is not null, since A is not null. Property † is therefore violated, meaning that it cannot be the case that $E' \succeq^C E \cup A$. Thus $E \cup A \succ^C E'$. ∎

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