Equilibrium Foundations of Continuous-Time Finance Lecture Notes

1 Preface

This is a set of Lecture Notes for a five-hour tutorial given at the Institute of Mathematical Sciences of the National University of Singapore in June 2005. Much of the material is extracted from a much longer set of lecture notes I developed for a forty-five hour Ph.D. class in continous-time finance in the Haas School of Business at UC Berkeley. That class made extensive use of Nielsen's excellent text [14]. In my view, Nielsen gives the best combined treatment of continuous-time Finance and the mathematics underlying it. Other authors provide a much broader range of applications, but skimp on the mathematics, or give elegant treatments of the mathematics but skimp on the applications. Nielsen proves many of the main mathematical results, and provides clear statements of the remaining ones, along with references to their proofs. His mathematical treatment is impeccable.

The motivational material on Brownian motion, and particularly the connection to random walks, comes largely from a wonderful set of lectures by Shizuo Kakutani that I was fortunate to attend at Yale in 1974. Those lectures played a critical role in the development of my nonstandard construction of Brownian motion and Itô integration.

The class at Berkeley, like Nielsen's text, focussed on a careful development of the pricing and replication of derivative securities, such as options (initiated by Black, Scholes and Merton), using the martingale method initiated by Harrison and Kreps [7]). This is the central material of continuous-time finance. In it, the stochastic process describing the evolution of prices of securities is exogenously specified. Taking the securities price process as given, the martingale method examines the relationship between the exogenously specified prices of the underlying securities and the prices of derivative securities. I strongly recommend that anyone interested in pursuing studies in continuous-time Finance read Nielsen's text.

The Singapore tutorial, by contrast, focussed on the *equilibrium foun-*

dations of continuous-time finance. The endowments and utility functions of the agents, and the dividends of the securities, are exogenously specified. The problem is to show that an equilibrium securities pricing process exists, and to characterize its properties. The discrete-time version of this problem has been solved, and was described in Felix Kubler's tutorial lectures. The continuous-time problem is very hard, and remains an area of active research. We describe a strategy for obtaining existence results using nonstandard analysis, and the results obtained to date using this strategy.

2 The Random Walk Model

The random walk model is a simple model of the evolution of a stock price. The increments in the random walk process are additive, while we normally think that changes in a stock price function multiplicatively. For this reason, we think of the random walk model as representing the natural logarithm of the stock price; equivalently, the stock price is the exponential of the random walk.

In the random walk model, information accrues in small discrete steps. Consider the time interval $[0, T]$. For $n \in \mathbf{N}$, divide the time interval into nT subintervals, each of length $\frac{1}{n}$. At time 0, the random walk process starts out at 0. At the beginning of each interval, toss a coin; if it comes out heads, the random walk process increases by $\frac{1}{\sqrt{n}}$ over the course of the interval; if the coin comes out tails, the random walk process decreases by $\frac{1}{\sqrt{n}}$ over the course of the interval.

Formally, the random walk model is specified as follows. The event space is $\Omega = \{-1, 1\}^{nT}$. Thus, every $\omega \in \Omega$ is a vector of $+1$s and -1s. Observe that Ω is finite, indeed $|\Omega| = 2^{nT}$. The collection of measurable events is $\mathcal{F}$, the collection of all subsets of Ω. The probability measure is $P(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{2^{nT}}$; thus, we assign equal probability $\frac{1}{2^{nT}}$ to every $\omega \in \Omega$. We consider two closely-related versions of the random walk process:

$$\begin{aligned} X_n(\omega, t) &= \sum_{k=1}^{\lfloor nt \rfloor} \frac{\omega_k}{\sqrt{n}} + \frac{(nt - \lfloor nt \rfloor)\omega_{\lfloor nt \rfloor + 1}}{\sqrt{n}} \\ \hat{X}_n(\omega, t) &= \sum_{k=1}^{\lfloor nt \rfloor} \frac{\omega_k}{\sqrt{n}} \end{aligned}$$

When dealing with a fixed n, we will typically omit the subscript n and write the random walk process as $X(\omega, t)$ or $\hat{X}(\omega, t)$.

Each $\omega \in \Omega$ corresponds one of the possible paths the random walk process might follow. $X(\omega, \cdot)$ denotes the function from $[0, T]$ to $\mathbf{R}$ defined by $X(\omega, \cdot)(t) = X(\omega, t)$; this is the *sample path* of the random walk process corresponding to ω. At time 0, we don't know which ω will occur, and thus we don't know which path $X(\omega, \cdot)$ the random walk will follow; we only know that one of the possible paths will occur. When we get to time T, we have been able to observe the full path of the random walk, and thus we know precisely which ω occurred.

The second term in the definition of X_n is a linear interpolation term which makes the paths $X_n(\omega, \cdot)$ into continuous functions; since the paths of Brownian motion are continuous functions, this has the mathematical advantage of putting X_n and Brownian motion into the same space. However, X_n has the disadvantage that for every small $\varepsilon > 0$, the evolution of the path $X_n(\omega, \cdot)$ over the interval $\left[\frac{k}{n}, \frac{k+1}{n}\right)$ is completely known at the time $\frac{k}{n} + \varepsilon$. The paths $\hat{X}_n(\omega, \cdot)$ of $\hat{X}_n$ are step functions, constant across time intervals of the form $\left[\frac{k}{n}, \frac{k+1}{n}\right)$ and discontinuous at the times $\frac{k}{n}$.

Suppose $t \in [0, T]$. The information revealed up to time t is $\omega_1, \omega_2, \ldots, \omega_{\lfloor nt \rfloor}$. Thus, the collection of measurable events at time t is

$$\mathcal{F}_t = \{A \in \mathcal{F} : \omega \in A, \omega'_k = \omega_k \text{ for } k \leq nt \Rightarrow \omega' \in A\}$$

$\hat{X}_n$ is *adapted* to the filtration $\{\mathcal{F}_t\}$, i.e. $\hat{X}_n(\cdot, t)$ is $\mathcal{F}_t$-measurable for all t; X_n is not adapted, because ω_{k+1} is revealed by $X_n\left(\omega, \frac{k}{n} + \varepsilon\right)$ for every positive ε.

The random walk has the following qualitative properties:

1. *Approximate Normality*: Fix $t = \frac{k}{n}$. Let $M(\omega, t)$ be the number of $+1$s in the first k coin tosses. $M(\omega, t)$ has the binomial distribution $b\left(k, \frac{1}{2}\right)$.

$$\begin{aligned} X(\omega, t) &= \frac{M(\omega, t) - (k - (M(\omega, t)))}{\sqrt{n}} \\ &= \frac{2\left(M(\omega, t) - \frac{k}{2}\right)}{\sqrt{n}} \end{aligned}$$

Since the expected value $E(M(\cdot,t)) = \frac{k}{2}$, $E(X(\cdot,t)) = 0$. Since the variance $\mathrm{Var}(M(\cdot,t)) = k\left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right) = \frac{k}{4}$, $\mathrm{Var}(X(\cdot,t)) = \frac{4\times\frac{k}{4}}{\left(\sqrt{n}\right)^2} = \frac{k}{n} = t$. By the Central Limit Theorem, the distribution of $X(\cdot,t)$ is very nearly $N(0,t)$, normal with mean zero and variance t, hence standard deviation $\sqrt{t}$.

2. *Independent Increments*: Suppose $t_1 < t_2 < \cdots < t_m$. Then

$$\{\hat{X}(\cdot,t_2) - \hat{X}(\cdot,t_1), \ldots \hat{X}(\cdot,t_m) - \hat{X}(\cdot,t_{m-1})\}$$

are independent random variables because they're determined by disjoint sets of coin tosses

$$\{\omega_{\lfloor nt_1\rfloor+1}, \ldots, \omega_{\lfloor nt_2\rfloor}\}, \ldots, \{\omega_{\lfloor nt_{m-1}\rfloor+1}, \ldots, \omega_{\lfloor nt_m\rfloor}\}$$

The same is true of the increments of X, provided we restrict the times to the form $t_i = \frac{k_i}{n}$.

3. *Tightness*: This is technical and you don't need a full understanding. The random walk paths are obviously continuous, since they are given by functions that are linear on each of the intervals $\left[\frac{k}{n}, \frac{k+1}{n}\right]$. However, as n increases, each of these linear functions, which has slope $\sqrt{n}$, becomes steeper. Roughly speaking, tightness says that the random walk paths nonetheless have continuous limits, with probability one. Technically, the condition is

$$\forall_{\varepsilon>0}\,\exists_{\delta>0}\,\forall_n\, P\left(\{\omega : \exists_{s,t}\, |s-t| < \delta, |X_n(\omega,t) - X_n(\omega,s)| > \varepsilon\}\right) < \varepsilon$$

4. *Variation of Paths*: Given a function $f : [0,T] \to \mathbf{R}$, the variation of f is

$$\sup_{m\in\mathbf{N}} \sup_{0=t_0<t_1<\cdots<t_m=T} \sum_{k=1}^{m} |f(t_k) - f(t_{k-1})|$$

f is said to be of bounded variation if the variation of f is finite. For all ω, the variation of the path $X(\omega,\cdot)$ is $nT\left(\frac{1}{\sqrt{n}}\right) = \sqrt{n}T \to \infty$ as $n \to \infty$. In other words, all of the random walk paths are of variation tending to infinity as $n \to \infty$.

5. *Quadratic Variation*: By analogy with the variation, it would be natural to try to define quadratic variation pathwise: given a function $f : [0,T] \to \mathbf{R}$, we could define the quadratic variation of f to be

$$\sup_{m\in\mathbf{N}} \sup_{0=t_0<t_1<\cdots<t_m=T} \sum_{k=1}^{m} \left(f(t_k) - f(t_{k-1})\right)^2$$

For all ω, if we take $t_k = \frac{k}{n}$, $\sum_{k=1}^{m} \left(f(t_k) - f(t_{k-1})\right)^2 = nT\left(\frac{1}{(\sqrt{n})^2}\right) = T$. As you will see in Problem Set 1, problems arise if we attempt to define the quadratic variation one path at a time, in particular if we are allowed to choose the partition $0 = t_0 < t_1 < \cdots < t_m = T$ as a function of ω. Thus, the quadratic variation needs to be defined taking the whole process into account, not one path at a time.

3 The Brownian Motion Model

The Brownian Motion Model is the limit of the random walk model as $n \to \infty$. This can be made precise in a number of ways.[1]

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{T}$ a time set, with either a finite time horizon (i.e. $\mathcal{T} = [0,T]$ for some $T \in \mathbf{R}$) infinite time horizon (i.e. $\mathcal{T} = [0,\infty)$).

A K-dimensional stochastic process is $X : \Omega\times\mathcal{T} \to \mathbf{R}^K$ such that $X(\cdot,t) : \Omega \to \mathbf{R}^K$ is measurable in ω for all $t \in \mathcal{T}$. $X(\omega,\cdot)$ is the function from $\mathcal{T}$ to $\mathbf{R}^K$ defined by $X(\omega,\cdot)(t) = X(\omega,t)$. $X(\omega,\cdot)$ is called a *sample path* of the process; it is one of the (usually infinitely) many possible paths the process could follow.

You will need to distinguish three different measures floating around:

[1] One of the natural ways is Donsker's Theorem. Let $X_n(\omega,t)$ denote the random walk model of Section 2 for a specific $n \in \mathbf{N}$. View $X_n(\omega,\cdot)$ as a random variable taking values in $C([0,T])$, the space of continuous functions from $[0,T]$ into $\mathbf{R}$, with the metric $d(f,g) = \sup_{t\in[0,T]} |f(t) - g(t)|$. Donsker's Theorem asserts that Brownian motion $B(\omega,t)$ is the limit in distribution of X_n as $n \to \infty$. The notion of convergence in distribution of random variables taking values in $C([0,T])$ is the following: for every bounded continous function $F : C([0,T]) \to \mathbf{R}$, $E(F(x_n(\omega,\cdot))) \to E(F(B(\omega,\cdot)))$. It is not hard to see that this is a generalization of the definition of convergence in distribution for random variables taking values in $\mathbf{R}$. For details, see Billingsley [3]. An alternative is to use nonstandard analysis to show that $B(\omega,t)$ can be constructed directly from a so-called "hyperfinite" random walk, as in Anderson [1].

1. P, the probability measure on Ω.
2. λ, Lebesgue measure on $\mathcal{T}$.
3. $P \otimes \lambda$, the product measure on $\Omega \times \mathcal{T}$ generated by P and λ.

For more information, see sections Appendices A.2 and B.2 of Nielsen.

Definition 3.1 A K-dimensional standard Brownian motion is a K-dimensional stochastic process B such that[2]

1. $B(\omega, 0) = 0$ almost surely (i.e. $P(\{\omega : B(\omega, 0) = 0\}) = 1$)
2. *Continuity*: $B(\omega, \cdot)$ is continuous almost surely. If Brownian motion is constructed as a limit of the random walk, this property comes from the tightness property of the random walk.
3. *Independent Increments*: If $0 \leq t_0 < t_1 < \cdots < t_m \in \mathcal{T}$,

   $$\{B(\cdot, t_1) - B(\cdot, t_0), \ldots, B(\cdot, t_m) - B(\cdot, t_{m-1})\}$$

   is an independent family of random variables. If Brownian motion is constructed as a limit of the random walk, this property comes from the independent increments property of the random walk.
4. *Normality*: If $0 \leq s \leq t$, $B(\cdot, t) - B(\cdot, s)$ is normal with mean $0 \in \mathbf{R}^K$ and covariance matrix $(t - s)I$, where I is the $K \times K$ identity matrix. If Brownian motion is constructed as a limit of the random walk, this property comes from the approximate normality of the random walk.

Theorem 3.2 *There is a probability space on which a K-dimensional standard Brownian motion exists.*

Example 3.3 *Time Change:* Given a K-dimensional standard Brownian motion B, let $Z(\omega, t) = B(\omega, \sigma^2 t)$. Thus, Z is obtained from B by speeding up time by a factor of σ^2. It is easy to see that Z satisfies all the properties of a standard Brownian motion, except that the covariance matrix of $B(\cdot, t) -$

[2]There is some redundancy among the conditions: continuity and independent increments imply normality (though not the specific mean and covariances given here), while independent increments and normality imply continuity.

$B(\cdot, s)$ is $\sigma^2(t-s)I$ when $s < t$. If we let $\hat{Z}(\omega, t) = \frac{Z(\omega,t)}{\sigma}$, then $\hat{Z}$ satisfies all the properties of standard Brownian motion. Thus, a constant time change of a standard Brownian motion is a scalar multiple of a (different) standard Brownian motion on the same probability space.

Theorem 3.4 *The sample paths of standard Brownian motion have the following qualitative properties:*

1. *Almost Sure Unbounded Variation*[3]*:*

$$P(\{\omega : \exists_{s<t}\ B(\omega, \cdot) \textit{ is of bounded variation on } [s,t]\}) = 0$$

2. *Almost Sure Nowhere Differentiability*[4]*:*

$$P(\{\omega : \exists_{t \in \mathcal{T}}\ B(\omega, \cdot) \textit{ is differentiable at } t\}) = 0$$

3. *Iterated Logarithm Laws*

   (a) *Long Run:*

$$P\left(\left\{\omega : \limsup_{t \to \infty} \frac{B(\omega,t)}{\sqrt{2t \ln \ln t}} = 1\right\}\right) = 1$$

   (b) *Short Run:*[5] *For all $t \in \mathcal{T}$*

$$P\left(\left\{\omega : \limsup_{s \searrow t} \frac{B(\omega,s) - B(\omega,t)}{\sqrt{2(s-t) \ln |\ln(s-t)|}} = 1\right\}\right) = 1$$

Remark 3.5 The Iterated Logarithm Laws are key to understanding the qualitative short-run and long-run behavior of Brownian motion. We will model stock prices by processes like $e^{(\mu - \sigma^2/2)t + \sigma B(\omega,t)}$. Consider first the short

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[3] This property can be derived from the variation of the random walk, but the argument is a bit subtle, as the variation is not continuous in $C([0,T])$.

[4] This property shows that, although Brownian motion paths are continuous, they are only barely continuous. A slightly weaker property (Brownian motion paths are almost surely not continuously differentiable on any open interval) follows immediately from almost sure unbounded variation.

[5] In Problem Set 1, you are asked to derive this from the Iterated Logarithm Law in the Long Run.

run. If s is close to t, then $\sqrt{s-t}$ is much bigger than $s-t$. $\ln|\ln s-t|$ goes to infinity as $s \to t$, but the growth rate is *very* slow. The Iterated Logarithm Law tells us that at times s arbitrarily close to t, $B(\omega, s)-B(\omega, t)$ will nearly hit both the upper and lower envelopes $\pm\sqrt{2(s-t)\ln|\ln(s-t)|}$ infinitely often. In particular, in the short run, only the the volatility matters; the drift term $e^{(\mu-\sigma^2/2)t}$ is completely unimportant. On the other hand, in the long run, as $t \to \infty$, $\sqrt{t} \to \infty$ much slower than t; $\ln \ln t \to \infty$, but *very* slowly. We will see that $E(Z(\cdot, t)) = e^{\mu t}$. This explains why we choose to write $\mu - \sigma^2/2$, rather than incorporate the $-\sigma^2/2$ into μ. In the long run, if $\mu > 0$, the volatility is overwhelmed in importance by the drift term $e^{\mu t}$.

You will see, in Problem Set 1, that the Quadratic Variation of the Random Walk cannot be defined pathwise; if the partition is allowed to depend in an arbitrary way on the path, the Quadratic Variation need not converge as $n \to \infty$. For the same reason, the Quadratic Variation of Brownian Motion is not defined pathwise. The following theorem says the the Quadratic Variation of Brownian Motion over every interval $[s, t]$ with $s < t$ is $t - s$:

Theorem 3.6 *Let B be a standard 1-dimensional Brownian Motion. Consider a sequence of partitions*

$$s = t_0^n < t_1^n < \cdots < t_{m_n}^n = t$$

indexed by n with

$$\max\left\{\left|t_k^n - t_{k-1}^n\right| : 1 \le k \le m_n\right\} \to 0$$

Then

$$\sum_{k=1}^{m_n}\left(B\left(\omega, t_k^n\right) - B\left(\omega, t_{k-1}^n\right)\right)^2 \to t - s \quad a.s.$$

as $n \to \infty$.

The theorem follows from the Strong Law of Large Numbers, using the fact that $B\left(\cdot, t_k^n\right) - B\left(\omega, t_{k-1}^n\right)$ is distributed as $N\left(0, t_k^n - t_{k-1}^n\right)$, so

$$E\left(\left(B\left(\cdot, t_k^n\right) - B\left(\cdot, t_{k-1}^n\right)\right)^2\right) = t_k^n - t_{k-1}^n$$

Proposition 3.7 *If B is standard Brownian motion, $\frac{B(\omega,t)}{t} \to 0$ almost surely, i.e.*

$$P\left(\left\{\omega : \lim_{t\to\infty} \frac{B(\omega,t)}{t} = 0\right\}\right) = 1$$

Proof: Notice that from the definition of standard Brownian motion,

$$\text{Var}\left(\frac{B(\cdot,t)}{t}\right) = \frac{\text{Var}B(\cdot,t)}{t^2} = \frac{t}{t^2} \to 0$$

so $\frac{B(\cdot,t)}{t}$ converges to zero in distribution. However, convergence almost surely is stronger than convergence in distribution. Thus, we apply the Iterated Logarithm Law in the Long Run.

$$\limsup_{t\to\infty} \frac{B(\omega,t)}{t} \leq \limsup_{t\to\infty} \frac{B(\omega,t)}{\sqrt{2t\ln\ln t}} \times \limsup_{t\to\infty} \frac{\sqrt{2t\ln\ln t}}{t} = 1 \times 0 = 0$$

almost surely. Since $-B$ is standard Brownian motion,

$$\liminf \frac{B(\omega,t)}{t} = -\limsup_{t\to\infty} -\frac{B(\omega,t)}{t} = 0$$

almost surely. Therefore, $\limsup_{t\to\infty} \frac{B(\omega,t)}{t} = 0$ almost surely. ∎

4 Information Structures

Recall that in the random walk, we defined $\mathcal{F}_t$ to be the collection of events definable in terms ofcoin tosses that had occurred up to time $\frac{\lfloor nt \rfloor}{n}$; it simply represents the information available at time t. Equivalently, $\mathcal{F}_t$ is the σ-algebra determined by $\hat{X}$ up to time t, or by X up to time $\frac{\lfloor nt \rfloor}{n}$. We need to extend this definition to continuous-time processes where the probability space is infinite.

Definition 4.1 A *filtration* is a family $(\mathcal{F}_t)_{t\in\mathcal{T}}$ of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$ whenever $s \leq t$. A filtration is *augmented* (sometimes called *complete*) if

$$C \subset B, P(B) = 0 \;\;\Rightarrow\;\; \forall_{t\in\mathcal{T}}\, C \in \mathcal{F}_t$$

A stochastic process Z is *adapted* to $(\mathcal{F}_t)_{t\in\mathcal{T}}$ if $Z(\cdot,t)$ is $\mathcal{F}_t$-measurable for all $t \in \mathcal{T}$. Every stochastic process Z generates a filtration: $\mathcal{F}_t$ is, roughly speaking, the σ-algebra of events revealed by Z up to and including time t. More formally, $\mathcal{F}_t$ is the smallest σ-algebra containing

$$\{\{\omega : Z(\cdot, s) \in (a, b)\}, s \leq t, a, b \in \mathbf{R}\}$$

Every stochastic process Z is adapted to the filtration it generates. $\hat{X}$ is adapted to the filtration we defined in the random walk model, but X is *not* adapted to that filtration.

Remark 4.2 $\mathcal{F}_t$ is interpreted as the information which has been revealed by time t. Suppose Z is a trading strategy, i.e. $Z(\omega, t)$ specified how many shares of each stock an individual will hold at (ω, t). Then Z must be adapted; the individual can't make decisions based on information that hasn't yet been revealed. There is another reason to insist that trading strategies must be adapted. If we allowed trading strategies that are not adapted, there would be arbitrage. An example of a non-adapted trading strategy would be "buy the stock today if its price will be higher tomorrow, but sell it short today if its price will be lower tomorrow." Its clear that this strategy guarantees a profit, and the profit can be made arbitrarily large by increasing the number of shares that are bought or sold short; thus, if nonadapted trading strategies were allowed, individuals would take actions that would force the price today to change to eliminate the arbitrage, and this price change would reveal the information on which the individuals were basing their trades, enlarging the filtration. Notice, however, that requiring that trading strategies be adapted imposes a fundamental limitation, because it does not allow us to study situations with asymmetric information. In reality, the information possessed varies considerably from one individual to another. Market microstructure focusses on how agents that are better informed than others use that information, and how the information is incorporated into prices as a result. The continuous-time formulation makes it difficult or impossible to address those kinds of questions; in effect, it is assumed that all agents see the same information at any given time.

Definition 4.3 Let $(\Omega, \mathcal{F}, P)$ be a probability space. A stochastic process Z is *measurable* if it is measurable with respect to the product σ-algebra on $\Omega \otimes \mathcal{T}$. Z is *integrable* if $Z(\cdot, t)$ is integrable for all $t \in \mathcal{T}$. Suppose Y is a random variable which is integrable and $\mathcal{G} \subset \mathcal{F}$. The *conditional expectation* of Y with respect to $\mathcal{G}$ is a random variable $W = E(Y|\mathcal{G})$ such that W is

$\mathcal{G}$-measurable, and $\int_G W dP(\omega) = \int_G Y dP(\omega)$ for all $G \in \mathcal{G}$. The existence of the conditional expectation is proven using the Radon-Nikodym Theorem; any two conditional expectations agree almost surely.

Remark 4.4 When Ω is finite, as in the random walk model, any σ-algebra $\mathcal{G}$ must be the collection of all unions of elements of a partition of Ω. For example, given $t < T$, we can define the partition of Ω determined by $\omega_1, \omega_2, \ldots, \omega_{\lfloor nt \rfloor}$. The partition sets are sets of the form

$$\{\omega' \in \Omega : \omega'_k = \omega_k \, (1 \leq k \leq nt)\}$$

Define $\omega' \sim_t \omega$ if $\omega'_k = \omega_k$ for $k \leq nt$. This partition generates the σ-algebra $\mathcal{F}_t$ in the sense that $\mathcal{F}_t$ consists precisely of all unions of partition sets. $E(W|\mathcal{F}_t)$ is computed by taking the average value of W over each of the partition sets:

$$E(W|\mathcal{F}_t)(\omega) = \frac{\sum_{\{\omega' : \omega' \sim_t \omega\}} W(\omega)}{|\{\omega' : \omega' \sim_t \omega\}|}$$

Definition 4.5 Z is a *martingale* with respect to a filtration $(\mathcal{F}_t)_{t \in \mathcal{T}}$ if

1. Z is integrable
2. Z is adapted to $(\mathcal{F}_t)_{t \in \mathcal{T}}$
3. For all $s, t \in \mathcal{T}$ with $s \leq t$

   $$E\left(Z(\cdot, t) | \mathcal{F}_S\right) = Z(\cdot, s)$$

   almost surely.

Example 4.6 If we let $\mathcal{T} = [0, T]$, the random walk $\hat{X}_n(\omega, t)$ is a martingale with respect to $(\mathcal{F}_t)_{t \in \mathcal{T}}$. To see this, compute

$$\begin{aligned} & E(\hat{X}_n(\cdot, t)|\mathcal{F}_s)(\omega_0) \\ = & \sum_{\omega \sim_s \omega_0} \frac{\hat{X}_n(\omega, t)}{2^{n(t-s)}} \\ = & \hat{X}_n(\omega_0, s) + \sum_{\omega \sim_s \omega_0} \sum_{k=ns+1}^{nt} \frac{\omega_k}{2^{n(t-s)}\sqrt{n}} \end{aligned}$$

$$
\begin{aligned}
&= \hat{X}_n(\omega_0, s) + \sum_{k=ns+1}^{nt} \sum_{\omega \sim_s \omega_0} \frac{\omega_k}{2^{n(t-s)}\sqrt{n}} \\
&= \hat{X}_n(\omega_0, s) + \sum_{k=ns+1}^{nt} \frac{0}{2^{n(t-s)}\sqrt{n}} \\
&= \hat{X}_n(\omega_0, s)
\end{aligned}
$$

since each ω_k is 1 exactly half the time and -1 exactly half the time. Note that X_n is not a martingale on $[0, T]$; it is not adapted to the filtration $\{\mathcal{F}_t\}_{t \in [0,T]}$, and it is not a martingale with respect to the filtration it generates. If we restrict the time set to the set of points $\left\{0, \frac{1}{n}, \ldots, T\right\}$, X_n is a martingale on this restricted time set.

Example 4.7 If B is standard Brownian motion, then B is a martingale with respect to the filtration it generates. This follows from the fact that the increment $B(\cdot, t) - B(\cdot, s)$ has mean zero and is independent of $\mathcal{F}_s$, while $B(\cdot, s)$ is measurable with respect to $\mathcal{F}_s$:

$$
\begin{aligned}
&E\left(B(\cdot, t)|\mathcal{F}_s\right)(\omega_0) \\
&\quad = E\left(B(\omega_0, s) + B(\cdot, t) - B(\cdot, s)|\mathcal{F}_s\right)(\omega_0) \\
&\quad = B(\omega_0, s) + E\left(B(\cdot, t) - B(\cdot, s)|\mathcal{F}_s\right)(\omega_0) \\
&\quad = B(\omega_0, s) + 0
\end{aligned}
$$

5 Stochastic Integrals and Capital Gains

Stochastic integrals are essential to defining the capital gains generated by a trading strategy. In this section, we motivate the stochastic integral by considering the random walk model.

As we have seen, the most common model for stock prices is the exponential of a generalized Brownian motion. Since we're just trying to motivate the stochastic integral, we pretend that the stock price is given by the random walk process X_n, rather than e^{X_n} or e^B. In particular, we assume there is only one stock. Let $\mathcal{T} = \{0, 1/n, \ldots, nT\}$.

Suppose an individual uses the trading strategy $\bar{\Delta}(\omega, t)$. In other words, $\bar{\Delta}$ is a stochastic process which tells the individual how many shares to hold at time t, when the state is ω. We require that $\bar{\Delta}$ be adapted with respect to the filtration $(\mathcal{F}_t)_{t\in\mathcal{T}}$. This will be true if and only if $\bar{\Delta}(\omega, t)$ depends only on $\omega_1, \ldots, \omega_{\lfloor nt \rfloor}$. We assume that $\bar{\Delta}(\omega, t)$ is constant on intervals of the form $\left[\frac{k}{n}, \frac{k+1}{n}\right)$; thus, the individual changes his/her portfolio holdings only at times $t \in \mathcal{T}$; this assumption does not alter the set of possible portfolio returns available to the individual.

What is the capital gain generated by the trading strategy $\bar{\Delta}(\omega, t)$? The capital gain between time k/n and $\frac{k+1}{n}$ is

$$\bar{\Delta}\left(\omega, \frac{k}{n}\right)\left(X\left(\omega, \frac{k+1}{n}\right) - X\left(\omega, \frac{k}{n}\right)\right)$$

so the capital gains process up to time $t \in \mathcal{T}$ is

$$\begin{aligned} G(\omega, t) &= \sum_{k=0}^{nt-1} \bar{\Delta}\left(\omega, \frac{k}{n}\right)\left(X\left(\omega, \frac{k+1}{n}\right) - X\left(\omega, \frac{k}{n}\right)\right) \\ &= \sum_{k=0}^{nt-1} \bar{\Delta}\left(\omega, \frac{k}{n}\right)\frac{\omega_{k+1}}{\sqrt{n}} \end{aligned}$$

This is called a Riemann-Stieltjes integral with respect to the integrator $X(\omega, \cdot)$; it is formed by taking values of the integrand $\bar{\Delta}$ and multiplying by changes in the value of the integrator X.

Riemann-Stieltjes integrals are normally defined provided that the integrand is continuous and the integrator is of bounded variation; the integrand $\bar{\Delta}$ is not continuous, but it is a step function, and the Riemann-Stieltjes integral is also defined in this case. The definition of $\int_a^b f(t)dg(t)$ begins with partitions: Suppose $a = t_0 < t_1 < \cdots < t_n = b$, then the Riemann-Stieltjes sum with respect to this partition is

$$\sum_{i=0}^{n-1} f(t_i)(g(t_{i+1}) - g(t_i))$$

The Riemann-Stieltjes integral is defined as the limit of the Riemann-Stieltjes sums as the partition gets finer and finer, provided the limit exists. Note that in defining the Riemann-Stieltjes integral with respect to the random walk, we need to consider partitions finer than the time points $\frac{k}{n}$ at which the random walk coins are tossed. The Riemann-Stieltjes integral makes perfect sense in the random walk model because the integrators (the paths of the random walk) are of bounded variation. It is true that the variation of the paths goes to infinity as n grows, but for each fixed n, every random walk path is piecewise linear, hence of bounded variation.

However, the paths of Brownian motion are almost surely *not* of bounded variation, so one cannot define the capital gain simply by taking a Riemann-Stieltjes integral. Itô finessed this problem by approximating the integrand by simple functions (functions which are piecewise constant over time). The Stieltjes integral makes sense if the integand is a simple function, even if the integrator is not of bounded variation. The properties of Brownian motion allowed Itô to extend this Stieltjes integral from adapted simple stochastic processes to adapted square-integrable stochastic processes. In the next section, we will give Itô's definition of the stochastic integral.[6]

6 Formal Definition of the Stochastic Integral

We begin with some preliminary material.

Suppose $(A, \mathcal{A}, \mu)$ is a measure space. We have three main examples in mind:

- $(\Omega, \mathcal{F}, P)$, the probability space representing the uncertainty
- $(\mathcal{T}, \mathcal{C}, \lambda)$, the Lebesgue measure space on the time set $\mathcal{T}$
- $(\Omega \otimes \mathcal{T}, \mathcal{F} \otimes \mathcal{C}, P \otimes \lambda)$, the product of the space of uncertainty and time.

Definition 6.1

$$L^1(A) = \{f : A \to \mathbf{R},\ f \text{ measurable},\ \int_A |f| d\mu < \infty\}$$

$$L^2(A) = \{f : A \to \mathbf{R},\ f \text{ measurable},\ \int_A f^2 d\mu < \infty\}$$

We identify two elements f, g of $L^1(A)$ or $L^2(A)$ if $f = g$ except on a set of μ measure zero. Note that if $A = \Omega$, L^2 is the set of random variables with

[6] An alternative approach is to use nonstandard analysis and construct Brownian motion as a hyperfinite random walk. The Itô integral with respect to the Brownian motion can be recovered from the Stieltjes integral with respect to the hyperfinite random walk; see Anderson [1] for details.

finite variances, and L^1 is the set of random variables with finite means. If $A = \Omega \otimes \mathcal{T}$, $L^1(A)$ and $L^2(A)$ are sets of stochastic processes. By Fubini's Theorem, if Z is a measurable process, then

$$\begin{aligned}
&\int_{\Omega\otimes\mathcal{T}} Z^2(\omega,t)d(P\times\lambda) \\
&\quad = \int_{\mathcal{T}}\left(\int_{\Omega} Z^2(\omega,t)dP\right)d\lambda \\
&\quad = \int_{\Omega}\left(\int_{\mathcal{T}} Z^2(\omega,t)d\lambda\right)dP
\end{aligned}$$

$L^1(A)$ and $L^2(A)$ are Banach spaces under the norms $\|f\|_1 = \int_A |f| d\mu$ on $L^1(A)$ and $\|f\|_2 = \left(\int_A f^2 d\mu\right)^{1/2}$ on $L^2(A)$. In other words, if we let $d_1(f,g) = \|f-g\|_1$ and $d_2(f,g) = \|f-g\|_2$ be the metrics induced by these norms, then $(L^1(A), d_1)$ and $(L^2(A), d_2)$ are complete metric spaces. A complete metric space is one with the property that every Cauchy sequence converges to an element of the metric space. Thus, if we have a sequence of functions $f_n \in L^2(A)$ and f_n is Cauchy, i.e.

$$\forall_{\varepsilon>0}\, \exists_N\, \forall_{m,n>N} \|f_m - f_n\|_2 < \varepsilon$$

then

$$\exists_{f\in L^2(A)}\, \|f_n - f\|_2 \to 0$$

The analogous property is true for $L^1(A)$.

We say that f_n converges to f in probability if, for every $\varepsilon > 0$,

$$P(\{\omega : |f_n(\omega) - f(\omega)| > \varepsilon\}) \to 0$$

We now turn to the definition of the Itô Integral. The physical interpretation of the Itô Integral arises from diffusion processes. The change dW of a Wiener process gives a standard diffusion, occurring at a constant rate. The integrand b specifies how fast the diffusion is occurring at a particular time s and state ω; for a smoke particle being bombarded by air molecules, the rate is a function of the temperature and pressure of the air, and the mass of the smoke particle. In finance, dW represents the volatility of the stock price, while the integrand b represents the portfolio holding. The class $\mathcal{L}^2$ in the following definition is the set of stochastic processes which can be Itô integrated.

Definition 6.2 Fix a filtration $(\mathcal{F}_t)_{t\in\mathcal{T}}$ and a K-dimensional Wiener process W with respect to $(\mathcal{F}_t)_{t\in\mathcal{T}}$. Let $\mathcal{L}^2$ denote the set of adapted, measurable processes $b : \Omega\otimes\mathcal{T} \to \mathbf{R}^m$ (where $\mathbf{R}^m$ may denote $\mathbf{R}^1$, $\mathbf{R}^K$, or $\mathbf{R}^{N\times K}$) such that

$$\int_0^t \|b(\omega, s)\|^2 ds < \infty$$

almost surely.[7] Let $\mathcal{H}^2 = \mathcal{L}^2 \cap L^2(\Omega\otimes\mathcal{T})$.

The definition of the Itô proceeds in stages, starting first with simple functions, then extending to $\mathcal{H}^2$, and finally extending to $\mathcal{L}^2$.

Step 1: First suppose $K = 1$, $b : \Omega \times \mathcal{T} \to \mathbf{R}$, and $T \in \mathcal{T}$. Fix $0 = t_0 < t_1 < \cdots < t_n = T$. Assume that $b \in \mathcal{H}^2$, and b is simple[8], i.e.

$$b(\omega, s) = b(\omega, t_k) \text{ for all } s \in [t_k, t_{k+1})$$

Define

$$\int_0^T b dW(\omega) = \sum_{k=0}^{n-1} b(\omega, t_k)(W(\omega, t_{k+1}) - W(\omega, t_k))$$

Observe that this is a Stieltjes integral; it makes sense, even though $W(\omega, \cdot)$ is not of bounded variation, because b is simple.

Lemma 6.3 (Itô Isometry) *If $b \in \mathcal{H}^2$ and b is simple, then*

$$\int_\Omega \left(\int_0^T b dW\right)^2 dP = \int_\Omega \int_0^T |b(\omega, s)|^2 \, ds \, dP$$

In other words,

$$\left\|\int_0^T b dW\right\|_2 = \|b\|_2$$

where the norm on the left side is the norm in $L^2(\Omega)$ and the norm on the right side is the norm in $L^2(\Omega \times [0, T])$.

---

[7] $\|b(\omega, s)\|$ denotes the Euclidean length of the scalar, vector, or matrix $b(\omega, s)$. For example, if $b(\omega, s)$ is an $N \times K$ matrix, $\|b(\omega, s)\|^2 = \sum_{ij} (b_{ij}(\omega, s))^2$.

[8] Our convention is different from that of Nielsen; his simple functions are left-continuous, while ours are right-continuous.

Proof:

$$
\begin{aligned}
&\int_\Omega \left(\int_0^T b\,dW\right)^2 dP \\
&= \int_\Omega \left(\sum_{k=0}^{n-1} b(\omega,t_k)\left(W(\omega,t_{k+1})-W(\omega,t_k)\right)\right)^2 dP \\
&= \int_\Omega \Bigg[\sum_{k=0}^{n-1} b^2(\omega,t_k)\left(W(\omega,t_{k+1})-W(\omega,t_k)\right)^2 \\
&\quad + 2\sum_{j<k} b(\omega,t_j)b(\omega,t_k)\left(W(\omega,t_{j+1})-W(\omega,t_j)\right)\left(W(\omega,t_{k+1})-W(\omega,t_k)\right)\Bigg] dP \\
&= \sum_{k=0}^{n-1}\left(\int_\Omega b^2(\omega,t_k)\,dP\right)\left(\int_\Omega \left(W(\omega,t_{k+1})-W(\omega,t_k)\right)^2 dP\right) \qquad (1)\\
&\quad +2\sum_{j<k}\left(\int_\Omega b(\omega,t_j)b(\omega,t_k)\left(W(\omega,t_{j+1})-W(\omega,t_j)\right) dP\right)\left(\int_\Omega \left(W(\omega,t_{k+1})-W(\omega,t_k)\right) dP\right) \\
&= \sum_{k=0}^{n-1}\left(\left(\int_\Omega b^2(\omega,t_k)\,dP\right)(t_{k+1}-t_k)\right)+0 \qquad (2)\\
&= \int_\Omega \sum_{k=0}^{n-1} b^2(\omega,t_k)(t_{k+1}-t_k))\,dP \\
&= \int_\Omega \int_0^T b^2(\omega,t)\,dt\,dP \\
&= \int_{\Omega\times[0,T]} \int_0^T b^2\,d(P\otimes\lambda)
\end{aligned}
$$

Equation (1) follows because $b(\cdot,t_j)$, $b(\cdot,t_k)$ and $W(\cdot,t_{j+1})-W(\cdot,t_j)$are independent of $W(\cdot,t_{k+1})-W(\cdot,t_k)$, while Equation (2) follows from the fact that $W(\cdot,t_{k+1})-W(\cdot,t_k)$ has mean zero and variance $t_{k+1}-t_k$. ■

Step 2: Extend the Itô Integral to $\mathcal{H}^2$. If $b\in\mathcal{H}^2$, fix n and let $t_k=\frac{k}{n}$, then define

$$b_n(\omega,t)=n\int_{t_{k-1}}^{t_k} b(\omega,s)\,ds \text{ if } t\in[t_k,t_{k+1})$$

For each time interval $[t_k,t_{k+1})$, $b_n(\omega,t)$ is the average of $b(\omega,\cdot)$ over the *previous* interval $[t_{k-1},t_k)$; this ensures that b_n is simple and adapted. Lusin's Theorem (which states, roughly speaking, that measurable functions are continuous functions on the complement of a set of arbitrarily small measure) then can be used to show that $\|b - b_n\|_2 \to 0$, so the sequence b_n is Cauchy in $L^2(\Omega \times [0,T])$. Thus, given $\varepsilon > 0$, there exists N such that if $m, n > N$, $\|b_m - b_n\|_2 < \varepsilon$. But by the Itô Isometry, if $m, n > N$

$$\begin{aligned} & \left\| \int_0^T b_m \, dW - \int_0^T b_n \, dW \right\|_2 \\ = & \left\| \int_0^T (b_m - b_n) \, dW \right\|_2 \\ = & \|b_m - b_n\|_2 \\ < & \varepsilon \end{aligned}$$

so the sequence $\int_0^T b_m \, dW$ is a Cauchy sequence in $L^2(\Omega)$, hence converges to a unique limit; we define $\int_0^T b \, dW$ to be this limit.

Step 3: Now suppose $b \in \mathcal{L}^2$, so $\int_0^T |b(\omega,s)|^2 \, ds < \infty$ almost surely in ω. Let

$$b_n(\omega,s) = \begin{cases} n & \text{if } b(\omega,s) > n \\ b(\omega,s) & \text{if } -n \le b(\omega,s) \le n \\ -n & \text{if } b(\omega,s) < -n \end{cases}$$

Then $b_n \in \mathcal{H}^2$ and that $\int_0^T |b_n - b|^2 \, ds \to 0$ almost surely (specifically, for each ω such that $b(\omega,\cdot) \in L^2([0,T])$). One can show that $\int_0^T b_m \, dW$ converges in probability; $\int_0^T b \, dW$ is defined to be the limit.

Step 4: If W is K-dimensional, and $b(\omega,s) \in \mathbf{R}^K$, define

$$\int_0^T b \, dW = \sum_{k=1}^K \int_0^T b_k \, dW_k$$

Notice that if we think of W as the price process of K stocks and b as the portfolio strategy, then $\int_0^T b \, dW$ is the capital gain from the portfolio, the sum of the capital gains on the individual stocks.

If W is K-dimensional, and $b(\omega,s) \in \mathbf{R}^{N \times K}$, define

$$\left(\int_0^T b \, dW \right)_j = \sum_{k=1}^K \int_0^T b_{jk} \, dW_k$$

Think of there being N stocks, each of whose price movements is determined by the components of the underlying Wiener process. b_{jk} give the coefficient of stock j on the k^{th} component of the Wiener process and $\int b \, dW$ gives the movement of the N-dimensional vector of stock prices. Note that if b is a K-dimensional vector process, the stochastic integral is a scalar process; if b is an $N \times K$ matrix process, the stochastic integral is an N-dimensional vector process.

The stochastic integral is better behaved mathematically for integrands $b \in \mathcal{H}^2$ than for integrands in $\mathcal{L}^2$. However, $\mathcal{H}^2$ is not closed under the manipulations we need to do in Finance, while $\mathcal{L}^2$ is; hence, we need to consider integrands in $\mathcal{L}^2$.

We have the following facts concerning the Itô Integral for integrands $b \in \mathcal{L}^2$:

- Our definition of $\int_0^T b\,dW$ was given for a single T, and is defined only up to a set of probability zero. Since the set of probability zero can be different for different choices of T, the paths of $\int_0^T b\,dW$ could be badly behaved. Fortunately, it is possible to choose a continuous version of the the integral, i.e. we may assume that except for a set of ω of probability zero, $\int_0^t b(\omega, s)\,dW(\omega, s)$ is continuous in t.

- Linearity:
$$\gamma \int_0^t a\,dW + \delta \int_0^t b\,dW = \int_0^t (\gamma a + \delta b)\,dW$$

- Time consistency: If $0 \leq s \leq t$, then
$$\int_0^s b\,dW = \int_0^t (\mathbf{1}_{\omega \times [0,s]} b)\,dW$$
where $\mathbf{1}_B$ denotes the indicator function of the set B.

- The Itô Integral is adapted, i.e. $\int_0^t b\,dW$ is an adapted process. This is easily seen to be true for simple processes in $\mathcal{H}^2$, and it is inherited as the integral is defined by limits.

- If Y is a $\mathcal{F}_s$-measurable random variable,
$$\int_s^t (Yb)\,dW = Y \int_s^t b\,dW$$
This would be trivial if the Itô were defined pathwise, but as we have seen, it is not. However, one can verify the property for simple processes in $\mathcal{H}^2$, and verify it is preserved when one takes limits.

The following proposition provides important additional properties of the Itô Integral when the integrand is in $\mathcal{H}^2$.

Proposition 6.4 *Let W be a K-dimensional Wiener process.*

1. *If $b : \Omega \times \mathcal{T} \to \mathbf{R}^K$ and $b \in \mathcal{H}^2$, then $\int_0^t b\, dW$ is a martingale.*[9]
2. *If $b, \beta : \Omega \times \mathcal{T} \to \mathbf{R}^K$, and $b, \beta \in \mathcal{H}^2$, then*

$$\begin{aligned}
\mathrm{Cov}\left(\int_s^t b\, dW, \int_s^t \beta\, dW \middle| \mathcal{F}_s\right) &= E\left(\left(\int_s^t b\, dW\right)\left(\int_s^t \beta\, dW\right) \middle| \mathcal{F}_s\right) \\
&= E\left(\int_s^t b \cdot \beta\, du \middle| \mathcal{F}_s\right) \\
&= \int_s^t E\left(b(u) \cdot \beta(u) | \mathcal{F}_s\right)\, du
\end{aligned}$$

3. *If $b : \Omega \times \mathcal{T} \to \mathbf{R}^{N\times K}$, and $b \in \mathcal{H}^2$, then*

$$\begin{aligned}
\mathrm{Cov}\left(\int_s^t b\, dW, \int_s^t b\, dW \middle| \mathcal{F}_s\right) &= E\left(\left(\int_s^t b\, dW\right)\left(\int_s^t b\, dW\right)^T \middle| \mathcal{F}_s\right) \\
&= E\left(\int_s^t bb^T\, du \middle| \mathcal{F}_s\right) \\
&= \int_s^t E\left(b(u)b(u)^T \middle| \mathcal{F}_s\right)\, du
\end{aligned}$$

   Thus, bb^T is called the instantaneous covariance matrix of the stochastic integral.

Corollary 6.5 *If $b, \beta : \Omega \times \mathcal{T} \to \mathbf{R}^K$, and $b, \beta \in \mathcal{H}^2$, and $0 \le s \le t \le u$, then*

$$\mathrm{Cov}\left(\int_s^t b\, dW, \int_t^u \beta\, dW \middle| \mathcal{F}_s\right) = 0$$

and

$$\mathrm{Cov}\left(\int_s^t b\, dW, \int_t^u \beta\, dW\right) = 0$$

[9]If $b \in \mathcal{L}^2$, it is not necessarily the case that $\int b\, dW$ is a martingale; indeed, there is no guarantee that $\int_0^t b\, dW \in L^1(\Omega)$, so the integrals in the definition of a martingale may not even be defined.

The previous Corollary shows that increments of stochastic integrals over disjoint time intervals are uncorrelated. As the following example shows, they are not generally independent.

Example 6.6 Let W be a 1-dimensional standard Wiener process, and

$$b(\omega,t) = \begin{cases} 1 & \text{if } W(\omega,s) < 1 \text{ for all } s < t \\ 0 & \text{otherwise} \end{cases}$$

Then $Z(\omega,t) = \int_0^t b(\omega,s)dW(\omega,s)$ follows the path $W(\omega,\cdot)$ up until the first time t at which $W(\omega,t) = 1$, at which point it stops. More formally, define $\tau(\omega) = \min\{t : X(\omega,t) = 1\}$; $\tau(\omega)$ is defined almost surely because $X(\omega,\cdot)$ is continuous almost surely. Then

$$Z(\omega,t) = W(\omega, t \wedge \tau(\omega))$$

where $t \wedge s$ denotes $\min\{t,s\}$. Notice that the increments of Z are *not* independent. Indeed, if $0 < s < t$ and $Z(\omega,s) = Z(\omega,s) - Z(\omega,0) = 1$, then the conditional probability that $Z(\omega,t) - Z(\omega,s) = 0$ is one. On the other hand, if $Z(\omega,s) = Z(\omega,s) - Z(\omega,0) < 1$, the conditional probability that $Z(\omega,t) - Z(\omega,s) = 0$ is zero.

Example 6.7 You found in the Problem Set that

$$\int_0^T \hat{X}_n \, dX_n = \frac{1}{2}\left(X_n^2(\omega,T) - T\right)$$

A slightly more elaborate argument shows that if W is a one-dimensional Wiener process,

$$\int_0^T W \, dW = \frac{1}{2}\left(W^2(\omega,T) - T\right)$$

Remember that the approximations to the integrand W used in defining the integral are always adapted. Because the increments in the Wiener process are normally distributed, whereas the increments in the random walk are always $\pm 1/\sqrt{n}$, the argument needs to rely on the Law of Large Numbers.

Theorem 6.8 (Martingale Representation Theorem) *Let W be a K-dimensional standard Brownian motion. If Z is a martingale with respect to the filtration generated by W, then there exists $b \in \mathcal{L}^2$ such that*

$$Z(\omega,t) = Z(\omega,0) + \int_0^t b dW(\omega,s)$$

If $Z(\cdot,T) \in L^2$, then $b \in \mathcal{H}^2$ on $\Omega \times [0,T]$.

Remark 6.9 This is a truly remarkable result.

1. It is hard to give a discrete intuition for it because, in essence, it is a theorem about the filtration generated by a Brownian motion. Note that the theorem implies that Z has a continuous version, so every martingale with respect to the filtration generated by a Brownian motion is continuous. There is something about the filtration that forces the release of new information to be done in a continuous way. You cannot capture sudden events (whether anticipated, such as the press release which follows each meeting of the Federal Reserve Open Market Committee changes the discount rate, or unanticipated, such as a large corporation announcing that it is retracting the last several years of its audited income statements) in a stock price model based on Brownian motion. It should be emphasized that the theorem assumes that Z is a martingale with respect to the filtration *generated* by W; it is not enough that Z be a martingale with respect to the filtration associated with a Wiener process, since that filtration may be larger than the filtration generated by the Wiener process.

2. It is very useful for Finance. Since Z is an Itô process, we can do Itô Calculus on Z. The theorem is essential in proving the Complete Markets Theorem, where it allows us to extract a trading strategy whose value process is a given martingale.

7 Itô Calculus

We want to study stock price processes of the form $e^{Z(\omega,t)}$ where Z is a generalized Brownian motion. In particular, we need to compute

$$\int_0^T \bar{\Delta}(\omega,t)\, de^{Z(\omega,t)}$$

the capital gain generated by a trading strategy $\bar{\Delta}$. Itô's Lemma gives us the key to defining the stochastic integral with respect to processes like $e^{Z(\omega,t)}$.

Fix a K-dimensional standard Wiener process W.

Definition 7.1 An N-dimensional *Itô process* is a stochastic process of the form

$$Z(\omega,t) = Z(\omega,0) + \int_0^t a(\omega,s)\, ds + \int_0^t b(\omega,s) dW(\omega,s) \tag{3}$$

where $a \in \mathcal{L}^1$ is an $N \times 1$ vector-valued process and $b \in \mathcal{L}^2$ is an $N \times K$ matrix-valued process. Note that a and b are allowed to depend on both ω and t. Itô processes are continuous and adapted. Every generalized Wiener process is an Itô process. a is called the *drift*, b the *dispersion*, and bb^T the *instantaneous covariance matrix* of Z. The following symbolic representations are all shorthand for Equation (3):

$$\begin{aligned} Z(t) &= X_0 + \int_0^t a\,ds + \int_0^t b\,dW \\ dZ(t) &= a(t)\,dt + b(t)\,dW(t) \\ dZ &= a\,dt + b\,dW \end{aligned}$$

If $D \subset \mathbf{R}^N$ is and $f : D \to \mathbf{R}$ is C^2, let

$$f_x(x) = \bigtriangledown f|_x = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N} \right)$$

denote the gradiant of f, viewed as a row vector, and let

$$f_{xx}(x) = Hf|_x = \left(\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right)$$

denote the Hessian matrix of f.

Theorem 7.2 (Itô's Lemma) *Let $D \subset \mathbf{R}^N$ be an open set, and Z an N-dimensional Itô process*

$$Z(t) = Z(0) + \int_0^t a\,ds + \int_0^t b\,dW$$

such that

$$P(\{\omega : Z(\omega, t) \in D \textit{ for all } t \in [0, T]\}) = 1$$

and $f : D \to \mathbf{R}$is C^2. Then $f(Z)$ is an Itô process, specifically $f(Z(t)) =$

$$f(Z(0)) + \int_0^t \left[f_x(Z)a + \frac{1}{2}\mathrm{tr}\left(b^T f_{xx}(Z) b \right) \right] ds + \int_0^t f_x(Z) b\,dW \qquad (4)$$

Remark 7.3 By analogy with the Fundamental Theorem of Calculus, the terms involving $f_x(Z)$ are expected, but the term involving $\mathrm{tr}\left(b^T f_{xx}(Z) b \right)$ is at first sight surprising. Note that

$$\mathrm{tr}(b^T f_{xx}(Z) b) = \sum_{i,j=1}^N \sum_{k=1}^K \frac{\partial^2 f}{\partial x_i \partial x_j}(Z) b_{ik} b_{jk}$$

b_{ik} is the coefficient of X_i on W_k, so $b_{ik}b_{jk}$ is the product of the coefficients of X_i and X_j on the *same* component k of the Wiener process W. Since

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(Z) b_{ik} b_{jk}$$

is integrated with respect to time t, the formula is saying, in effect that $(dW_k)^2 = dt$, but this is just reasserting that the quadratic variation of a Wiener process grows linearly with time. The term arises because the quadratic variation of the Wiener process is not zero, and hence the second order terms in the Taylor expansion of f matter. There are no terms corresponding to $b_{ik}b_{j\ell}$ with $k \neq \ell$, so the formula is saying, in effect, that $(dW_k)(dW_\ell) = 0$ if $k \neq \ell$. Itô's Lemma is often summarized by saying

$$(dW_k)(dW_\ell) = \delta_{k\ell} dt$$

where

$$\delta_{k\ell} = \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell \end{cases}$$

Example 7.4 [Black-Scholes Stock Price] The stock price in the Black-Scholes model is

$$S(t) = S(0) e^{(\mu - \sigma^2/2)t + \sigma W(t)}$$

Let

$$\begin{aligned} Z(t) &= \ln S(t) \\ &= \ln S(0) + \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W(t) \\ &= \ln S(0) + \int_0^t \left(\mu - \frac{\sigma^2}{2}\right) ds + \int_0^t \sigma \, dW(s) \end{aligned}$$

so Z is an Itô process, and

$$\begin{aligned} dZ &= \left(\mu - \frac{\sigma^2}{2}\right) ds + \sigma \, dW \\ S(t) &= e^{Z(t)} \\ dS &= de^Z \end{aligned}$$

$$\begin{aligned} &= \left[e^Z\left(\mu - \frac{\sigma^2}{2}\right) + \frac{\sigma^2}{2}e^Z\right] dt + e^Z \sigma\, dW \\ &= e^Z \mu\, dt \ + \ e^Z \sigma\, dW \\ &= S\mu\, dt \ + \ S\sigma\, dW \end{aligned}$$

so

$$\frac{dS}{S} = \mu\, dt \ + \sigma\, dW$$

$\frac{dS}{S}$ is the proportional change in S, so the proportional change in S has drift μ and instantaneous variance σ. This provides another explanation of why we write $S = S(0)e^{(\mu-\sigma^2/2)t+\sigma W(t)}$; the $-\sigma^2/2$ is needed to cancel out a $\sigma^2/2$ that comes from Itô's Lemma, resulting in instantaneous drift μ in the proportional change of S.

Plausibility Argument for Itô's Lemma: Here, we give a conceptually simple, but admittedly notationally messy, calculation verifying Itô's Lemma for Stieltjes integrals of simple integrands integrated with respect to a random walk. The intuition behind the standard proof of Itô's Lemma is very close to this argument, but complications arise because the Itô Integral is defined for general integrands by approximation, and because the relation $(\Delta W_j)(\Delta W_k) = \delta_{jk} dt$ is not true over finite time intervals. However, it is easy to see this relation holds for the random walk, The argument given here can be turned into a rigorous proof of Itô's Lemma using nonstandard analysis (Anderson [1]). Let

$$Y(t) = Y(0) + \int_0^t a\, ds + \int_0^T b\, dX$$

where X is a 2-dimensional n-step random walk. In other words, $\Omega = \{-1,1\}^{nT} \times \{-1,1\}^{nT}$, $\omega = (\omega_{\ell k} : \ell = 1,2,\ k = 1,\ldots,nT)$, $N = 2$ and $K = 2$. Suppose also that a and b are simple processes which are measurable in the filtration generated by the random walk; for simplicity, we assume here that a and b are uniformly bounded as $n \to \infty$. $O(h)$ denotes a quantity which is a bounded multiple of h as $h \to 0$, while $o(h)$ denotes a quantity which goes to zero faster than h as $h \to h$. For example, Taylor's Theorem says that if $g : \mathbf{R} \to \mathbf{R}$ is a C^2 function,

$$g(x+h) = g(x) + g'(x)h + \frac{1}{2}g''(x)h^2 + o(h^2)$$

Let

$$\begin{aligned}
\Delta Y\left(\omega,\frac{k}{n}\right) &= Y\left(\omega,\frac{k+1}{n}\right)-\left(\omega,\frac{k}{n}\right)\\
&= \left(\begin{array}{c} \frac{a_1(\omega,k/n)}{n}+\frac{b_{11}(\omega,k/n)\omega_{1(k+1)}+b_{12}(\omega,k/n)\omega_{2(k+1)}}{\sqrt{n}} \\ \frac{a_2(\omega,k/n)}{n}+\frac{b_{21}(\omega,k/n)\omega_{1(k+1)}+b_{22}(\omega,k/n)\omega_{2(k+1)}}{\sqrt{n}} \end{array}\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
&f(Y(\omega,T))-f(Y(\omega,0))\\
&= \sum_{k=0}^{nT-1}\left(f\left(Y\left(\omega,\frac{k+1}{n}\right)\right)-f\left(Y\left(\omega,\frac{k}{n}\right)\right)\right)\\
&= \sum_{k=0}^{nT-1}\bigtriangledown f|_{Y(\omega,k/n)}\cdot\Delta Y\left(\omega,\frac{k}{n}\right)+\frac{1}{2}\sum_{k=0}^{nT-1}\left(\Delta Y\left(\omega,\frac{k}{n}\right)\right)^T Hf|_{Y(\omega,k/n)}\Delta Y\left(\omega,\frac{k}{n}\right)\\
&\quad +no\left(\frac{1}{n}\right)
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=0}^{nT-1}\bigtriangledown f|_{Y(\omega,k/n)}\Delta Y\left(\omega,\frac{k}{n}\right)\\
&= \sum_{k=0}^{nT-1}\left.\frac{\partial f}{\partial x_1}\right|_{Y(\omega,k/n)}\left(\frac{a_1(\omega,k/n)}{n}+\frac{b_{11}(\omega,k/n)\omega_{1(k+1)}+b_{12}(\omega,k/n)\omega_{2(k+1)}}{\sqrt{n}}\right)\\
&\quad +\sum_{k=0}^{nT-1}\left.\frac{\partial f}{\partial x_2}\right|_{Y(\omega,k/n)}\left(\frac{a_2(\omega,k/n)}{n}+\frac{b_{21}(\omega,k/n)\omega_{1(k+1)}+b_{22}(\omega,k/n)\omega_{2(k+1)}}{\sqrt{n}}\right)\\
&= \int_0^T\left.\frac{\partial f}{\partial x_1}\right|_{Y(\omega,t)}a_1(\omega,t)\,dt+\int_0^T\left.\frac{\partial f}{\partial x_1}\right|_{Y(\omega,t)}b_{11}(\omega,t)\,dX_1(\omega,t)\\
&\quad +\int_0^T\left.\frac{\partial f}{\partial x_1}\right|_{Y(\omega,t)}b_{12}(\omega,t)\,dX_2(\omega,t)+\int_0^T\left.\frac{\partial f}{\partial x_2}\right|_{Y(\omega,t)}a_2(\omega,t)\,dt\\
&\quad +\int_0^T\left.\frac{\partial f}{\partial x_2}\right|_{Y(\omega,t)}b_{21}(\omega,t)\,dX_1(\omega,t)+\int_0^T\left.\frac{\partial f}{\partial x_2}\right|_{Y(\omega,t)}b_{22}(\omega,t)\,dX_2(\omega,t)\\
&= \int_0^T\bigtriangledown f|_{Y(\omega,t)}\,a(\omega,t)\,dt
\end{aligned}$$

$$+\int_0^T \left(\frac{\partial f}{\partial x_1}\bigg|_{Y(\omega,t)} b_{11}(\omega,t) + \frac{\partial f}{\partial x_2}\bigg|_{Y(\omega,t)} b_{21}(\omega,t) \right) dX_1(\omega,t)$$

$$+\int_0^T \left(\frac{\partial f}{\partial x_1}\bigg|_{Y(\omega,t)} b_{12}(\omega,t) + \frac{\partial f}{\partial x_2}\bigg|_{Y(\omega,t)} b_{22}(\omega,t) \right) dX_2(\omega,t)$$

$$= \int_0^T \bigtriangledown f|_{Y(\omega,t)}\, a(\omega,t)\, dt + \int_0^T \bigtriangledown f|_{Y(\omega,t)}\, b(\omega,t)\, dX(\omega,t)$$

Since ω_{1k} and ω_{2k} are independent, the product $\omega_{1k}\omega_{2k}$ equals $+1$ with probability $1/2$ and -1 with probability $1/2$, so we can form a random walk

$$\bar{X}\left(\omega, \frac{k}{n}\right) = \sum_{j=1}^{k} \frac{\omega_{1j}\omega_2 j}{\sqrt{n}}$$

$\bar{X}$ is a standard random walk, which in the limit is standard Brownian motion.

$$\frac{1}{2}\sum_{k=0}^{nT-1} \left(\Delta Y\left(\omega, \frac{k}{n}\right)\right)^T Hf|_{Y(\omega,k/n)} \Delta Y\left(\omega, \frac{k}{n}\right)$$

$$= \frac{1}{2}\sum_{k=0}^{nT-1} \left(\Delta Y_1\left(\omega, \frac{k}{n}\right), \Delta Y_2\left(\omega, \frac{k}{n}\right)\right) \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}\Big|_{Y(\omega,t)} & \frac{\partial^2 f}{\partial x_1 \partial x_2}\Big|_{Y(\omega,t)} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}\Big|_{Y(\omega,t)} & \frac{\partial^2 f}{\partial x_2^2}\Big|_{Y(\omega,t)} \end{pmatrix} \begin{pmatrix} \Delta Y_1\left(\omega, \frac{k}{n}\right) \\ \Delta Y_2\left(\omega, \frac{k}{n}\right) \end{pmatrix}$$

$$= \frac{1}{2}\sum_{k=0}^{nT-1} \frac{\partial^2 f}{\partial x_1^2}\bigg|_{Y(\omega,t)} \left(\Delta Y_1\left(\omega, \frac{k}{n}\right)\right)^2 + \frac{\partial^2 f}{\partial x_1^2}\bigg|_{Y(\omega,t)} \left(\Delta Y_2\left(\omega, \frac{k}{n}\right)\right)^2$$

$$+2\,\frac{\partial^2 f}{\partial x_1 \partial x_2}\bigg|_{Y(\omega,t)} \left(\Delta Y_1\left(\omega, \frac{k}{n}\right)\right)\left(\Delta Y_2\left(\omega, \frac{k}{n}\right)\right)\Bigg)$$

$$= \frac{1}{2}\sum_{k=0}^{nT-1} \frac{\partial^2 f}{\partial x_1^2}\bigg|_{Y(\omega,t)} \left(\frac{a_1(\omega,k/n)}{n} + \frac{b_{11}(\omega,k/n)\omega_{1(k+1)} + b_{12}(\omega,k/n)\omega_{2(k+1)}}{\sqrt{n}}\right)^2$$

$$+ \frac{\partial^2 f}{\partial x_1^2}\bigg|_{Y(\omega,t)} \left(\frac{a_2(\omega,k/n)}{n} + \frac{b_{21}(\omega,k/n)\omega_{1(k+1)} + b_{22}(\omega,k/n)\omega_{2(k+1)}}{\sqrt{n}}\right)^2$$

$$+2\,\frac{\partial^2 f}{\partial x_1 \partial x_2}\bigg|_{Y(\omega,t)} \left(\frac{a_1(\omega,k/n)}{n} + \frac{b_{11}(\omega,k/n)\omega_{1(k+1)} + b_{12}(\omega,k/n)\omega_{2(k+1)}}{\sqrt{n}}\right)$$

$$\times \left(\frac{a_2(\omega,k/n)}{n} + \frac{b_{21}(\omega,k/n)\omega_{1(k+1)} + b_{22}(\omega,k/n)\omega_{2(k+1)}}{\sqrt{n}}\right)\Bigg)$$

$$
\begin{aligned}
&\frac{1}{2}\sum_{k=0}^{nT-1}\left.\frac{\partial^2 f}{\partial x_1^2}\right|_{Y(\omega,t)}\left(\frac{a_1(\omega,k/n)}{n}+\frac{b_{11}(\omega,k/n)\omega_{1(k+1)}+b_{12}(\omega,k/n)\omega_{2(k+1)}}{\sqrt{n}}\right)^2\\
&= \frac{1}{2}\sum_{k=0}^{nT-1}\left.\frac{\partial^2 f}{\partial x_1^2}\right|_{Y(\omega,t)}\left(\frac{O(1)}{n^2}+\frac{O(1)}{n^{3/2}}\right.\\
&\quad\left.+\frac{(b_{11}(\omega,k/n)\omega_{1(k+1)})^2+(b_{12}(\omega,k/n)\omega_{2(k+1)})^2+2b_{11}(\omega,k/n)b_{12}(\omega,k/n)\omega_{1(k+1)}\omega_{2(k+1)}}{n}\right)\\
&= O\left(\frac{1}{n^{1/2}}\right)+\frac{1}{\sqrt{n}}\sum_{k=0}^{nT-1}\left.\frac{\partial^2 f}{\partial x_1^2}\right|_{Y(\omega,t)}\frac{b_{11}(\omega,k/n)b_{12}(\omega,k/n)\omega_{1(k+1)}\omega_{2(k+1)}}{\sqrt{n}}\\
&\quad+\frac{1}{2}\sum_{k=0}^{nT-1}\left.\frac{\partial^2 f}{\partial x_1^2}\right|_{Y(\omega,t)}\frac{b_{11}(\omega,k/n)^2+b_{12}(\omega,k/n)^2}{n}\\
&= O\left(\frac{1}{n^{1/2}}\right)+\frac{1}{\sqrt{n}}\int_0^T\left.\frac{\partial^2 f}{\partial x_1^2}\right|_{Y(\omega,t)} b_{11}(\omega,k/n)b_{12}(\omega,k/n)d\bar{X}\\
&\quad+\frac{1}{2}\int_0^T\left.\frac{\partial^2 f}{\partial x_1^2}\right|_{Y(\omega,t)}\left(b_{11}(\omega,t)^2+b_{12}(\omega,t)^2\right)\,dt\\
&= O\left(\frac{1}{\sqrt{n}}\right)+\frac{1}{2}\int_0^T\left.\frac{\partial^2 f}{\partial x_1^2}\right|_{Y(\omega,t)}\left(b_{11}(\omega,t)^2+b_{12}(\omega,t)^2\right)\,dt
\end{aligned}
$$

because the stochastic integral with respect to $\bar{X}$ is finite almost surely. Similarly,

$$
\begin{aligned}
&\frac{1}{2}\sum_{k=0}^{nT-1}\left.\frac{\partial^2 f}{\partial x_1^2}\right|_{Y(\omega,t)}\left(\frac{a_2(\omega,k/n)}{n}+\frac{b_{21}(\omega,k/n)\omega_{1(k+1)}+b_{22}(\omega,k/n)\omega_{2(k+1)}}{\sqrt{n}}\right)^2\\
&= O\left(\frac{1}{\sqrt{n}}\right)+\frac{1}{2}\int_0^T\left.\frac{\partial^2 f}{\partial x_2^2}\right|_{Y(\omega,t)}\left(b_{21}(\omega,t)^2+b_{22}(\omega,t)^2\right)\,dt
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\frac{1}{2}\sum_{k=0}^{nT-1}2\left.\frac{\partial^2 f}{\partial x_1\partial x_2}\right|_{Y(\omega,t)}\left(\left(\frac{a_1(\omega,k/n)}{n}+\frac{b_{11}(\omega,k/n)\omega_{1(k+1)}+b_{12}(\omega,k/n)\omega_{2(k+1)}}{\sqrt{n}}\right)\right.\\
&\qquad\left.\times\left(\frac{a_2(\omega,k/n)}{n}+\frac{b_{21}(\omega,k/n)\omega_{1(k+1)}+b_{22}(\omega,k/n)\omega_{2(k+1)}}{\sqrt{n}}\right)\right)\\
&= \sum_{k=0}^{nT-1}\left.\frac{\partial^2 f}{\partial x_1\partial x_2}\right|_{Y(\omega,t)}\left(O\left(\frac{1}{n^{3/2}}\right)+\frac{b_{11}(\omega,k/n)b_{21}(\omega,k/n)(\omega_{1(k+1)})^2}{n}\right.
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{b_{12}(\omega,k/n)b_{22}(\omega,k/n)(\omega_{2(k+1)})^2}{n} \\
& +\frac{(b_{11}(\omega,k/n)b_{22}(\omega,k/n)+b_{12}(\omega,k/n)b_{21}(\omega,k/n))\omega_{1(k+1)}\omega_{2(k+1)}}{n}\Bigg) \\
= \;& O\left(\frac{1}{\sqrt{n}}\right)+\sum_{k=0}^{nT-1}\frac{\partial^2 f}{\partial x_1\partial x_2}\bigg|_{Y(\omega,t)}\frac{b_{11}(\omega,k/n)b_{21}(\omega,k/n)+b_{12}(\omega,k/n)b_{22}(\omega,k/n)}{n} \\
+ \;& \frac{1}{\sqrt{n}}\sum_{k=0}^{nT-1}\frac{\partial^2 f}{\partial x_1\partial x_2}\bigg|_{Y(\omega,t)}\frac{(b_{11}(\omega,k/n)b_{22}(\omega,k/n)+b_{12}(\omega,k/n)b_{21}(\omega,k/n))\omega_{1(k+1)}\omega_{2(k+1)}}{\sqrt{n}} \\
= \;& O\left(\frac{1}{\sqrt{n}}\right)+\int_0^T\frac{\partial^2 f}{\partial x_1\partial x_2}\bigg|_{Y(\omega,t)}(b_{11}(\omega,t)b_{21}(\omega,t)+b_{12}(\omega,t)b_{22}(\omega,t))\,dt \\
+ \;& \frac{1}{\sqrt{n}}\int_0^T\frac{\partial^2 f}{\partial x_1\partial x_2}\bigg|_{Y(\omega,t)}(b_{11}(\omega,t)b_{22}(\omega,t)+b_{12}(\omega,t)b_{21}(\omega,t))\,d\bar{X} \\
= \;& O\left(\frac{1}{\sqrt{n}}\right)+\int_0^T\frac{\partial^2 f}{\partial x_1\partial x_2}\bigg|_{Y(\omega,t)}(b_{11}(\omega,t)b_{21}(\omega,t)+b_{12}(\omega,t)b_{22}(\omega,t))\,dt
\end{aligned}
$$

Combining the above calculations, and taking the limit as $n \to \infty$, we have

$$
\begin{aligned}
f(Y(\omega,T)) \;=\; & f(Y(\omega,0)) \\
& +\int_0^T \bigtriangledown f|_{Y(\omega,t)}\,a(\omega,t)\,dt+\int_0^T \bigtriangledown f|_{Y(\omega,t)}\,b(\omega,t)\,dX(\omega,t) \\
& +\frac{1}{2}\int_0^T\frac{\partial^2 f}{\partial x_1^2}\bigg|_{Y(\omega,t)} b_{11}(\omega,t)^2+b_{12}(\omega,t)^2\,dt \\
& +\frac{1}{2}\int_0^T\frac{\partial^2 f}{\partial x_2^2}\bigg|_{Y(\omega,t)} b_{21}(\omega,t)^2+b_{22}(\omega,t)^2\,dt \\
& +\frac{1}{2}\int_0^T 2\,\frac{\partial^2 f}{\partial x_1\partial x_2}\bigg|_{Y(\omega,t)}(b_{11}(\omega,t)b_{21}(\omega,t)+b_{12}(\omega,t)b_{22}(\omega,t))\,dt \\
= \;& \int_0^T\left(\bigtriangledown f|_{Y(\omega,t)}\,a(\omega,t)+\frac{1}{2}\mathrm{tr}(b(\omega,t)^T(Hf|_{Y(\omega,t)})b(\omega,t))\right)dt \\
& +\int_0^T \bigtriangledown f|_{Y(\omega,t)}\,b(\omega,t)\,dW(\omega,t)
\end{aligned}
$$

Proposition 7.5 (Uniqueness of Coefficients of Itô Processes) *Let $a,\alpha \in \mathcal{L}^1$ be N-dimensional, $b,\beta \in \mathcal{L}^2$ be $N\times K$-dimensional, and $X_0, Y_0 \in L^2$ be*

N-dimensional. If

$$X_0 + \int_0^t a\,ds + \int_0^t b\,dW = Y_0 + \int_0^t \alpha\,ds + \int_0^t \beta\,dW$$

for all t, almost surely in ω, then

$$\begin{aligned}
X_0(\omega) &= Y_0(\omega) \ \ P \textit{ almost surely} \\
a(\omega, t) &= \alpha(\omega, t) \ \ \lambda\otimes P \textit{ almost everywhere} \\
b(\omega, t) &= \beta(\omega, t) \ \ \lambda\otimes P \textit{ almost everywhere}
\end{aligned}$$

Proof: If $b, \beta \in \mathcal{H}^2$, this follows immediately from the Itô Isometry. Since we need the result when $b, \beta \in \mathcal{L}^2$, we use Itô's Lemma. It is sufficient to consider the case $N = 1$. Let

$$\begin{aligned}
Z(\omega, t) &= X_0 + \int_0^t a\,ds + \int_0^t b\,dW - \left(Y_0 + \int_0^t \alpha\,ds + \int_0^t \beta\,dW\right) \\
&= X_0 - Y_0 + \int_0^t (a - \alpha)\,ds + \int_0^t (b - \beta)\,dW \\
&= X_0 - Y_0 + \int_0^t \gamma\,ds + \int_0^t \delta\,dW
\end{aligned}$$

where $\gamma = a - \alpha$ and $\delta = b - \beta$. $Z(\omega, \cdot) = 0$ almost surely, so $X_0(\omega) - Y_0(\omega) = Z(\omega, 0) = 0$ almost surely. We show $\gamma = 0$ and $\delta = 0$ $\lambda\otimes P$-almost everywhere.

$$\begin{aligned}
0 &= e^{Z(\omega,t)} - 1 \\
&= e^{Z(\omega,0)} + \int_0^t \left[e^0\gamma + \frac{1}{2}e^0\delta^T\delta\right] ds + \int_0^t e^0\delta\,dW - 1 \\
&= \int_0^t \left[\gamma + \frac{1}{2}\delta^T\delta\right] ds + \int_0^t \delta\,dW \\
&= \int_0^t \gamma\,ds + \int_0^t \delta\,dW + \frac{1}{2}\int_0^t \delta^T\delta\,ds \\
&= Z(\omega, t) - Z(\omega, 0) + \frac{1}{2}\int_0^t \delta^T\delta\,ds \\
&= \frac{1}{2}\int_0^t \delta^T\delta\,ds
\end{aligned}$$

which implies that $\delta = 0$ ($P\otimes\lambda$-almost everywhere), so $\int_0^t \delta\,dW = 0$ for all t, so $\int_0^t \gamma\,ds = 0$ for all t, so $\gamma = 0$ ($P\otimes\lambda$-almost everywhere). ■

Corollary 7.6 (Proposition 2.7) *If the Itô process*

$$X(t) = X(0) + \int_0^t a\,ds + \int_0^t b\,dW$$

is a martingale with respect to the filtration generated by W*, then* $a = 0$ $P\otimes\lambda$*-almost everywhere.*

Remark 7.7 [Caution] The converse is true if $b \in \mathcal{H}^2$, but it is not generally true if $b \in \mathcal{L}^2$.

8 Integrals with respect to Itô Processes

Our basic model for a stock price will be geometric Brownian motion, which is an Itô Process but not a Wiener process. In order to compute the capital gain generated by a portfolio strategy, we need to be able to take Itô integrals with respect to Itô processes. Let

$$Z(t) = Z_0 + \int_0^t a\,ds + \int_0^t b\,dW$$

where

$$\begin{array}{rl} Z_0 \text{ is } \mathcal{F}_0\text{-measurable} & Z_0(\omega) \in \mathbf{R}^N \\ a \in \mathcal{L}^1 & a(\omega,t) \text{ is } N \times 1 \\ b \in \mathcal{L}^2 & b(\omega,t) \text{ is } N \times K \end{array}$$

Suppose that we replace the K-dimensional standard Wiener process W with the random walk X_n, and assume that a, b and γ are simple and adapted with respect to the random walk filtration. To simplify notation, we take $K = 1$. Then if $Z = Z_0 + \int a\,ds + \int b\,dX_n$,

$$Z\left(\omega, \frac{j}{n}\right) = \sum_{k=0}^{j-1} \frac{a\left(\omega, \frac{k}{n}\right)}{n} + \sum_{k=0}^{j-1} \frac{b\left(\omega, \frac{k}{n}\right)\omega_{k+1}}{\sqrt{n}}$$

so

$$\begin{aligned} \Delta Z\left(\omega, \frac{k}{n}\right) &= Z\left(\omega, \frac{k+1}{n}\right) - Z\left(\omega, \frac{k}{n}\right) \\ &= \frac{a\left(\omega, \frac{k}{n}\right)}{n} + \frac{b\left(\omega, \frac{k}{n}\right)\omega_{k+1}}{\sqrt{n}} \end{aligned}$$

$$\begin{aligned}
\int_0^T \gamma dZ &= \sum_{k=0}^{nT} \gamma\left(\omega, \frac{k}{n}\right) \Delta Z\left(\omega, \frac{k}{n}\right) \\
&= \sum_{k=0}^{nT} \gamma\left(\omega, \frac{k}{n}\right) \left(\frac{a\left(\omega, \frac{k}{n}\right)}{n} + \frac{b\left(\omega, \frac{k}{n}\right)\omega_{k+1}}{\sqrt{n}}\right) \\
&= \sum_{k=0}^{nT} \frac{\gamma\left(\omega, \frac{k}{n}\right) a\left(\omega, \frac{k}{n}\right)}{n} + \sum_{k=0}^{nT} \frac{\gamma\left(\omega, \frac{k}{n}\right) b\left(\omega, \frac{k}{n}\right)\omega_{k+1}}{\sqrt{n}} \\
&= \int_0^T \gamma(\omega, s)a(\omega, s)\, ds + \int_0^T \gamma(\omega, s)b(\omega, s)\, dX_n \\
&= \int_0^T \gamma a\, ds + \int_0^T \gamma b\, dX_n
\end{aligned}$$

Now, return to the situation in which W is a K-dimensional standard Wiener process. We see that we want

$$\int_0^t \gamma\, dZ = \int_0^t \gamma a\, ds + \int_0^t \gamma b\, dW$$

In order for this to make sense, we need to know that γa is integrable with respect to time and γb is Itô integrable with respect to W. This motivates the following definition:

Definition 8.1 Suppose

$$Z(t) = Z(0) + \int_0^t a\, ds + \int_0^t b\, dW$$

where W is a standard K-dimensional Wiener process, $a \in \mathcal{L}^1$ is N-dimensional and $b \in \mathcal{L}^2$ is $N \times K$-dimensional. Let

$$\mathcal{L}(Z) = \{\gamma : \gamma \text{ is adapted, measurable, } M \times N, \gamma a \in \mathcal{L}^1, \gamma b \in \mathcal{L}^2\}$$

If $\gamma \in \mathcal{L}(Z)$, define

$$\int_0^t \gamma\, dZ = \int_0^t \gamma a\, ds + \int_0^t \gamma b\, dW$$

Remark 8.2 γ may be in $\mathcal{L}(Z)$ even if it is not in $\mathcal{L}(Z_i)$ for some i. This has economic significance. It may be that that the Itô coefficients of the securities price process become linearly dependent at some node (ω, t). Near such a point, it is possible to construct trading strategies with large holdings of the securities but little risk. For example, if Z_1 and Z_2 are perfectly correlated at time t_0, they will be nearly perfectly correlated at times near t_0, so a large long position in Z_1 and a large short position in Z_2 entails little risk. The realization of the capital gains in the two securities can well be infinite in each security, but of opposite sign. When the capital gains are computed considering the whole position in the pair of securities, the infinities cancel out and generate a well-defined finite capital gain. Such trading strategies can well be optimal in this situation.

9 Securities and Trading Strategies

We assume there are $N+1$ long-lived securities indexed $n = 0, \ldots, N$; often but not always, the zeroth security is a Money-Market Account, which is instantaneously riskless. The security price process is an Itô Process

$$\bar{S}(t) = \bar{S}(0) + \int_0^t \bar{\mu}\, ds + \int_0^t \bar{\sigma}\, dW$$

where W is a K-dimensional standard Wiener process, $\bar{\mu} \in \mathcal{L}^1$ is $(N+1) \times 1$, and $\bar{\sigma} \in \mathcal{L}^2$ is $(N+1) \times K$.

A *trading strategy* is an adapted, measurable $1 \times (N+1)$ process $\bar{\Delta}$; $\bar{\Delta}_n(\omega, t)$ denotes the holding of security n at node (ω, t). The *value process* is

$$\bar{\Delta}\bar{S} = \bar{\Delta}_0 \bar{S}_0 + \cdots + \bar{\Delta}_N \bar{S}_N$$

The set of trading strategies for which the capital gain process is well-defined is

$$\mathcal{L}(S) = \{\bar{\Delta} : \bar{\Delta}\bar{\mu} \in \mathcal{L}^1 \text{ and } \bar{\Delta}\bar{\sigma} \in \mathcal{L}^2\}$$

The *Cumulative Gain Process* of $\bar{\Delta}$ with respect to $\bar{S}$ is

$$\begin{aligned} \mathcal{G}(\bar{\Delta}; \bar{S})(t) &= \bar{\Delta}(0)\bar{S}(0) + \int_0^t \bar{\Delta}\, d\bar{S} \\ &= \bar{\Delta}(0)\bar{S}(0) + \int_0^t \bar{\Delta}\bar{\mu}\, ds + \int_0^t \bar{\Delta}\bar{\sigma}\, dW \end{aligned}$$

Implicitly, this definition assumes the securities pay no dividends.

$\bar{\Delta}$ is *self-financing* if it satisfies the budget constraint

$$\bar{\Delta}\bar{S} = \mathcal{G}(\bar{\Delta}; \bar{S})$$

i.e.

$$\bar{\Delta}(t)\bar{S}(t) = \bar{\Delta}(0)\bar{S}(0) + \int_0^t \bar{\Delta}\, d\bar{S}$$

almost surely, for all t. In other words, after $\bar{\Delta}(0)\bar{S}(0)$ is invested to buy the initial portfolio $\bar{\Delta}(0)$, no additional money goes in to buy stocks and no money is withdrawn. Writing the self-financing condition in differential form,

$$d(\bar{\Delta}\bar{S}) = \bar{\Delta}\, d\bar{S}$$

i.e. the instantaneous rate of change of the value process is the security holding times the instantaneous rate of change of the security prices. Heuristically, this is saying that

$$\bar{S}\, d\bar{\Delta} = 0$$

i.e. the instantaneous change in the portfolio is orthogonal to the vector of securities prices, which just says the value of shares bought equals the value of shares sold. This is heuristic, rather than precise, because the trading strategy $\bar{\Delta}$ is not required to be an Itô Process, so we may not be able to assign formal meaning to $d\bar{\Delta}$.

It is important to understand that self-financing is a property of the trading strategy and not of the value process. The following is a simpler (but perhaps fairly stupid) example. Suppose there are two assets

$$\bar{S}_0(t) = \bar{S}_0(0)e^{r_0 t} \text{ and } \bar{S}_1(t) = \bar{S}_1(0)e^{r_1 t}$$

with $r_1 > r_0$. The buy-and-hold strategy

$$\bar{\Delta} = (0, 1)$$

(hold one unit of S_2 no matter what) is self-financing and yields the value process

$$\bar{\Delta}(t)\bar{S}(t) = \bar{S}_1(0)e^{r_1 t}$$

The strategy

$$\bar{\Delta}' = \left(\frac{S_1(0)e^{(r_1-r_0)t}}{S_0(0)}, 0\right)$$

has the same value process

$$\bar{\Delta}'(t)\bar{S}(t) = \bar{S}_0(0)e^{r_0 t}\frac{\bar{S}_1(0)e^{(r_1-r_0)t}}{\bar{S}_0(0)} = \bar{S}_1(0)e^{r_1 t}$$

but it is *not* self-financing; money is constantly going in to increase the holding of security zero and this is not balanced by any sale of security one.

A *numeraire* for $\bar{S}$ is a self-financing trading strategy $\bar{b}$ such that $\bar{b}\bar{S} = 1$ for all t, almost surely. S is said to be *normalized* if there is a numeraire for S. Consider the following examples:

1. Suppose $N = 0$ and $\bar{S}_0$ is a money-market account. The only self-financing trading strategies are buy-and-hold strategies; $\bar{b}(t) = \bar{b}(0)$. If b is a numeraire, then $\bar{S}_0(t)$ must be a constant, independent of t; any increase in value due to interest is incorporated into the units of account in which the security price is measured. Said another way, the currency is not dollars but units of the security, and each unit of the security buys more real goods as time passes.

2. Assuming no new shares are issued or redeemed, and no mergers occur, one self-financing strategy is buy and hold *all* the shares outstanding. If this is a numeraire, then $\bar{\Delta}\bar{S} = 1$ means that prices are deflating at the rate of growth of the market porfolio.

3. At each node (ω, t) we can multiply all the security prices by an arbitrary scalar $\alpha(\omega, t)$ without changing the opportunities to make a self-financed change of portfolio at time t. As long as we also multiply the price of real goods by the same scalar, nothing has changed; the set of goods that can be bought with the value of the portfolio is unchanged. If we multiply security prices at each node by an arbitrary scalar $\alpha(\omega, t)$, the Itô integrals needed to define capital gains may no longer be defined. A surprising fact is that, if the needed stochastic integrals are defined, they compute the capital gains correctly, and the set of self-financing portfolios is invariant to these price changes.

10 Nonstandard Analysis

Nonstandard analysis is a a treatment of real analysis (and other areas in mathematics) using the intuitive concept of an infinitesimal. It has had extensive applications in probability theory. The Loeb measure construction [13] is a way of taking a so-called hyperfinite probability space (a space with infinitely many points, but which behaves formally exactly like a discrete probability space) and constructing a measure-theoretic probability space in the usual standard sense. Anderson [1] showed that a hyperfinite random walk (which behaves formally exactly like a finite random walk) becomes a standard Brownian motion when it is viewed on the Loeb probability space. He also showed that Itô integrals in the standard world can be calculated by computing Stieltjes integrals on the hyperfinite random walk; in other words, the capital gain generated by a trading strategy can be computed in exactly the same way it is calculated in a finite discrete model. Keisler [8] showed that this methodology can be used to solve stochastic differential equations, and gave the first proofs of several new standard theorems concerning them. Yeneng Sun of the National University of Singapore is a leading expert in the use of nonstandard analysis in measure theory and mathematical economics, including finance.

Nonstandard analysis permits us to analyze the existence of equilibria in continuous-time finance models starting from discrete-time finance models. Because time does not permit us to develop the techniques of nonstandard analysis, our treatment of the existence question in continuous time will be described in the hyperfinite world, where everything behaves formally exactly as in the familiar discrete world. We will, however, describe the regularity conditions that must be satisfied by the hyperfinite equilibria to permit us to extract equilibria in the continuous-time model.

11 Raimondo's Single-Agent Model

This section is taken from Raimondo [15]. Raimondo's single-agent model is defined as follows:

1. Trade and consumption occur over a compact time interval $[0, T]$, endowed with a measure λ which agrees with Lebesgue measure on $[0, T)$ and such that $\lambda(\{T\}) = 1$.

2. The information structure is represented by a filtration $\{\mathcal{F}_t : t \in [0, T]\}$ on a probability space $(\Omega, \mathcal{F}, \mu)$. There is a standard d-dimensional

Brownian motion $\beta = (\beta_1, \ldots, \beta_d)$ such that β_i is independent of β_j if $i \neq j$ and such that the variance of $\beta_i(t, \cdot)$ is t and $\beta_i(t, \cdot) = E(\beta_i(T, \cdot)|\mathcal{F}_t)$.

3. There is exactly one representative agent. The endowment of the agent satisfies
$$e(\omega, t) = 1 \text{ for all } (\omega, t) \in \Omega \times [0, T)$$
The endowment $e(\omega, T)$ in period T satisfies
$$e(\omega, T) = \rho(\beta_1(\omega, T), \ldots, \beta_d(\omega, T))$$
where $\rho : \mathbf{R}^d \to \mathbf{R}$ is continuous and satisfies
$$0 \leq \rho(x) \leq r + e^{r|x|}$$
for some $r \in \mathbf{R}_+$. The endowment in period $t \in [0, T)$ is interpreted as a rate of flow of endowment, while the endowment in period T is interpreted as a stock or lump. Given a measurable consumption function $c : \Omega \times [0, T] \to \mathbf{R}$, the utility function of the agent is
$$U(c) = E_\mu \left[\int_0^T \varphi_1(c_t) dt + \varphi_2(c_T) \right]$$
where the twice differentiable functions $\varphi_i : \mathbf{R}_{++} \to \mathbf{R}$ $(i = 1, 2)$ satisfy
$$\begin{cases} \varphi_i'(c) & > & 0 \text{ for } i = 1, 2 \\ \varphi_i''(c) & < & 0 \text{ for } i = 1, 2 \\ \varphi_i(z) & & \text{is bounded below} \end{cases}$$
Examples of utility functions satisfying the conditions on φ_i are the CARA utilities $\varphi_i(z) = \gamma e^{\alpha z}$ for $\alpha, \gamma < 0$ and the CRRA utilities $\varphi_i(z) = \gamma x^\alpha$ $(0 < \alpha < 1,\ \gamma > 0)$. The assumption that φ_i is bounded below is used at only point in the proof; we conjecture that it can be weakened to $\varphi_2'(z) = O\left(\frac{1}{z^r}\right)$ as $z \to 0$ for some $r \in \mathbf{R}$. If so, the CRRA utility function $\varphi_i(z) = \gamma \ln z$ $(\gamma > 0)$ and the CARA utility $\varphi_i(z) = \gamma x^\alpha$ $(\alpha < 0,\ \gamma < 0)$ would be covered by the theorem.

4. There are $J+1$ tradable assets, with $0 \leq J \leq d$: J stocks $A_1, \ldots, A_J$ which pay off[10]

$$A_j(\omega, t) = \begin{cases} 0 & \text{if } t \neq T \\ e^{\beta_j(\omega, T)} & \text{if } t = T \end{cases}$$

and a bond B which pays off

$$B(\omega, t) = \begin{cases} 0 & \text{if } t \neq T \\ 1 & \text{if } t = T. \end{cases}$$

Observe that the payoffs of different stocks are independent. The agent is initially endowed with security holdings $z(\omega, 0) = ((1, \ldots, 1), 0)$: one unit of each stock and zero units of the bond. If $J = 0$ (i.e. there are no stocks in the model), we assume that $\rho(x) \geq e^{\alpha \cdot x}$ for some $\alpha \in \mathbf{R}^d$; this will ensure that the income in the terminal period T is not too small.

5. There is a short-sale constraint, i.e. there is some $M > 0$ such that the agent is not permitted to hold less than $-M$ units of either the stock or the bond.

6. In order to define the budget set of an agent, we need to have a way of calculating the capital gain the agent receives from a given trading strategy. In other words, we need to impose conditions on prices and strategies that ensure that the stochastic integral of a trading strategy with respect to a price process is defined. The essential requirements are that the trading strategy at time t not depend on information which has not been revealed by time t, and the trading strategy times the variation in the price yields a finite integral. Specifically,

   (a) A stochastic process X is said to be adapted if, for all t, $X(t, \cdot)$ is measurable with respect to $\mathcal{F}_t$.

[10] The functional form $A_j(\omega, T) = e^{\beta_j(\omega, T)}$ is not essential. All but one portion of the proof works if $A_j(\omega, T)$ is an arbitrary continuous function of $\beta_j(\omega, T)$ satisfying an exponential growth condition, and that one part works for a large class of functions of β_j, but we have not identified the exact class. Of course, changing the payoff will alter the pricing formula.

(b) A security price process is a pair of stochastic processes $p = (p_A, p_B)$, where $p_A = (p_{A_1}, \ldots, p_{A_J})$, and p_{A_j} and p_B are continuous square-integrable martingales with respect to $\{\mathcal{F}_t\}$. p_{A_j} and p_B are priced *cum dividend* at time T. A consumption price process is a stochastic process $p_C(\omega, t)$.

(c) Given a security price process p, a trading strategy is a pair $(z_A, z_B) : \Omega \times [0, T] \to [-M, \infty) \times [-M, \infty)^d$ such that z_A and z_B are adapted and $z_{A_j} \in L^2(\Omega \times [0, T], \mathcal{P}, q_{A_j})$, $z_B \in L^2(\Omega \times [0, T], \mathcal{P}, q_B)$.

7. Given a security price process p and a consumption price process p_C, the budget set is the set of all consumption plans c which satisfy the budget constraint

$$\begin{aligned}
&\mathbf{1} \cdot p_A(0) + \int_0^t z dp + \int_0^t p_C(\omega, s)(e(\omega, s) - c(\omega, s))ds = p(\omega, t) \cdot z(\omega, t) \\
&\text{for almost all } \omega \text{ and all } t < T \\
&\mathbf{1}_J \cdot p_A(0) + \int_0^T z dp + \int_0^T p_C(\omega, s)(e(\omega, s) - c(\omega, s))ds \\
&+(e(\omega, T) + z_A(\omega, T)e^{\beta(\omega, T)} + z_B(\omega, T) - c(\omega, T))p_C(\omega, T) \\
&= p(\omega, T) \cdot z(\omega, T) \text{ for almost all } \omega
\end{aligned}$$

for some trading strategy z. We follow standard notation in writing $\mathbf{1} = (1, \ldots, 1)$ and

$$\int z dp = \sum_{j=1}^{J} \int z_{A_j} dp_{A_j} + \int z_B dp_B$$

Observe that it is implicit in the definition that $p_C(\omega, \cdot)(e(\omega, \cdot) - c(\omega, \cdot)) \in L^1([0, T])$.

8. Given a price process p, the demand of the agent is a consumption plan and a trading strategy which satisfy the budget constraint and such that the consumption plan maximizes utility over the budget set.

9. An equilibrium for the economy is a price process p, a trading strategy z and a consumption plan c which lies in the demand set so that the

securities and goods markets clear, i.e. for almost all ω

$$\begin{array}{rcl} z_A(\omega,t) & = & \mathbf{1} \text{ for all } t \in [0,T] \\ z_B(\omega,t) & = & 0 \text{ for all } t \in [0,T] \\ c(\omega,t) & = & 1 \text{ for all } t \in [0,T) \\ c(\omega,T) & = & e(\omega,T) + \mathbf{1}_J \cdot e^{\beta(\omega,T)} \end{array}$$

where $e^{\beta(\omega,t)}$ denotes the vector

$$\left(e^{\beta_1(\omega,t)}, \ldots, e^{\beta_d(\omega,t)}\right)$$

and $\mathbf{1}_J = (1, \ldots, 1, 0, \ldots, 0) \in \mathbf{R}^d$ is the vector with J 1's followed by $d - J$ 0's.

Theorem 11.1 (Raimondo) *There is a standard probability space $(\Omega, \mathcal{F}, \mu)$, a filtration $\mathcal{F}_t$, and a d-dimensional Brownian motion $\beta = (\beta_1, \ldots, \beta_d)$ such that the continuous time finance model just described has an equilibrium. At equilibrium, the short-sale constraint is not binding. The pricing process is given by*

$$\begin{array}{rcl} p_{A_j}(\omega,t) & = & e^{\beta_j(\omega,t)} \int_{-\infty}^{\infty} \varphi_2'\left(F(t,\omega,x)\right) e^{\sqrt{T-t}x_j} d\Phi(x) \\ p_B(\omega,t) & = & \int_{-\infty}^{\infty} \varphi_2'\left(F(t,\omega,x)\right) d\Phi(x) \\ p_C(\omega,t) & = & \varphi_1'(1) \textit{ for } t < T \\ p_C(\omega,T) & = & \varphi_2'\left(F(T,\omega,0)\right) \\ \frac{p_{A_j}(\omega,t)}{p_B(\omega,t)} & = & e^{\beta_j(\omega,t)} \frac{\int_{-\infty}^{\infty} \varphi_2'(F(t,\omega,x)) e^{\sqrt{T-t}x_j} d\Phi(x)}{\int_{-\infty}^{\infty} \varphi_2'(F(t,\omega,x)) d\Phi(x)} \end{array} \qquad (5)$$

where

$$F(t,\omega,x) = \rho\left(\beta(\omega,t) + \sqrt{T-t}x\right) + \mathbf{1}_J \cdot \left(e^{\beta(\omega,t)+\sqrt{T-t}x}\right)$$

and Φ is the cumulative distribution function of the standard d-dimensional normal.

Outline of Proof: Construct a hyperfinite economy as follows:

1. Choose $n \in {}^*\mathbf{N} \setminus \mathbf{N}$. For $t \in [0,T]$, define $\hat{t} = \frac{[nt]}{n}$; in particular, $\hat{T} = \frac{[nT]}{n}$. Define a hyperfinite random walk $\hat{\beta}$ on this hyperfinite time axis. and hyperfinite probability space $\hat{\Omega}$.

2. Let Ω be the (complete) Loeb measure generated by $\hat{\Omega}$ (Loeb [13]) and $\beta(\omega, t) = {}^\circ\beta(\omega, \hat{t})$. β is a standard Brownian motion on Ω (Anderson [1].)

3. For all $\omega \in \hat{\Omega}$, define $\hat{e}(\omega, t) = e(\omega, t) = 1$ for all $t \in \mathcal{T}$, $t < \hat{T}$ and $\hat{e}(\omega, \hat{T}) = {}^*\rho(\beta(\omega, \hat{T}))$.

4. For all $\omega \in \hat{\Omega}$, define $\hat{A}(\omega, t) = A(\omega, t) = \hat{B}(\omega, t) = B(\omega, t) = 0$ for all $t < \hat{T}$, and $\hat{A}(\omega, \hat{T}) = e^{\hat{\beta}(\omega, \hat{T})}$ (i.e. $\hat{A}_j(\omega, \hat{T}) = e^{\hat{\beta}_j(\omega, \hat{T})}$, $i = 1, \ldots m$) $A(\omega, T) = e^{\beta(\omega, T)}$, $\hat{B}(\omega, \hat{T}) = B(\omega, T) = 1$. Note that $A(\omega, T) = {}^\circ\hat{A}(\omega, \hat{T})$ for μ-almost all ω.

5. Given an internal consumption plan $\hat{c}$, the agent's utility is

$$\hat{U}(\hat{c}) = E_{\hat{\mu}} \left(\left(\Delta T \sum_{s \in \mathcal{T}, s < \hat{T}} {}^*\varphi_1(\hat{c}(\omega, t)) \right) + {}^*\varphi_2(\hat{c}(\omega, \hat{T})) \right)$$

6. Existence of equilibrium follows immediately from Radner [16]. Note that the short-sale constraint is not binding.

7. The market clearing conditions on $\hat{c}$ guarantee that consumption is positive in every period. Since the security payoffs are nonnegative for all (ω, t) and strictly positive for (ω, T) for all ω, the absence of arbitrage guarantees that $p_A(\omega, t) \gg 0$ and $p_B(\omega, t) > 0$ for all (ω, t).

8. The pricing formulas for the hyperfinite model (analogous to those for the continuous-time model) come from the first order conditions, as discussed in Felix Kubler's lectures.

9. Use the growth condition on security dividends to show that the dividends in the hyperfinite model are SL^1 (analogous to uniform integrability for sequences of finite economies). Therefore, the expectations defining prices in the hyperfinite model are infinitely close to the corresponding expectations in the continuous-time model (Anderson [1]).

10. Verify that the hyerfinite prices are almost surely S-continuous, hence the induced prices in the continuous model are almost surely continuous.

11. Verify that the hyperfinite prices are SL^2 (analogous to uniformly L^2 for sequences of finite economies) martingales. This implies the prices in the continuous-time economy are square-integrable martingales. (This is much harder in the multi-agent model).

12. Using results of Anderson [1] and Lindström [10] on stochastic integration, show that the induced consumptions in the continuous-time model are in the budget set using the single agent's buy-and-hold strategy (This is much harder in the multi-agent model).

13. If the prices and consumptions do not form an equilibrium for the continuous-time model, there must be a continuous-time trading strategy which finances a consumption plan that delivers strictly higher utility.

14. "Lift" the continuous-time consumption and trading strategy to a hyperfinite consumption and trading strategy. Show that the lifted trading strategy is in the hyperfinite budget set, and delivers strictly higher utility than the hyperfinite equilibrium consumption, contradiction.

12 Multiple-Agent Economies with Endogenously Dynamically Complete Markets

This section is based on Anderson and Raimondo [2].

1. Trade and consumption occur over a compact time interval $[0, T]$, endowed with a measure ν which agrees with Lebesgue measure on $[0, T)$ and such that $\nu(\{T\}) = 1$.

2. The information structure is represented by a filtration $\{\mathcal{F}_t : t \in [0, T]\}$ on a probability space $(\Omega, \mathcal{F}, \nu)$. A stochastic process $X(t, \omega)$ is said to be adapted if, for all t, $X(t, \cdot)$ is measurable with respect to $\mathcal{F}_t$.

3. There is a standard K-dimensional Brownian motion $\beta = (\beta_1, \ldots, \beta_K)$ such that $\beta_k(t, \cdot) = E(\beta_k(T, \cdot)|\mathcal{F}_t)$.

4. There are I agents $i = 1, \ldots, I$. The endowment of the agent i is a process
$$e_i(t,\omega) = \begin{cases} f_i(\beta(t,\omega),t) & \text{if } t \in [0,T) \\ F_i(\beta(T,\omega)) & \text{if } t = T \end{cases}$$
where $f_i : \mathbf{R}^K \times [0,T] \to \mathbf{R}_{++}$ and $F_i : \mathbf{R}^K \to \mathbf{R}_{++}$ are analytic functions (i.e. at every point in their stated domain, they can be represented locally by power series). Let $e(t,\omega) = \sum_{i=1}^I e_i(t,\omega)$ denote the aggregate endowment.

5. There are $J+1 = K+1$ tradable securities (indexed by $j = 0, \ldots, J$) which pay dividends
$$A_j(t,\omega) = \begin{cases} g_j(\beta(t,\omega),t) & \text{if } t \in [0,T) \\ G_j(\beta(T,\omega)) & \text{if } t = T \end{cases}$$
where $g_j : \mathbf{R}^K \times [0,T] \to \mathbf{R}_{+}$ and $G_j : \mathbf{R}^K \to \mathbf{R}_{++}$ are analytic functions. The net supply of security j is $\eta_j \in \{0,1\}$; thus, securities may be in net supply zero or net supply one. We assume that the social endowment plus security dividends (of the stocks in net supply one) are uniformly bounded below, and that the security dividends satisfy a mild growth condition:
$$\begin{aligned} \exists_{m>0}\ e(t,\omega) + \sum_{j=0}^J \eta_j A_j(t,\omega) &\geq m \\ \exists_{r>0}\ e(t,\omega) + \sum_{j=0}^J A_j(t,\omega) &\leq r + e^{r|\beta(t,\omega)|} \end{aligned}$$
Let $R_T : \mathbf{R}^K \to \mathbf{R}^K$ be defined by
$$R_T(\beta) = \frac{(G_1(\beta), \ldots, G_J(\beta))}{G_0(\beta)}$$
Since $G_0(\beta) \neq 0$ for all β, R_T is an analytic function.
We asssume the following Nondegeneracy Condition: for some $\beta \in \mathbf{R}^K$, the Jacobian matrix of R_T has rank K.

6. Agent i is initially endowed with deterministic security holdings $e_{iA} = (e_{iA_0}, \ldots, e_{iA_J}) \in \mathbf{R}_+^{J+1}$ satisfying

$$\sum_{i=1}^{I} e_{iA_j} = \eta_j$$

Note that the initial holdings are independent of the state ω. Moreover, the initial security holdings are required to be nonnegative; without this restriction, there might be an agent who cannot make good on his/her initial short position, and hence no equilibrium would exist.

7. Given a measurable consumption function $c_i : [0,T] \times \Omega \to \mathbf{R}_{++}$, the utility function of the agent is

$$U_i(c) = E_\nu \left[\int_0^T h_i(c_i(t,\cdot), \beta(t,\cdot), t)dt + H_i(c_i(T,\cdot), \beta(T,\cdot)) \right]$$

where the functions $h_i : \mathbf{R}_+ \times \mathbf{R}^K \times [0,T] \to \mathbf{R} \cup \{-\infty\}$ and $H_i : \mathbf{R}_{++} \times \mathbf{R}^K \to \mathbf{R} \cup \{-\infty\}$ are analytic on $\mathbf{R}_{++} \times \mathbf{R}^K \times [0,T]$ and $\mathbf{R}_{++} \times \mathbf{R}^K$ respectively (i.e. at each point, they are represented locally by a power series) and satisfy

$$\begin{array}{lccl}
\lim_{c \to 0_+} \frac{\partial h_i}{\partial c} & = & \infty & \text{uniformly in } (\beta, t) \\
\lim_{c \to 0_+} \frac{\partial H_i}{\partial c} & = & \infty & \text{uniformly in } \beta \\
\lim_{c \to \infty} \frac{\partial h_i}{\partial c} & = & 0 & \text{uniformly in } (\beta, t) \\
\lim_{c \to \infty} \frac{\partial H_i}{\partial c} & = & 0 & \text{uniformly in } \beta \\
\lim_{c \to 0_+} h_i(c, \beta, t) & = & h_i(0, \beta, t) & \text{uniformly in } (\beta, t) \\
\lim_{c \to 0_+} H_i(c, \beta) & = & H_i(0, \beta) & \text{uniformly in } \beta \\
\left.\frac{\partial h_i}{\partial c}\right|_{(c,\beta,t)} & > & 0 & \text{for } c \in \mathbf{R}_{++} \\
\left.\frac{\partial^2 h_i}{\partial c^2}\right|_{(c,\beta,t)} & < & 0 & \text{for } c \in \mathbf{R}_{++} \\
\forall_{c>0} \exists_{M \in \mathbf{R}} \forall_{(\beta,t)} \left.\frac{\partial h_i}{\partial c}\right|_{(c,\beta,t)} & \leq & M & \\
\forall_{c>0} \exists_{M \in \mathbf{R}} \forall_{\beta} \left.\frac{\partial H_i}{\partial c}\right|_{(c,\beta)} & \leq & M &
\end{array}$$

Note that these conditions are satisfied by all state-independent utility functions in the CARA and CRRA classes. Note also that we allow quite general state-dependence of the utility function, as long as the state-dependence enters through the Brownian motions that represent the uncertainty in the economy. If the state-dependence were not measurable in the Brownian motions, there would be no hope of obtaining effective dynamic completeness with securities whose dividends *are* measurable with respect to the Brownian filtration.

8. In order to define the budget set of an agent, we need to have a way of calculating the capital gain the agent receives from a given trading strategy. In other words, we need to impose conditions on prices and strategies that ensure that the stochastic integral of a trading strategy with respect to a price process is defined. The essential requirements are that the trading strategy at time t not depend on information which has not been revealed by time t, and the trading strategy times the variation in the price yields a finite integral. Specifically,

   (a) A consumption price process is an Itô process $p_C(t, \omega)$.

   (b) A securities price process is an Itô process $p_A = (p_{A_0}, \ldots, p_{A_J}) : \Omega \times [0, T] \to \mathbf{R}^{J+1}$ such that the associated cumulative gains process
   $$\gamma_j(t, \omega) = p_{A_j}(t, \omega) + \int_0^t p_C(s, \omega) A_j(s, \omega)\, ds$$
   is a continuous square-integrable Itô martingale. Securities are priced *cum dividend* at time T.

   (c) Given a securities price process p_A, a trading strategy for agent i is an Itô process z_i where

      i. $z_i : [0, K) \times \Omega \to \mathbf{R}^{J+1}$
      ii. $z_i(t, \cdot)$ is $\mathcal{F}_t$-measurable for all $t \in [0, T)$
      iii. z_i is measurable in $[0, K) \times \Omega$
      iv. $z_i \in \mathcal{H}^2(\gamma)$.[11]

9. Given a securities price process p_A and a consumption price process p_C, the budget set for agent i is the set of all consumption plans c_i such

[11] If σ is the instantaneous volatility matrix of γ (the matrix of Itô coefficients of γ with respect to the Brownian motion), then $z_i \in \mathcal{H}^2(\gamma)$ means that $z_i \cdot \gamma \in L^2([0, T) \times \Omega)$. This implies that z_i is Itô integrable with respect to $\gamma = (\gamma_0, \ldots, \gamma_J)$. It also rules out arbitrage strategies like the doubling strategy of Harrison and Kreps [7].

that there exists a trading strategy so that c_i and t_i satisfy the budget constraint

$$
\begin{aligned}
& p(t,\omega) \cdot z_i(t,\omega) \\
& \quad = \quad e_{iA}(\omega) \cdot p_A(0,\omega) + \int_0^t z_i \, d\gamma + \int_0^t p_C(s,\omega)(e_i(s,\omega) - c_i(s,\omega))ds \\
& \qquad\quad \text{for almost all } \omega \text{ and all } t \in [0,T) \\
& 0 \quad = \quad e_{iA}(0,\omega) \cdot p_A(0,\omega) + \int_0^T z_i \, d\gamma + \int_0^T p_C(s,\omega)(e_i(s,\omega) - c_i(s,\omega))ds \\
& \qquad\quad +(e_i(T,\omega) - c_i(T,\omega))p_C(T,\omega)) \\
& \qquad\quad \text{for almost all } \omega
\end{aligned}
$$

10. Given a price process p, the demand of the agent is a consumption plan and a trading strategy which satisfy the budget constraint and such that the consumption plan maximizes utility over the budget set.

11. An equilibrium for the economy is a securities price process p_A, a consumption price process p_C, a trading strategy z and a consumption plan c which lies in the demand set so that the securities and goods markets clear, i.e. for almost all ω

$$
\begin{aligned}
\sum_{i=1}^{I} z_{iA_j}(t,\omega) &= \eta_j \text{ for } j = 0, \ldots, J \text{ and almost all } (t,\omega) \\
\sum_{i=1}^{I} c_i(t,\omega) &= \sum_{i=1}^{I} e_i(t,\omega) + \sum_{j=0}^{J} \eta_j A_j(t,\omega) \text{ for almost all } (t,\omega)
\end{aligned}
$$

Theorem 12.1 *The continuous-time finance model just described has an equilibrium. The equilibrium securities prices and consumption prices are given by analytic functions of $(\beta(t,\omega),t)$ for $t \in [0,T)$, and as analytic functions of $\beta(T,\omega)$ for $t = T$. There is an analytic function of $(\beta(t,\omega),t)$ such that the equilibrium trading strategies equal this function except on a closed set of measure zero in $[0,T) \times \Omega$. The equilibrium prices are effectively dynamically complete: any integrable function which is adapted to the* Brownian *filtration can be replicated by a trading strategy. The equilibrium consumptions are Pareto optimal.*

Outline of Proof:

1. As in Raimondo's single agent model, let the time axis be $\mathcal{T} = \{0, \Delta T, 2\Delta T, \ldots, \hat{T}\}$.

2. If we used the usual nonstandard construction of Brownian motion, each node would have 2^K successor nodes, ruling out dynamic completeness if $K > 1$. So instead, construct a random walk $\hat{\beta}$ in $\mathbf{R}^K$ such that each node has $K + 1$ successor nodes and

$$\begin{aligned} E(\hat{\beta}(t + \Delta T, \cdot)|(t, \omega_0)) &= \hat{\beta}(t, \omega_0) \\ E((\Delta\hat{\beta}_i(t, \omega))(\Delta\hat{\beta}_j(t, \omega)) &= \frac{\delta_{ij}}{\Delta T} \end{aligned}$$

   Show that $\beta(t, \omega) = {}^\circ\hat{\beta}(\hat{t}, \omega)$ is a standard Brownian motion (this is not quite covered in the earlier papers in nonstandard probability).

3. Use the analytic functions to induce endowments, utility functions, and security payoffs in the hyperfinite economy.

4. An equilibrium for the economy is a security price process $\hat{p}$, a consumption price process $\hat{p}_C$, trading strategies $\hat{z}_i$ and consumption plans $\hat{c}_i$ which lies in the demand sets of the agents so that the securities and goods markets clear, i.e. for all $t \in \mathcal{T}$ and all $\omega \in \hat{\Omega}$

$$\begin{aligned} \sum_{i=1}^{I} \hat{z}_i(t, \omega) &= (\eta_0, \ldots, \eta_J) \\ \sum_{i=1}^{I} \hat{c}_i(t, \omega) &= \sum_{i=1}^{I} \hat{e}_i(t, \omega) + \sum_{j=0}^{J} \eta_j \hat{A}_j(\omega, t) \end{aligned}$$

5. Use the Duffie-Shafer Theorem [4, 5] to perturb the endowments and security dividends by at most $(\Delta T)^2$ to ensure the existence of a Pareto optimal equilibrium with dynamically complete securities prices. Note that this does not rule out the possibility that the determinant which determines dynamic completeness is infinitesimal; since such nodes will become Hart points in the continuous-time model, we will need to show that the set of nodes where the prices are "infinitesimal Hart points" has Loeb measure zero.

6. Since the marginal utility of consumption is infinite at zero, and the aggregate consumption is strictly positive at each node, the equilibrium consumptions of all agents are strictly positive at each node. Let Δ be the open $I-1$-dimensional simplex in $\mathbf{R}^I_{++}$. Pareto optimality implies that there exists $\lambda = (\lambda_1, \ldots, \lambda_I) \in {}^*\Delta$ such that at each node (t, ω), there is a positive constant $\mu(t, \omega)$ such that

$$\begin{array}{ll} \lambda_1 {}^*\frac{\partial h_1}{\partial c}(\hat{c}_i(t,\omega), \hat{\beta}(t,\omega), t) = \cdots = \lambda_I {}^*\frac{\partial h_I}{\partial c}(\hat{c}_I(t,\omega), \hat{\beta}(t,\omega), t) = \mu(t,\omega) & \text{for } t < \hat{T} \\ \lambda_1 {}^*\frac{\partial H_1}{\partial c}(\hat{c}_i(\hat{T},\omega), \hat{\beta}(t,\omega)) = \cdots = \lambda_I {}^*\frac{\partial H_I}{\partial c}(\hat{c}_I(\hat{T},\omega), \hat{\beta}(t,\omega)) = \mu(\hat{T},\omega) & \end{array}$$

Let $\hat{c}(t,\omega) = \sum_{i=1}^I \hat{c}_i(t,\omega)$. By the analytic implicit function theorem, there exist standard analytic functions such that

$$\begin{array}{rcl} \mu(t,\omega) &=& {}^*\hat{\pi}((\lambda_1, \cdots, \lambda_I), \hat{c}(t,\omega), \hat{\beta}(t,\omega), t) \text{ for } t < \hat{T} \\ \mu(\hat{T},\omega) &=& {}^*\hat{\Pi}((\lambda_1, \cdots, \lambda_I), \hat{c}(\hat{T},\omega), \hat{\beta}(\hat{T},\omega)) \\ \hat{c}_i(t,\omega) &=& {}^*\hat{\psi}_i((\lambda_1, \cdots, \lambda_I), \hat{c}(t,\omega), \hat{\beta}(t,\omega), t) \text{ for } t < \hat{T} \\ \hat{c}_i(\hat{T},\omega) &=& {}^*\hat{\Psi}_i((\lambda_1, \cdots, \lambda_I), \hat{c}(\hat{T},\omega), \hat{\beta}(\hat{T},\omega)) \end{array}$$

7. $\hat{p}_C(t,\omega) = \mu(t,\omega)$ are the Arrow-Debreu prices of consumption. Since total supply (from endowments and dividends) is uniformly bounded below, aggregate consumption is uniformly bounded below, so p_C is uniformly bounded above by a standard number.

8. The first order conditions imply that the securities prices $\hat{p}_{A_j}$ and the total gains processes $\hat{\gamma}_j$ are given by the nonstandard extensions of standard analytic functions evaluated at $\hat{\lambda}, \hat{\beta}$, and the perturbed endowments and dividends.

9. Let
$$\hat{R}(t,\omega) = \frac{(\hat{p}_{A_1}(t,\omega), \ldots \hat{p}_{A_J}(t,\omega))}{\hat{p}_{A_0}(t,\omega)}$$
There is a standard analytic function ρ such that
$$\begin{array}{rcl} \hat{p}_A(t,\omega) &=& {}^*\rho(\lambda, \hat{\beta}(t,\omega), t) + O(\Delta T) \\ \hat{R}(t,\omega) &=& \dfrac{{}^*\rho_{1,\ldots,J}(\lambda, \hat{\beta}(t,\omega), t)}{\rho_0(\lambda, \hat{\beta}(t,\omega), t)} + O(\Delta T) \end{array}$$

The condition for markets to be dynamically complete in the continuous-time model will be that the Jacobian matrix of R (the analogue in the continuous-time model of $\hat{R}$) with respect to the Brownian motion is nonsingular.

10. Let $\hat{\gamma}_j(t_0, \omega_0)$ be the total gains process of security j. It is clear that each $\hat{\gamma}_j$ is a hypermartingale. It is SL^2 from the growth condition on dividends.

11. Now, we extract the equilibrium prices and trading strategies for the Loeb measure economy generated by the hyperfinite economy.

$$\begin{aligned} p_C(t,\omega) &= {}^\circ\hat{p}_C\left(\hat{t},\omega\right) \\ p_A(t,\omega) &= {}^\circ\hat{p}_A\left(\hat{t},\omega\right) \\ \gamma(t,\omega) &= {}^\circ\hat{\gamma}\left(\hat{t},\omega\right) \end{aligned}$$

Then $p_C(t,\omega)$ and $p_A(t,\omega)$ are standard analytic functions of $(\beta(t,\omega),t)$ for $t < T$ and of $\beta(T,\omega)$ for $t = T$. The $O((\Delta T)^2)$ perturbations wash out. The parameter λ is replaced by ${}^\circ\lambda$ and disappears as a variable. $R(t,\omega) = \frac{(p_{A_1}(t,\omega),\ldots,p_{A_J}(t,\omega))}{p_{A_0}(t,\omega)}$ is analytic because p_{A_0} is never zero.

12. Since the Jacobian matrix $\left.\frac{\partial R_T}{\partial \beta}\right|_{\beta}$ is nonsingular at some β_0, $\det\left(\left.\frac{\partial R_T}{\partial \beta}\right|_{\beta_0}\right) \neq 0$, so $\det\left(\left.\frac{\partial R}{\partial \beta}\right|_{\beta)}\right) \neq 0$. By continuity, there is an open set of (β,t) in $\mathbf{R}^K \times [0,T]$ on which $\det\left(\left.\frac{\partial R}{\partial \beta}\right|_{(\beta,t)}\right) \neq 0$. But the determinant of a matrix is a polynomial function of the entries of the matrix, so $\det\left(\left.\frac{\partial R}{\partial \beta}\right|_{(\beta,t)}\right)$ is an analytic function of $(\beta,t) \in \mathbf{R}^K \times [0,T]$. If $\det\left(\left.\frac{\partial R}{\partial \beta}\right|_{(\beta,t)}\right) = 0$ on a set of positive measure, it must be identically zero. We conclude that $B = \{(\beta,t) \in \mathbf{R}^K \times [0,T] : \det\left(\left.\frac{\partial R}{\partial \beta}\right|_{(\beta,t)}\right) = 0\}$ is a null set.

13. Since the distribution of β is absolutely continuous with respect to Lebesgue measure, $\{(t,\omega) : (\beta(t,\omega),t) \in B\}$ is a null set. Therefore, the securities price process p_A is essentially dynamically complete ([14], Theorem 5.6).

14. Since $c(t,\omega)$ and $\psi_i(c,\beta,t)$ are analytic functions for $t \in [0,T)$, each $c_i(t,\omega)$ is an analytic function of $(\beta(t,\omega),t)$. Since $c(T,\omega)$ and $\Psi(c,\beta)$ are analytic functions, each $c_i(T,\omega)$ is an analytic function of $\beta(T,\omega)$.

15. Since $\gamma(t,\omega)$ is an analytic function of $(\beta(t,\omega),t)$, Itô's Lemma implies it is an Itô process. The fact that p_C is uniformly bounded and the growth condition on the endowments and dividends implies that the Itô coefficients of γ lie in $\mathcal{H}^2$.

16. The hyperfinite equilibrium trading strategies $\hat{z}_i$ satisfy a linear equation, where the coefficients of the linear equation are given by standard analytic functions of $(\hat{\beta}(t,\omega),t)$. Therefore, except at the infinitesimal Hart points (which we found are contained in $\mathrm{st}^{-1}(B)$, where B is a closed set of measure zero), $\hat{z}_i$ is a standard analytic function of $(\hat{\beta}(t,\omega),t)$. In particular, it does not chatter. Define $z_i(t,\omega) = {}^\circ\hat{z}_i(t,\omega)$.

17. As in Raimondo's Single Agent Model, show that the prices, consumptions and trading strategies form an equilibrium.

18. Since everything is expressed as analytic functions of $(\beta(t,\omega),t)$, we can move the prices, consumptions and trading strategies to the original model and verify they form an equilibrium there.

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Problems

1. Consider a one-dimensional random walk process like the random walk process X discussed in the tutorial, except that every step is of the form $\frac{\pm\sigma}{\sqrt{n}}$ rather than $\frac{\pm 1}{\sqrt{n}}$. What is the limit of this random walk as $n \to \infty$ (you may argue informally)? Show that the limit is a standard one-dimensional Brownian motion with a time change.

2. Let X be the one-dimensional random walk discussed in class. We noted in the tutorial that, with respect to the partition $t_k = \frac{k}{n}$, the quadratic variation

$$\sum_{k=0}^{nT-1} \left(X(\omega, t_{k+1}) - X(\omega, t_k)\right)^2 = T$$

for all ω.

   (a) Show that if $t_k = \frac{2k}{n}$ for $k = 1, \ldots, \lfloor \frac{nT}{2} \rfloor$, then

$$\sum_{k=0}^{\lfloor \frac{nT}{2} \rfloor - 1} \left(X(\omega, t_{k+1}) - X(\omega, t_k)\right)^2 \to T$$

   in probability as $n \to \infty$.

   (b) Show that, if one is allowed to choose t_k as a function of ω ($k = 1, \ldots, m(n, \omega)$), then one can find a choice of $t_k(\omega)$ such that

$$\sum_{k=0}^{m(n,\omega)} \left(X(\omega, t_{k+1}(\omega)) - X(\omega, t_k(\omega))\right)^2 \to \frac{3T}{2}$$

   in probability as $n \to \infty$.

   (c) Conjecture what assumption is needed on the $t_k(\omega)$ to ensure that

$$\sum_{k=0}^{m(n,\omega)} \left(X(\omega, t_{k+1}(\omega)) - X(\omega, t_k(\omega))\right)^2 \to T$$

   in probability as $n \to \infty$.