Notes on asset pricing in discrete time

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Chapter 1

Introduction

> "If we take the Keynesian construction seriously, that is, as of a world with a past as well as a future and in which contracts are made in terms of money, no equilibrium may exist...... From all this, as well as from our existence discussions, we conclude that the Keynesian revolution cannot be understood if proper account is not taken of the powerful influence exerted by the future and the past on the present and by the large modifications that must be introduced into both value theory and stability analysis, if the requisite future markets are missing."
> Kenneth Arrow and Frank Hahn, General Competitive Analysis, 1971.

The Arrow-Debreu model of an economy seems to describe a static world without time or uncertainty. Hicks (1939), in his 'Value and Capital' already criticized the steady-state analysis of Walras as a highly deceptive way of analyzing the intertemporal economy we live in. Arrow and Debreu, it seems, merely formalized the highly deceptive model developed by Walras. However, as Chapter 7 of Debreu's 'Theory of Value' shows a simple redefinition of a commodity allows one to obtain a theory of time and uncertainty which is formally identical to the static general equilibrium theory.

Modeling time in the Walrasian tradition seems fairly straightforward. Even before Arrow and Debreu's work, Irving Fisher (1930) wrote about the theory of interest and developed (a somewhat imprecise) dynamic general equilibrium model. The main idea is that a tomato in summer is a different good than a tomato in winter. A commodity is not only identified by its physical characteristics (i.e. a tomato) but also by the date. The relative price between goods today and goods tomorrow is the interest rate. Fisher's model becomes interesting if one makes additional assumptions on preferences and if one examines models with infinite time horizon.

As early as 1953 Arrow found a brilliant way to incorporate uncertainty into the model. He introduced 'states of the world' as a complete description of a date-event. A contract for the transfer of a commodity specifies now, in addition to its physical properties, its location and its date, an event of occurrence (state of the world) of which the transfer is conditional (so a tomato when it rains tomorrow is a different good than a tomato when the sun shines). Once uncertainty and time is incorporated into the model one has to ask, however, if it is realistic to assume that at time zero there are markets for all commodities and agents buy their consumption bundles for the (possibly infinite) future. Instead, Arrow suggested that agents trade financial assets instead. This was the birth of modern asset pricing.

In 1978 Lucas published a paper on asset pricing that proved highly influential in macro-economics mainly because it provided a simple way to bring the basic general equilibrium model to data. In his model, there is only one agent. This has the advantage of tractability, but obviously is not satisfactory from the point of view of general equilibrium analysis. In the late 70's and early 80's economists then understood how to generalize his model to several agents.

I will introduce the general equilibrium model with incomplete asset markets (GEI model). In this model households have access to financial markets but cannot write contracts for state contingent consumption directly. The number of securities is taken to be exogenous.

I discuss the Lucas asset pricing model and then a model with heterogeneous agents but finite time horizon. Finally, I will describe how to generalize this model to infinite horizon.

Chapter 2

Time, uncertainty and assets

We examine an economy over $t = 0, \ldots, T \leq \infty$ with uncertainty over the state of nature in periods $t = 1, \ldots, T$.

We assume that uncertainty can be modeled as an event tree Σ. We denote a generic node of the event tree as $\sigma \in \Sigma$. There is a unique root node σ_0 which does not have a predecessor. Each other node $\sigma \in \Sigma$ has a unique direct predecessor σ_-. Each non-terminal node σ has one or several (but finitely many) direct successors which we collect in a set $\Im(\sigma)$. A terminal node has no successors, i.e. $\Im(\sigma)$ is empty if σ is a terminal node. We write $\sigma \succ_\Sigma \sigma'$ if σ is a (no necessarily direct) successor of σ' in the tree Σ (and $\sigma \succeq_\Sigma \sigma'$ if we want to allow for the possibility that $\sigma = \sigma'$.

Figure 2.1 shows a simple event tree for a 3 period model (T=2)

Let M denote the total number of states, that is the number of nodes in the event tree. We collect all nodes at time t, i.e. all nodes which have t predecessors in a set $\mathcal{N}_t$.

2.1 Stationary trees

It is often useful to restrict the nature of uncertainty a little bit beyond this very general formulation and to assume that each non-terminal node has the same number of successor nodes (as is the case in Figure 2.1) and that all terminal nodes are in $\mathcal{N}_T$. We call an event tree which satisfies these two requirements a 'stationary tree'.

In Chapter 3 we will go even further and assume stationarity of all exogenous variables. This will enable us to represent an infinite economy with finitely many utility

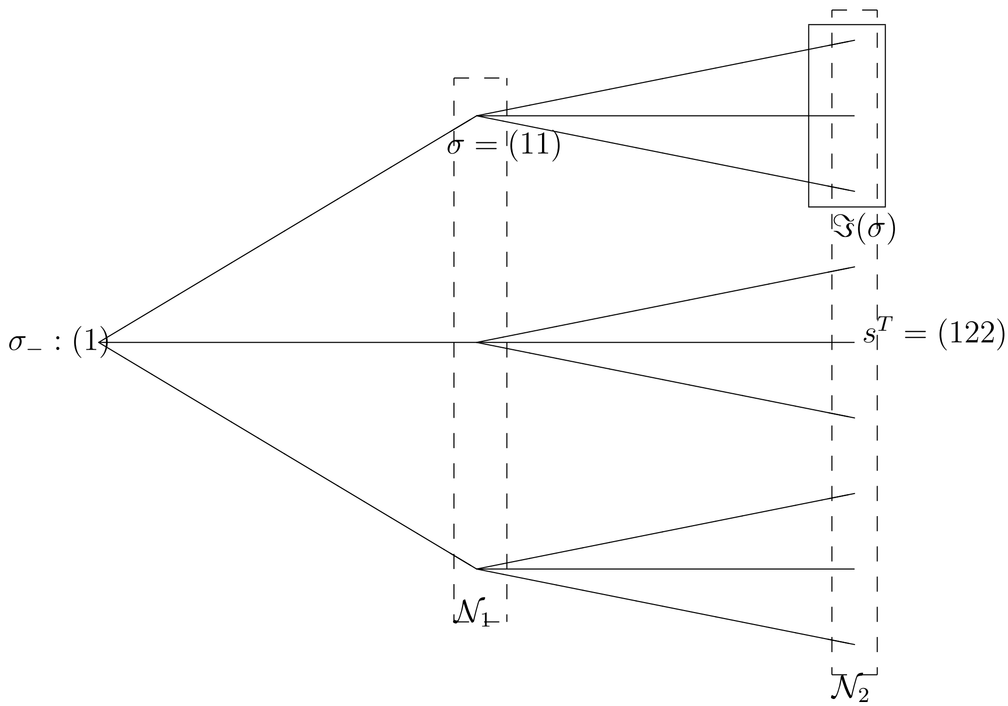


Figure 2.1: Example of an event tree Σ with $T = 2$

functions and real parameters. For this the following notation will be particularly useful.

We assume that in each period t one of S possible shocks $s_t \in \mathcal{S}$ realizes. With this assumption we can associate with each node in a stationary event tree a history of shocks. For a typical node σ which occurs at time t, we often write

$$\sigma = s^t = (s_0, s_1, ..., s_t)$$

If a node s^t is followed by a shock $\bar{s}$ we write $(s^t\bar{s})$ for the new node.

With this notation we can refer to each node simply as a series of shocks, for example $s^T = (122)$ is a node in the tree in Figure 2.1. In a slight abuse of notation we will write s^{t-1} for s^t's predecessor and s^{t+1} for a generic direct successor. I will try to avoid the use of ‘state’ in a multiple period setting and will instead use ‘shock’ on one hand and date-event or node on the other. The term ’state’ will be reserved to the collection of endogenous variables sufficient for the maximization problem of an agent in a recursive economy.

2.1.1 Infinite event trees: Paths versus nodes

In Chapters 3 and 5 of these notes we assume that $T = \infty$. This raises some technical issues. For example, what is an appropriate commodity space for this economy ? Are there countably many commodities or uncountably many ? One way to look at this, is to realize that there will be uncountably many paths: Suppose for simplicity that $S = 10$ – then we will be able to associate with each paths a real number in the interval $[0, 1)$. If we take this interpretation, we will have to require that everything we says holds almost surely (since there can always be one path with has probability zero and about which we do not care). In the next section I will explain how this view fits nicely into probability theory.

On the other hand, looking at the economy as an infinite event tree, we realize that the number of nodes is countable. Each node in the tree Σ is a *finite* history of shocks and we can therefore just enumerate all nodes.

$$s^t = (s_0, \ldots, s_t).$$

In these notes I will take the latter interpretation since it is somewhat more convenient. The set of nodes Σ remains countable and in this interpretation the commodity space is ℓ^∞ - the space of all bounded functions from Σ.

In parts of the notes we will assume that s_t is a stochastic process that follows a Markov chain. However, for now this is irrelevant, the only point is that a date-event (or node) in the event tree can be associated with a finite history of exogenous shocks.

2.2 A view from probability theory

In finance (and sometimes in Macro) uncertainty is often modeled using concepts from probability theory. The underlying idea is that there are some true probabilities for the states of the world. For a finite event tree these concepts are equivalent to our notation (as mentioned above for an infinite horizon model it makes a difference because the set of paths in not countable in this model). The advantage of the finance formulation is that it can easily be extended to continuous shocks (i.e. the shocks can be normally distributed) and to continuous time. Since models with continuous shocks or time are not well understood at this point (certainly not by me) there is no advantage to using the finance notation in these notes. Throughout the notes I will stick to the math econ formulation, but it is worth pointing out the differences (for a good introduction into the finance formulation see e.g. Duffie (2001) – this is in general a good reference for dynamic asset pricing).

There is an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ denotes the tribe (also called σ-algebra or σ-field depending on the context) of subsets of the set of possible states of the world Ω. $\mathbb{P}$ denotes the probability measure that assigns to any event $A \in \mathcal{F}$ a probability $\mathbb{P}(A)$.

At each date T a tribe $\mathcal{F}_t \subset \mathcal{F}$ denotes the set of events corresponding to the information available at date t. We assume $\mathcal{F}_t \subset \mathcal{F}_s$ whenever $s > t$. The filtration $\mathbb{F} = \{\mathcal{F}_0, \ldots, \mathcal{F}_T\}$ represents how information is revealed through time. If $\mathcal{F}_T = \mathcal{F}$ we can take each $\omega \in \Omega$ to be a sample path of the event tree. In the example in Figure 2.1 the set Ω has 9 elements altogether, if we number them (say from top to bottom) 1, ..., 9, we can write $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and

$$\mathcal{F}_1 = \{\Omega, \emptyset, \{1,2,3\}, \{4,5,6\}, \{7,8,9\}, \{1,2,3,4,5,6\}, \{1,2,3,7,8,9\}, \{4,5,6,7,8,9\}\}$$

We can then identify a node of the event tree by a date and a state of the world, ω_t. For two different states ω and ω' we must have that ω_t and ω'_t denote the same node of the event tree if and only if there is no $A \in \mathcal{F}_t$ such that $\omega \in A$ and $\omega' \notin A$. Endowments, dividends, individual consumption etc. then have to be adapted processes, i.e. some $x(\omega_t)$ is $\mathcal{F}_t$ measurable.

This notation is only really useful if one wants to examine continuous shocks or continuous time. For this, it is important to understand that with finitely many periods the two notations describe exactly the same things.

2.3 Physical commodities and financial assets

We assume that at each node of the event tree there is a single perishable commodity available for consumption. Generalizing the analysis to multiple commodities changes nothing in the basic results (just makes notation even more cumbersome).

We assume that there are no markets for contingent claims, i.e. the agents face a separate budget constraint at every node σ.

In order to transfer wealth between time periods and states of nature, agents have to hold a portfolio of assets. At each node $\sigma \in \Sigma$ there are J assets $j \in \mathcal{J}$, each asset $j \in \mathcal{J}$ generates payoff at all direct successors $(a_j(\zeta))_{\zeta \in \Im(\sigma)}$. Assuming that the number of assets is the same at all nodes is without loss of generality since $a_j(\zeta)$ could be zero for all successors ζ. The crucial assumption is that assets pay off in both assets and the single commodity, i.e. $a_j(\sigma) \in \mathbb{R}^{J+1}$ for all $\sigma \in \Sigma$. We assume that $a_{j0}(\sigma)$ is the payoff in node σ commodity (and sometime write $d_j(\sigma)$ to refer to this payoff as a dividend) while $\tilde{a}_j = (a_{j1}, \ldots, a_{jJ})$ is the payoffs in node σ assets. The price of assets at node σ is denoted by $q(\sigma)$, a row vector. We collect payoffs of all assets at σ in a $J+1 \times J$ matrix

$$A(\sigma) = (a_1(\sigma), \ldots, a_J(\sigma)).$$

An agent must trade in assets in order to transfer wealth across time and states. At each non-terminal node σ agent h chooses a portfolio $\theta^h(\sigma) \in \mathbb{R}^J$. At each node except the root node the agent receives dividend payments from the assets he has bought at the previous node, $(1, q(\sigma))A(\sigma)\theta^h(\sigma_-)$.

We will refer to a process of portfolio-holdings $(\theta(\sigma))_{\sigma \in \Sigma}$ as a 'trading strategy'. Later on we impose restrictions on admissible trading strategies, but for now it is just an arbitrary map for Σ to portfolios.

It will be useful to associate with a trading strategy $(\theta(\sigma))_{\sigma \in \Sigma}$ a so-called 'gain-process':

$$D^\theta(\sigma) = (1, q(\sigma))A(\sigma)\theta(\sigma_-) - \theta(\sigma)q(\sigma) \tag{2.1}$$

We denote the initial portfolio of an agent by $\theta(s^{-1})$. Later on we will allow assets to be in unit net supply or in zero net supply and we will require that the initial portfolios of agents add up to the asset's net supply.

2.3.1 Trees, stocks and bonds

We want to examine two special cases of assets. Trees (or stocks) are infinitely lived assets which pay non-negative dividends $d(\sigma)$ at all date events σ. One period assets, for example one period bonds, just pay dividends, but not in other assets.

In finance and macroeconomics it is often assumed that there are only long-lived assets such as stocks and a single bond. In these models, stocks pay dividends each period and are traded each period. Note that a stock is a special case of our more general asset structure: Asset 1 is a stock if for all (non-terminal) nodes σ, $a_1(\sigma) = (d_1(\sigma), 1, 0, \ldots, 0)$ where $d_1(\sigma)$ is simply the dividend process of the stock. A stock is therefore just an asset that pays some dividends and one unit of the new stock next period. So called short-lived (or one period) assets on the other hand pay only in commodities and not in other assets.

2.3.2 The absence of arbitrage

As it is well known, if utility-functions are increasing the agents problem to maximize utility over the budget set can only have a solution if there are no arbitrage opportunities in the price system.

Definition 1 *Prices and payoffs* $(q(\sigma), A(\sigma))_{\sigma\in\Sigma}$ *preclude arbitrage if there is no trading strategy* $(\theta(\sigma))_{\sigma\in\sigma}$ *with* $\theta(s^{-1}) = 0$ *such that* $D^\theta(\sigma) \geq 0$ *for all* $\sigma \in \Sigma$ *and* $D^\theta(\sigma) \neq 0$ *for at least one* $\sigma \in \Sigma$.

What the definition rules out is that it is possible to get adopt a portfolio strategy over the tree that ensures positive payoffs in some states, while never paying anything. We will see later that there might be limited arbitrage opportunities in the presence of transaction costs or trading constraints. However, for now we assume that there are no trading constraints and security-prices must satisfy the absence of arbitrage.

One can show that the absence of arbitrage is equivalent to the existence of a strictly positive pricing vector.

Theorem 1 *Prices and payoffs* $(q(\sigma), A(\sigma))_{\sigma\in\Sigma}$ *preclude arbitrage if and only if there exists a strictly positive state-price process* $(\alpha(\sigma))_{\sigma\in\Sigma}$ *such that for all non-terminal* $\sigma \in \Sigma$,

$$q(\sigma) = \frac{1}{\alpha(\sigma)} \sum_{\zeta\in\Im(\sigma)} \alpha(\zeta)(1, q(\zeta))A(\zeta) \tag{2.2}$$

Note that, if T is finite, one can obtain an expression for asset prices as a linear function of all future dividends (i.e. commodity payoffs) by substituting out all future prices in Equation 2.2. For example, the price at node σ_0 of a stock with dividend process $(d_j(\sigma))_{\sigma\in\Sigma}$ is simply

$$q_j(\sigma_0) = \sum_{\sigma\in\Sigma} \alpha(\sigma) d_j(\sigma).$$

If T is infinite, we can generally only get an inequality in equation (2.2) since we cannot rule out bubbles. It is beyond the scope of this class to get into details there. A very good reference is K.X.D. Huang and J. Werner 'Valuation bubbles and sequential bubbles' which is published in *Economic Theory*.

The theorem can be proved using a separating hyperplane argument. The following lemma follows pretty easily from the separating hyperplane theorem (see for example Duffie (2001))

Lemma 1 *Suppose M and K are closed convex cones in $\mathbb{R}^n$ that intersect precisely at zero. If K is not a linear subspace, then there is a nonzero linear functional F such that $F(x) < F(y)$ for each $x \in M$ and each nonzero $y \in K$.*

If we define a set $\mathcal{M} \subset \mathbb{R}^M$ as $\mathcal{M} = \{(D^\theta(\sigma))_{\sigma\in\Sigma} : \theta \text{ is a trading strategy}\}$ the absence of arbitrage means that $\mathcal{M} \cap \mathbb{R}^M_+ = \{0\}$. It is clear that if there exist strictly positive supporting prices, then there cannot be a positive element in $\mathcal{M}$ because if it were we could multiply it by the state prices and obtain a contradiction (by definition of a trading strategy we would get that both $-x$ and $+x$ must be strictly positive for some x). Conversely, the above implies that there exists a linear function F that separates $\mathbb{R}^M_+$ and $\mathcal{M}$. The functional must be strictly positive because for all $x \in \mathbb{R}^M_+$ we have $F(x) > 0$. Furthermore for any $D^\theta \in \mathcal{M}$, $F(D^\theta) \leq 0$, but since $\mathcal{M}$ is a linear subspace this must imply that $F(D^\theta) = 0$. For any asset j traded at node σ, we can pick a trading strategy to be $\theta_j(\sigma) = 1$ and $\theta_{j'}(\zeta) = 0$ for all $\zeta \neq \sigma$ and $j' \neq j$ and get Equation (2.2). This proves the theorem.

The absence of arbitrage has a nice geometric interpretation. Given J assets at some node σ the set of arbitrage free prices is the interior of a cone with origin at zero. The cone is spanned by the extreme vectors of the assets' payoffs in commodities across assets for given states.

Chapter 3

Lucas asset pricing

> "Dynamics studies sequences of vectors of random variables indexed by time, called time series."
> Lars Ljungquist and Thomas Sargent, Recursive Macroeconomic Theory.

In this chapter I will present my version of Lucas' famous asset pricing model. I will present some basic dynamic programming and I will talk about the equity premium puzzle.

3.1 The basic Lucas model

The easiest model arises if one assumes that there is only one consumer in the economy. Since there are no firms either, there is no trade, the single consumer eats aggregate endowments and one can easily say something about the equilibrium asset prices. The only available assets in the economy are trees (Lucas-trees) and a bond.

Lucas (1978) was the first to write down this model, his contribution (in my view) was to make uncertainty tractable by assuming that all exogenous variables follow a Markov chain.

3.1.1 Markov uncertainty

The event tree is generated by a Markov chain (s_t) with finite[1] support $\mathcal{S} = \{1, ..., S\}$. Each node of the event tree can be associated with a finite history of shocks $\sigma = s^t =$

[1] In Lucas' original model it is assumed that this is an $AR(1)$ process with log-normal innovation. This assumption has the advantage that it is easier to take the model to data. However, in models with heterogeneous agents things become incredibly complicated if we do not assume finite support.

$(s_0, ..., s_t)$. The Markov chain follows a transition π.

There is a single agent with time-separable expected utility

$$U(c) = E_0 \sum_{t=0}^{\infty} \beta^t u(c(s^t)),$$

where the expectation is taken under the Markov-chain probabilities.

We will assume throughout that $0 < \beta < 1$ and that $u : \mathbb{R}_{++} \to \mathbb{R}$ is strictly increasing, strictly concave, C^2 and satisfies the Inada condition $u'(c) \to \infty$ as $c \to 0$.

The agent has individual endowments $e(s^t) \geq 0$ which are time-invariant functions of the shock alone, i.e. $e(s^t) = \bar{e}(s_t)$. Each assets' dividends are time-invariant functions of the shock alone $A(s^t) = \bar{A}(s_t)$.

The agent is endowed with an initial portfolio of assets $\theta(s^{-1})$.

3.2 Equilibrium, necessary first order conditions and asset prices

Before defining a competitive equilibrium in this economy, we need to discuss the agent's maximization problem.

3.2.1 The individual's problem

We assume that the single agent takes prices as given[2] and chooses portfolios and consumption to maximize his utility

At each node the faces the budget constraint

$$c(s^t) - e(s^t) \leq (1, q(s^t))A(s^t)\theta(s^{t-1}) - q(s^t)\theta(s^t).$$

We collect the set of all non-negative consumption processes and portfolio processes which satisfy these constraints at all nodes in a budget set $\mathcal{B}(q)$.

The agent chooses consumption and portfolios at all nodes, (c, θ) to solve

$$U(c) \text{ subject to } (c, \theta) \in \mathcal{B}(q) \tag{3.1}$$

I will give references below which show how one can approximate the original model by finitely many shocks.

[2]One could think of this as a continuum of identical agents, this might make it easier to assume price-taking...

Unfortunately, this maximization problem will almost never have a solution because we have not ruled out Ponzi schemes yet. Given any candidate solution to (3.1) which yields finite utility, one can always improve by consuming one unit more today, borrowing that unit and rolling over the debt until infinity. Of course, this might involve infinite debt at infinity. So in order to rule out Ponzi-schemes, we need to **impose additional restrictions** on the agent's choices. There are several possible ways to do this, see Levine and Zame (1996) or Magill and Quinzii (1996) for a discussion. For now I want to impose as an additional constraint, the so-called implicit debt constraint

$$\inf_{\sigma \in \Sigma} q(\sigma)\theta(\sigma) > -\infty \tag{3.2}$$

This constraint implies that along all paths of the event tree, the agent can never get so much into debt that he cannot pay it off in finite time (maintaining non-negative consumption).

3.2.2 Equilibrium

A competitive equilibrium in this economy consists of processes for asset prices and individual choices $(q(\sigma), c(\sigma), \theta(\sigma))_{\sigma \in \Sigma}$ such that the agent solves (3.1) subject to (3.2) and markets clear, i.e. at all $\sigma \in \Sigma$,

$$\theta(\sigma) = \theta(s^{-1}), \quad c(\sigma) = e(\sigma) + \theta(s^{-1}) \cdot d(\sigma)$$

Note that we define market clearing relative to the agent's initial endowments in the assets. In Lucas' model, the trees are in unit net supply and the agent's consumption in fact consists entirely of the fruits (dividends) of the tree. On the other hand, one could consider a model where all assets are in zero net supply and the agent only eats his individual endowments.

3.2.3 Equilibrium prices

The following condition, called Euler-equation in macroeconomics, is a necessary condition for an interior (finite) optimum

$$-q(s^t)u'(c(s^t)) + \sum_s \pi(s|s_t)(1, q(s^t))A(s^t)u'(c(s^{t+1})) = 0 \tag{3.3}$$

To see this, suppose (c^*, θ^*) is a solution to (3.1), (3.2) with strictly positive consumption. For any portfolio θ, because c^* is positive, there must exist an $\alpha > 0$

such that $c^*(s^t) - \alpha q \cdot \theta > 0$ and $c^*(s^{t+1}) + \alpha(1, q(s^{t+1}))A(s^{t+1}) \cdot \theta > 0$ for all $s^{t+1} \succ s^t$. define

$$g(\alpha) = u(c^*(s^t) - \alpha q \cdot \theta) + \beta \sum_s \pi(s|s_t) u(c^*(s^{t+1} + \alpha(1, q(s^{t+1}))A(s^{t+1})))$$

By optimality of c^*, $g'(\alpha) = 0$ at $\alpha = 0$. This implies (3.3).

In equilibrium, by market clearing we must have that the agent consumption equals endowments plus dividends, that is they only depend on the current shock s ! Using the Euler equations of the first agent we can compute the price function of any asset (knowing the consumption of the agent). Taking advantage of the time invariant consumption, we define (probability deflated) prices by $p_s = u'(c(s))$. Here it helps that the shock space is finite. In Lucas original model, asset prices where the solution to a linear Fredholm integral equation. However, note that one way to solve linear Fredholm equations is by quadrature based methods which discretize the continuous state space (see Tauchen and Hussey (1991)).

For a long-lived asset (tree) j the Euler equations for agent 1 are:

$$q_j(s_t) p_{s_t} = \beta E \left\{ p_{s_{t+1}} (q_j(s_{t+1}) + d_j(s_{t+1}) | s_t \right\}$$

These equations are a system of S linear equations in S unknowns. In the following, we will use a circle, $\circ$, to denote element-wise multiplication of vectors. Specifically, if $x, y \in \mathbb{R}^S$ then

$$x \circ y = \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \\ \vdots \\ x_S y_S \end{pmatrix} \in \mathbb{R}^S$$

We denote the identity matrix of size S by I_S.

The solution for asset prices is therefore

$$q_j \circ p = [I_S - \beta \Pi]^{-1} \beta \Pi (p \circ d_j).$$

For a short-lived asset j the Euler equations for agent 1 are:

$$q_j(s_t) p_{s_t} = \beta E \left\{ p_{s_{t+1}} d_j(s_{t+1}) | s_t \right\}$$

To summarize, in this model, if an equilibrium exists, the equilibrium prices follow directly from the agent's marginal utilities and aggregate consumption. With this observation (still believing that equilibrium exists, which basically means here that the first order conditions are also sufficient) one can easily bring the model to data. One can for example assume that there is a single tree which one identifies with the aggregate stock market and a bond and one can then check what preferences one needs to satisfy (3.3). This leads to the equity premium puzzle of Mehra and Prescott (1985). I do not have time to go into this here, but Kocherlakota (1996) is a good reference.

3.3 Sufficient conditions

Under which additional assumptions can we show that (3.3) are also sufficient conditions for optimality ? If we can establish this, then it is clear that an equilibrium exists and equilibrium prices are simply determined as described in the section above.

The following lemma helps.

Lemma 2 *Assume that Bernoulli utility* $u(.)$ *is bounded above. Suppose asset prices are bounded, i.e.* $\sup_\sigma q(\sigma) < \infty$. *A process* $(\bar{c}(\sigma), \bar{\theta}(\sigma))$, *with* $\sup_\sigma q \cdot \bar{\theta}(\sigma) < \infty$ *and with* $\sup_\sigma u'(c(\sigma)) < \infty$ *solves an agent* h*'s optimization problem (3.1),(3.2) if for all* s^t *the Euler equation holds, i.e.*

$$-q(s^t)u'(\bar{c}(s^t)) + \sum_s \pi(s|s_t)(1, q(s^{t+1}))A(s_{t+1})u'(\bar{c}(s^{t+1})) = 0$$

Proof. Let $(c(\sigma), \theta(\sigma))$ be an arbitrary budget feasible process, satisfying (3.2). Note that concavity of u implies that for all σ,

$$u(\bar{c}(\sigma)) - u(c(\sigma)) \geq u'(\bar{c}(\sigma))(\bar{c}(\sigma) - c(\sigma)).$$

Since consumption in 0 only differs by the value of the new portfolio, we have that $u(\bar{c}_0) \geq u(c_0) + u'(\bar{c}_0)q(s_0)(\theta(s_0) - \bar{\theta}(s_0))$.

For any T, by induction and concavity we will show that

$$E\sum_{t=0}^{\infty} \beta^t u_h(\bar{c}(s^t)) \geq \quad E\textstyle\sum_{t=0}^{T} \beta^t u_h(c(s^t)) + E\sum_{t=T+1}^{\infty} \beta^t u_h(\bar{c}(s^t)) +$$
$$\beta^T E\left(u'(\bar{c}(s^t))q(s^t)(\theta(s^t) - \bar{\theta}(s^t))\right)$$

We have already shown that the inequality holds for $T = 0$. To understand the induction step, use the first order conditions to substitute $\beta \sum_s \pi(s|s_t)(1, q(s^{t+1}))A(s_{t+1})$ for $u'(\bar{c}(s^t)q(s^t)$. The budget constraint and the law of iterated expectations then imply the induction step.

Since u is bounded above the second term on the right hand side will converge to zero as $T \to \infty$. The third term will converge to zero (or something positive), because consumption is bounded below and asset prices are bounded at equilibria we look at. $\square$

Chapter 4

GEI over multiple periods

In this chapter we want to modify the previous model, assume that the time horizon is finite but that there are heterogeneous agents in the economy. In contrast to the analysis in Chapters 3 and 5 we will not rely on any Markov-structure of the tree here and we do not need to assume that agents have time-separable expected utility.

4.1 An exchange economy

We consider a T periods economy where finitely many households make consumption decisions over the entire time horizon and trade financial securities to share risk. We take as given an event tree Σ which we assume to be finite in this chapter. Let M denote the number of nodes in the tree.

Households

There are H agents – agent $h \in \mathcal{H}$ is characterized by an initial endowment process $e^h = (e^h(\sigma))_{\sigma \in \Sigma} \in \mathbb{R}^M_{++}$ and his or her preferences. These are represented by a utility function over consumption processes, $c = (c(\sigma))_{\sigma \in \Xi} \in \mathbb{R}^M_{++}$, $u^h : \mathbb{R}^M_+ \to \mathbb{R}$. While we usually assume that agents have time-separable expected utility with beliefs being derived from π, most of the analysis in this chapter does actually not need any other assumption than concavity and monotonicity of u^h.

Assets

Without loss of generality we assume in this chapter of the notes that all assets are in zero net supply. By altering individual endowments appropriately, we can always reformulate the model into having assets in unit net supply. For example, if there is an asset in unit net supply which pays dividends $d(\sigma)$ at state σ and is initially owned by agent 1 the model is equivalent to a model where this asset is in zero net supply and agent 1's endowments are $\bar{e}^1$ with $\bar{e}^1(\sigma) = e^1(\sigma) + d(\sigma)$. If a stock is in unit net supply, it simply means that for all $\sigma \in \Sigma$ we have to subtract the stock's dividends from the initial owners individual endowments.

4.1.1 Walrasian equilibrium

The definition of a Walrasian equilibrium is standard - a collection of Debreu prices $\rho \in \Delta^{M-1}$ and a consumption allocation $(c^h)_{h\in\mathcal{H}}$ such that markets clear and all agents maximize their utility.

Even more so than in a two period model, it is clear in this model what it makes little sense to assume that in the beginning of all times agent trade node-contingent commodities and that markets never reopen after this. Instead we want to assume that agents trade in financial markets.

4.1.2 Arrow equilibria and dynamic completeness

It is clear that given endowments and utility functions a Walrasian equilibrium exists. We can decentralize this equilibrium if we assume that there exists an Arrow security for each node of the event tree. However, in general, one would think that one needs much fewer securities than that. In particular, if at any node of the tree one had one Arrow security for each direct successor markets would be complete. Nevertheless at the root node there are only markets for the Arrow securities which pay at direct successors.

Following Kreps(1982) we call a model dynamically complete if the markets can be completed through trading over time, i.e. for each consumption process $(c(\sigma)_{\sigma\in\Sigma}) \in \mathbb{R}^M_+$ there exists a trading strategy θ such that $c(\sigma) = D^\theta(\sigma)$ for all $\sigma \neq \sigma_0$.

This is equivalent to saying that the dimension of the marketed subspace $\mathcal{M}$ is M.

For example, if we had a binomial event tree, i.e. each non-terminal node has exactly two successors and we have two assets which pay positive dividends at each node chances are that trading these two assets at each node suffices to achieve the complete markets allocation. If assets only pay in commodities this is straightforward - all we need is that the payoffs are linearly independent.

However, if assets pay both in other assets and in commodities there might be a problem. If for example at some σ $a_1(\zeta) = (d, 1, 0)$ and $a_2(\zeta) = (d, 0, 1)$ for both $\zeta \in \Im(\sigma)$, we will not have complete markets if $q_1(\zeta) = q_2(\zeta)$ at both $\zeta \in \Im(\sigma)$.

Therefore the definition of dynamically complete markets depends on the price system q as well as on payoffs $(a(\sigma))$. Given a stochastic finance economy E one cannot tell (without looking at the equilibrium prices) if markets are complete or not. We will return to this issue below after we formally defined an equilibrium.

4.1.3 Equilibrium

We assume that agents maximize their utility functions over their budget set. Like before, since portfolio-choices uniquely determine consumption choices we can substitute for the consumption process and obtain an indirect utility function. The first order conditions for agents' optimality are the following equations which must hold for each non-terminal nodes $\sigma \in \Sigma$,

$$-qD_{c(\sigma)}u^h(c) + \sum_{\zeta \in \Im(\sigma)} (1, q(\zeta))A(\sigma)D_{c(\zeta)}u^h(c) = 0$$

For inexplicable reasons macroeconomists call these equations Euler-equations - we will do so as well.

If utility is strictly increasing this first order condition implies directly possible state prices $\alpha \in \mathbb{R}^M_{++}$. If markets are complete these state prices will coincide with the Debreu equilibrium prices.

GEI Equilibrium

Just like before equilibrium is characterized by market clearing and agents' optimality.

Definition 2 *A GEI equilibrium for a 'Stochastic Finance Economy' is defined as a collection of consumption processes $(\bar{c}^h)_{h=1,\ldots,H}$, portfolio holdings $(\bar{\theta}^h)_{h=1,\ldots,H}$ and an asset price process $\bar{q}$ that satisfy the following conditions:*

(1) For all agents $h = 1, \ldots, H$:

$$(\bar{c}^h, \bar{\theta}^h) \in \arg\max_{c,\theta} \; u^h(c) \; s.t. \; (c, \theta) \in \mathcal{B}^h(q)$$

(2) $\sum_{h \in \mathcal{H}} \bar{\theta}^h(\sigma) = 0$ at all nodes $\sigma \in \Sigma$.

By Walras' law, market-clearing $\sum_{h\in\mathcal{H}}\bar{\theta}^h = 0$ in the financial markets also implies market-clearing in all spot markets, that is, $\sum_{h\in\mathcal{H}}(\bar{c}^h - e^h) = 0$.

As before we can show that if at each node, there is an Arrow security for each direct successor node, a GEI equilibrium is equivalent to a Walrasian equilibrium. For general asset structures, the equivalence only holds if *in equilibrium* markets turn out to be dynamically complete. We will show now that this might cause existence problems.

4.1.4 Conditions for complete markets

As mentioned above, whether markets are complete or not might depend on the actual GEI equilibrium whenever the payoff of assets depend on prices (which is the case if we have stocks, for example).

Markets are dynamically complete, if and only if for all non-terminal nodes $\sigma \in \Sigma$ there are S assets with linearly independent payoffs (in numéraire commodity terms) at direct successor nodes. I.e. if a given node σ has S direct successors which we denote by $(\sigma 1), (\sigma 2), \ldots, (\sigma S)$, it must hold that

$$\text{rank}\begin{pmatrix} (1, q(\sigma 1))A(\sigma 1) \\ \vdots \\ (1, q(\sigma S))A(\sigma S) \end{pmatrix} = S$$

In order to understand this, note that markets are complete if and only if for each node (except the root node) σ there exists a trading strategy θ^σ such that $D^{\theta^\sigma}(\sigma) = 1$ and $D^{\theta^\sigma}(\sigma') = 0$ for all $\sigma' \neq \sigma$, $\sigma' \neq \sigma_0$. But if the above rank condition is satisfied a trading strategy like this can be easily constructed as follows:

$$\theta^\sigma(\sigma') = 0 \text{ unless } \sigma \succ \sigma'$$

Suppose a given σ' has S direct successors. If $(\sigma' s)$ is a predecessor of σ or if it is σ itself (i.e. if $\sigma' = \sigma_-$) then let

$$\theta^\sigma(\sigma') = \begin{pmatrix} (1, q(\sigma' 1))A(\sigma' 1) \\ \vdots \\ (1, q(\sigma' S))A(\sigma' S) \end{pmatrix}^{-1} \iota_s$$

where ι_s is the s'th unit vector in $\mathbb{R}^S$ (i.e. 0 everywhere except in the s'th component). If we construct this strategy recursively, we can guarantee that it pays one unit at node

σ, while at all intermediate nodes it pays zero. At the root node, σ_0 one obviously has to give up some consumption in order to finance this strategy.

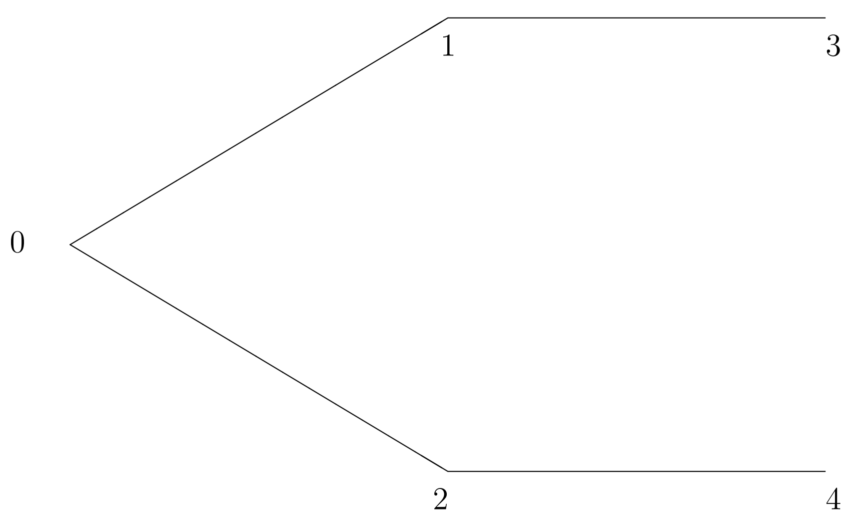
As mentioned above, if assets only pay in commodities, it can be easily checked if markets are complete or not. But if there are long-lived assets in the economy, this is going to depend on the specific GEI equilibrium we consider. Unfortunately, the GEI equilibrium allocation will certainly depend on whether markets are complete or not. This loop might cause existence problems.

4.2 Existence of GEI equilibrium

The following example shows that equilibria might fail to exist in this model.

The example is constructed to be extremely simple. Examples of non-existence can also be constructed for stationary trees and longer time horizons.

Suppose there are three periods $t = 0, 1, 2$.



There are two possible states at $t = 1$ which we denote by 1 and 2 and each of these states has exactly one successor (i.e. all uncertainty is resolved at $t = 1$). We denote the period $t = 2$ nodes by 3 (1's successor) and 4 (2's successor)[1]. There are two assets at each node. The first asset pays 1 unit of the good at all direct successors and zero in all other assets and the second asset pays 1 unit of the first asset at 1 and 2 and zero otherwise (in particular zero commodities). i.e.

$$a_1(\sigma) = (1, 0, 0) \text{ for all } \sigma \text{ and } a_2(\sigma) = (0, 1, 0) \text{ for } \sigma = 1, 2.$$

There are two agents who have Cobb-Douglas utility functions

$$u^1(c) = \log(c_0) + \log(c_1) + \log(c_2) + 1/2(\log(c_3) + \log(c_4))$$

$$u^2(c) = \log(c_0) + \log(c_1) + \log(c_2) + \log(c_3) + \log(c_4)$$

The individual endowments are given by

$$e^1(0, 1, 2, 3, 4) = (1.5, 2, 1, 1.5, 1.5) \text{ and } e^2(0, 1, 2, 3, 4) = (1.5, 1, 2, 1.5, 1.5).$$

The following argument shows that there cannot be an equilibrium: If $q_1(1) \neq q_1(2)$ markets will be dynamically complete and the resulting equilibrium can be implemented as an Arrow-Debreu equilibrium In this case, by the first welfare theorem and efficiency $c^1(3) = c^1(4)$ and $c^1(1) = c^1(2)$ But then, by the agents' Euler equations $q_1(1) = q_1(2)$. But then the two assets at 0 are collinear and markets cannot be complete – a contradiction.

On the other hand, if $q_1(1) = q_1(2)$, the Euler equation, together with market clearing, imply that

$$1/2\frac{c^1(1)}{c^1(3)} = 1/2\frac{c^1(2)}{c^1(4)} = \frac{3 - c^1(1)}{3 - c^1(3)} = \frac{3 - c^1(2)}{3 - c^1(4)}$$

But

$$1/2\frac{c^1(1)}{c^1(3)} = \frac{3 - c^1(1)}{3 - c^1(3)} \text{ implies } c^1(3) = 3\frac{c^1(1)}{6 - c^1(1)}$$

and

$$1/2\frac{c^1(2)}{c^1(4)} = \frac{3 - c^1(2)}{3 - c^2(4)} \text{ implies } c^1(4) = 3\frac{c^1(2)}{6 - c^1(2)}.$$

Therefore $1/2c^1(1)/c^1(3) = 1/2c^1(2)/c^1(4)$ must imply that $c^1(1) = c^1(2)$ and that $c^1(3) = c^1(4)$.

[1]Since the tree is not stationary, we deviate from our notation, associating a node with a history of shocks.

However, this consumption cannot be achieved (cannot be in the budget constraint) if there is only one asset in period 0 which pays the same dividends in both states 1 and 2. Therefore an equilibrium cannot exist.

The example also shows that whether markets are complete or not depends on endogenous variables.

4.2.1 Restoring existence: Generic existence ?

Economists realized quite early that the examples of non-existence like the one above are not robust. If we change individual endowments a tiny little bit and equilibrium suddenly exists. In the 1980's several papers showed that GEI equilibria exist 'generically' in endowments, i.e. examples of non-existence can never be robust. Geanakoplos (1990) provides a number of good references for these proofs. While these results are not uninteresting and provided some new insights it seems to me that it is not worth introducing a large mathematical machinery necessary for these proofs.

However, I do want to point out that the above example of non-existence is clearly not robust to a small perturbation in asset payoffs. If one perturbs the payoffs of any asset in commodities by arbitrarily small ϵ, a GEI equilibrium exist and markets are dynamically complete in equilibrium. This argument relies critically on the fact that in the perturbed economy/equilibrium markets turn out to be complete. When markets are still incomplete in the perturbed economy (e.g. fewer assets than states) the general existence proof is quite involved.

Unfortunately the entire complicated argument of generic existence breaks down if there are non-linear payoff structures. I want to illustrate this in the following example.

Options

Assume that in the above example, asset 2 would not always pay in asset 1 but would actually be an option. call option on asset 1, i.e. at Consider the following variation: We have the same endowments as above but this time utility functions are given by

$$u^1(c) = \log(c_0) + \log(c_1) + \log(c_2) + 1/2\log(c_3) + \log(c_4)$$

$$u^2(c) = \log(c_0) + \log(c_1) + \log(c_2) + \log(c_3) + 1/2\log(c_4)$$

Assume that asset 1 payoff in the second period is as before but that in period 1 it pays one unit of asset 1, i.e. $a_1(1) = a_1(2) = (0, 1, 0)$ and that asset 2 is a put-option, with a strike-price depending on the state (this is just to keep the example simple - it will be easy to see that we can construct examples where this is not necessary). We model a put option by assuming that asset two only pays in commodities, but that the amount commodities paid depends on the price of asset 1. Let

$$d_2(1) = \max(0, K_1 - q_1(1)) \text{ and } d_2(2) = \max(0, K_2 - q_1(2))$$

with $K_1 = 0.7$ and $K_2 = 0.71$. The claim is that there exist no equilibrium for a set of endowments and dividends with positive measure. To see this, consider first the complete markets outcome: By symmetry, we must have that $c^1(1) = c^1(2) = 1.5$. By the same argument as above, market clearing implies that $c^1(3) = \frac{3c^1(1)}{6-c^1(1)} = 1$. Therefore the Euler-equation implies that $q_1(1) = 0.75$. By symmetry $q_1(1) = q_1(2) = 0.75$. Sadly, with this price, the option will pay zero in both states and markets are not dynamically complete. This will be true for a set of dividends and endowments of positive measure. On the other hand, with the option paying zero in both states, by symmetry $\theta_2^1(0) = \theta_2^2(0) = 0$ and $\theta_1^1(0) = \theta_2^1(0) = 0$. Therefore the price of the bond in $t = 1$ must satisfy

$$q_1(1) = 1/2 \frac{2 - q_1(1)\theta_2^1(1)}{1.5 + \theta_2^1(1)} = \frac{1 + q_1(1)\theta_2^1(1)}{1.5 - \theta_2^1(1)}$$

It is easy to see that $\theta = 0$ is the unique solution to this. Therefore $q_1(1) = 2/3$. By symmetry $q_1(2) = 2/3$. But now (and robustly so) the option is in the money and markets are dynamically complete. There exits no equilibrium for an open set of endowments and asset payoffs.

There is no good solution to this problem (in the 80's a lot of people came up with quite obscure assumptions to rule this out but I will not go into this). It seems to me that the only solution is to assume either short-sale constraints or transaction costs. In these cases, the set of possible θ will be compact and standard arguments show that equilibria always exist.

4.2.2 Restoring existence II: Radner

The most obvious way to ensure that demand for assets remains continuous is to restrict agents to choose portfolios from a compact set. If we assume that $|\theta| < K$

for some constant K, agents' best responses will be continuous by construction. A bit more interesting it to assume that agents face short sale constraints only in assets which pay off in other assets. The option in the above example can be interpreted as such an asset since it pays off in the stock. Since we can restrict portfolios of short lived assets a priori by the absence of arbitrage this should be sufficient to obtain existence. I will go through a proof in some detail now. This has already been noted by Radner (1972). I will present a modified version of his argument.

A formal existence proof

We make some formal assumptions

Assumption 1

1. *For each agent h we have*
   (a) *The utility function u^h is strictly increasing and strictly concave*
   (b) *Individual endowments are strictly positive, $e^h \in \mathbb{R}^M_{++}$*
   (c) *All asset payoffs are non-negative, i.e. $A(\sigma) \geq 0$ for all $\sigma \in \Sigma$.*
2. *Assets' payoffs are non-negative, $a_j(\sigma) > 0$ for all $\sigma \in \Sigma$. Whenever an asset pays off in other assets all agents face constraints on short selling the asset, i.e. there exists some number $K > 0$ such that*

$$\tilde{a}_j(\sigma) > 0 \Rightarrow \theta^h_j(\sigma_-) \geq -K$$

Under this assumption a GEI equilibrium always exists. In order to prove this it is useful to choose a different price normalization than before in this chapter. We denote by $p(\sigma)$ the price of the commodity at node σ and normalize

$$\sum_{j \in \mathcal{J}} q_j(\sigma) + p(\sigma) = 1 \text{ for all } \sigma \in \Sigma.$$

We restrict attention to normalized prices with $p(\sigma_t) \geq \epsilon > 0$; and we impose an upper bound $\kappa > 0$, on trades in assets by individuals. We then have the following lemma which you will prove as an exercise.

Lemma 3 *For each $\kappa > 0$ and each $\epsilon > 0$, for each individual h the best response correspondence, subject to the bounds,*

$$\phi^h(q,p) = \arg \max_{(c,\theta)\in\mathcal{B}^h(p,q)} u^h(c) \text{ s.t. } |\theta_j(\sigma)| \le \kappa \text{ for all } j,\sigma$$

is uhc, non-empty and convex valued for all (q,p) with $p(\sigma) \ge \epsilon > 0$ for all $\sigma \in \Sigma$.

We introduce a price player for each node. With the lemma it then follows from a standard fixed point argument that there exist for every individual, $c^h_{\epsilon,\kappa}, \theta^h_{\epsilon,\kappa}$, and prices of commodities and assets $p_{\epsilon,\kappa}$ and $q_{\epsilon,\kappa}$, such that individuals optimize, subject to the additional bound on net trades, while, at every date-event,

$$(p_{\epsilon,\kappa}(\sigma), q_{\epsilon,\kappa}(\sigma_t)) \in \arg\max\{p(\sigma)\sum_{h\in\mathcal{H}}(c^h_{\epsilon,k}(\sigma) - e^h(\sigma)) + q(\sigma)\sum_{h\in\mathcal{H}}\theta^h_{\epsilon,k}(\sigma) : p(\sigma) \ge \epsilon\}.$$

By the budget constraints, whenever $\sum_{h\in\mathcal{H}}\theta^h_{\epsilon,\kappa}(\sigma_-) \ge 0$, we must have

$$p_{\epsilon,\kappa}(\sigma)\sum_{h\in\mathcal{H}}(c^h_{\epsilon,\kappa}(\sigma) - e^h(\sigma)) + q_{\epsilon,\kappa}(\sigma)\sum_{h\in\mathcal{H}}\theta^h_{\epsilon,\kappa}(\sigma) \le 0.$$

By strict monotonicity of utility and since $\sum_{h\in\mathcal{H}}\theta^h_{\epsilon,\kappa}(\sigma_0) = 0$, we must have that the equation holds with equality and by the optimality of the price player we must have for all $\sigma \in \Sigma$,

$$\sum_{h\in\mathcal{H}}(c^h_{\epsilon,\kappa}(\sigma) - e^h(\sigma)) = 0 \quad \sum_{h\in\mathcal{H}}\theta^h_{\epsilon,\kappa}(\sigma) = 0$$

For fixed k, demands and prices for commodities and assets lie in a compact set, and, as a consequence, as $\epsilon \to 0$, they converge, up to a subsequence. Strict monotonicity implies that the price of the commodity remain positive: at the limit, $p_k(\sigma) > 0$, at all date-events. Since for all terminal $\bar{\sigma}$, $q(\bar{\sigma}) = 0$ we have $p_k(\bar{\sigma}) = 1$. If prices of assets at intermediate nodes are positive, monotonicity and market clearing implies that a sufficient small price of the commodity will lead to at least one agent's demand for this commodity to explode. The limit, as $\epsilon \to 0$ is therefore a GEI equilibrium with constraints on all asset trades.

Since the argument works for any finite $\tau > 0$, if we would impose any short sale constraints on all assets, we would be done. However, we want to allow for unlimited short-sales in short-lived assets. So it remains to be shown that as $\kappa \to \infty$ equilibrium portfolio holdings remain bounded. By assumption (and market clearing) holdings of assets which pay in other assets are bounded below by K and above by $(H-1)K$. Since consumption is bounded demand for one period assets is bounded if (and only if) the prices of these assets do not allow for arbitrage (this can be proved as an exercise.)

But if there is an arbitrage opportunity, markets for commodities cannot clear: For sufficiently large κ each agent will hold an arbitrage portfolio. By market clearing this is impossible. Therefore, for sufficiently large κ prices will preclude arbitrage and eventually the constraints on trades in short lived assets can no longer be binding.

From this proof, it should also be clear why we need short-sale constraints on assets that pay in other assets. Since their payoffs (in terms of the numéraire commodity) are exogenous, it is impossible to derive a bound for individuals' holdings in these assets.

4.3 What does the representative agent represent?

In macroeconomics and finance, researchers often consider dynamic general equilibrium models in which there is a single agent. This obviously simplifies solving the model enormously. Prices are not a solution to a fixed point problem anymore but can just be read off the representative agents' consumption (which in the case of a pure exchange economy is aggregate endowments, in the case of a production economy the solution to a simple optimization problem). The use of a single agent in macro has often led general equilibrium theorists to not think of macro as applied general equilibrium theory. I will try to argue that part of the confusion comes from the fact that people read to much into the representative agent. For example, Kirman (1992) writes:

> ... there is no plausible formal justification for the assumption that the aggregate of individuals, even maximizers, acts itself like an individual maximizer.

In this section of the notes I will try to show under which conditions the assumption of a representative agent is not as ridiculous as it sounds.

4.3.1 A price representative agent

For me, the main justification for assuming that there exists a representative agent is that when markets are complete (or slightly more general when the equilibrium allocation is Pareto optimal) possible equilibrium prices for an economy with a single agent are the same as possible equilibrium prices for an economy with several agents if they all have identical beliefs and identical impatience. The following theorem makes this clear.

Theorem 2 *Given an economy $((u^h, e^h)_{h=1}^H, A)$ where all agents maximize time-separable expected utility of the form*

$$U^h(c) = E \sum_{t=0}^{T} \beta^t v^h(c(s^t))$$

(the expectation is taken under some common probability measure). If the GEI equilibrium is Pareto efficient there exists a utility function $v^R : \mathbb{R}_+ \to \mathbb{R}$ such that the economy with only one agent, $((v^R, \sum_{h \in \mathcal{H}} e^h), A)$ has an GEI equilibrium with identical equilibrium asset prices.

Proof. Let $(c^h)_{h \in \mathcal{H}}$ be the GEI equilibrium allocation for the original economy. By the definition of Pareto efficiency, there must exist $\lambda \in \mathbb{R}_+^H$ such that

$$(c^h) = \arg \max_{(c^h) \in \mathbb{R}_+^{MH}} \sum_{h \in \mathcal{H}} \lambda^h u^h(c) \text{ s.t. } \sum_{h \in \mathcal{H}} (c - e^h) = 0$$

This implies that for all agents h and h' and all nodes σ

$$\lambda^h v_h'(c^h(\sigma)) = \lambda^{h'} v_{h'}'(c^{h'}(\sigma)).$$

Saying that state prices α support an equilibrium for the GEI economy means that for all σ, σ'

$$\frac{\alpha(\sigma)}{\alpha(\sigma')} = \frac{v_h'(c^h(\sigma))}{v_h'(c^h(\sigma'))}$$

But by definition this then implies that

$$\frac{\alpha(\sigma)}{\alpha(\sigma')} = \frac{\sum_{h \in \mathcal{H}} \lambda^h v_h'(c^h(\sigma))}{\sum_{h \in \mathcal{H}} \lambda^h v_h'(c^h(\sigma'))}$$

Define $v^R(c) = \sum_{h \in \mathcal{H}} \lambda^h v^h(c)$. The prices α must then support an equilibrium in the representative agent economy. □

The theorem shows that for asset pricing purposes the assumption of a representative agent might be justified if we think that markets are dynamically complete and that all agents have identical beliefs. Note that if markets are complete but agents do not have time-separable expected utility with identical beliefs, there will still be a representative agent, but unfortunately we can say nothing or little about his preferences. In particular they will, of course, not be time and state separable which makes it generally impossible to say anything sensible about asset pricing.

However, even when the assumptions of Theorem 2 hold it is in general difficult to say anything about how the risk aversion of the representative agent relates to the risk attitudes of the agents in the economy.

Even if we assume that all agents have identical utility (i.e. they only differ through their individual endowments), it is not obvious how the wealth distribution affects the risk-aversion of the representative consumer. Gollier (2001) considers a model with only two periods (the analysis can be easily generalized to T periods) and shows that the representative agent is more risk averse than the individual agents if and only if the absolute risk tolerance of all agents is concave. If all agents have HARA utility functions, the absolute risk tolerance is linear and hence the representative agent's attitude to risk is identical to that of all agents in the economy. Therefore, under the common assumption of HARA utility there is no problem as long as one assumes identical utility functions. Without this assumption, no results are known that relate the risk attitude of the representative agent to that of the agents in the economy.

4.3.2 Gorman aggregation

Microeconomists, when they criticize the concept of a representative agent are of course aware of the above theorem. However, what they have in mind if they talk about representative agent is that the economy on the aggregate 'behaves' as an economy with a single agent. Behaves means here that it shows the same comparative statics. This, of course, is generally not true. One needs extremely strong assumptions (in addition to complete markets) as was already pointed out by Gorman (1953).

4.3.3 Incomplete markets

Besides the fact that one might want to allow for heterogeneous beliefs across agents, the main shortcoming of Theorem 2 is obviously that it assumes complete markets (or more precisely efficient allocations).

When markets are incomplete, the concept of a representative agent really makes no formal sense. However, there is no a a large applied literature which argues that quantitatively the effects of incomplete markets are very small. Levine and Zame

(2002) offer a formal explanation for this in a model with infinitely lived agents. It is beyond the scope of this class to get into this in detail. It is true that for a finite horizon model one can show fairly easily that generally (generically in endowments) GEI equilibrium prices can never be supported by a representative agent.

Chapter 5

Infinite time and several agents

It seems fairly straightforward to extend the model from last chapter to a model with infinite time horizon. However, to make things formally correct is not quite so trivial. For example, what is the commodity space, what can we say about preferences over infinite commodity bundles etc. What assumptions do we impose on short-sales, bounded debt etc ?

In this chapter we will consider a very simple version of the infinite horizon version of the model. We assume that each node is a finite history of shocks and that all agents maximize time separable expected utility. Later, when we consider incomplete markets we will also impose restrictions on short-sales.

There are H infinitely lived agents, $h \in \mathcal{H}$, and a single commodity in a pure exchange economy. Each agent $i \in \mathcal{I}$ has endowments $e^i(\sigma) > 0$ at all nodes $\sigma \in \Sigma$ which are time-invariant functions of the shock alone, i.e. there exist functions $\mathbf{e}^h : \mathcal{S} \to \mathbb{R}_+$ such that $e^h(s^t) = \mathbf{e}^h(s_t)$. Agent h has von Neumann-Morgenstern utility over infinite consumption streams

$$U^h(c) = E_0 \sum_{t=0}^{\infty} \beta^t u_h(c_t)$$

for a differentiable, strictly increasing and concave Bernoulli function u_h which satisfies an Indada condition.

There are J infinitely lived assets in unit net supply. Each asset j pays shock dependent dividends $d_j(s)$, we denote its price at node s^t by $q_j(s^t)$. Agents trade these assets but are restricted to hold non-negative amounts of each asset. We denote portfolios by $\theta^h \in \mathbb{R}^J$. At the root node s_0 agents hold initial shares $\theta^{\mathcal{H}}(s^{-1})$.

A competitive equilibrium is a collection $((c^h(\sigma), \theta^h(\sigma))_{h \in \mathcal{H}}, q(\sigma))_{\sigma \in \Sigma}$ such that market clear,

$$\sum_{h \in \mathcal{H}} \theta_j^h(\sigma) = \sum_{h \in \mathcal{H}} \theta_j^h(s^{-1}) \text{ for all } \sigma \in \Sigma, j \in \mathcal{J},$$

and such that agents optimize

$$\begin{aligned} c^h \in \arg\max\nolimits_{c \geq 0} U^h(c) \quad \text{s.t.} \quad & \forall s^t \in \Sigma \\ & c^h(s^t) = \mathbf{e}^h(s_t) + \theta^h(s^{t-1})(q(s^t) + d(s_t)) - \theta^h(s^t)q(s^t), \\ & \inf\nolimits_\sigma q(\sigma)\theta^h(\sigma) > -\infty. \end{aligned}$$

Note that as before, we need to add some constraint that rules out Ponzi schemes. As before, I added the implicit debt constraint. From the previous chapter we can guess that this will not always be sufficient to guarantee existence, so later we will also impose short-sale constraints.

5.1 Complete markets

As long as security markets are complete and preferences are time-separable von-Neumann-Morgenstern, the model remains tractable even if households are heterogeneous. In fact, it is fairly easy to show that both individual consumptions and asset prices are time-invariant function of the shock to endowments and dividends alone. Portfolio holdings are constant across time and states.

By the first welfare theorem the Arrow-Debreu equilibrium allocation will be efficient. The following lemma is fairly trivial but surprisingly very useful and not very well known.

Lemma 4 *Given an efficient allocation $(c^h(\sigma))_{\sigma \in \sigma}^{h \in \mathcal{H}}$, the individual consumptions must be time-invariant functions of the shock alone, i.e. there exist $\bar{c} : \mathcal{S} \to \mathbb{R}_{++}^{\mathrm{H}}$ such that for all s^t and all $h \in \mathcal{H}$, $c^h(s^t) = \bar{c}^h(s_t)$. This fact immediately implies that marginal utilities are collinear among agents.*

Proof. Denote the period 0 probability of event s^t by $\pi(s^t)$. Suppose that there is an equilibrium where for two date-event nodes $s^t, s^{t'}$ with $s_t = s_{t'}$ we have $c^{h'}(s^t) \neq c^{h'}(s^{t'})$ for some agent $h' \in \mathcal{H}$. Then we could improve everybody's utility by redistributing consumption at these nodes as follows, let

$$\tilde{c}^h(s^t) = \frac{\beta^t \pi(s^t) c^h(s^t) + \beta^{t'} \pi(s^{t'}) c^h(s^{t'})}{\beta^t \pi(s^t) + \beta^{t'} \pi(s^{t'})}$$

for all $h \in \mathcal{H}$. This convex combination, $\tilde{c}$, is clearly a feasible allocation and by strict concavity agent h' will derive higher utility. Therefore, $c^{h'}(s^t) \neq c^{h'}(s^{t'})$ contradicts efficiency. □

Of course, as before, when there are long-lived assets, completeness of markets is endogenous – we will return to this in Section 5.1.2 below.

To simplify things, we will assume for the rest of this section that the only assets traded are one-period securities (e.g. bond) and trees.

5.1.1 Characterizing efficient equilibria

In this section, we first want to assume that markets are complete. By the first welfare theorem, equilibrium allocations are efficient. Lemma (4) implies that agents' consumptions are time-invariant functions of the exogenous shock alone. This allows us to write $c(s)$ as functions of the shock s alone.

Prices

Using the Euler equations of the first agent we can compute the price function of any asset (knowing the consumption of the agent). Taking advantage of the time invariant consumption, we define (probability deflated) prices by $p_s = u_1'(c(s))$. Don't get confused. These are not Arrow-Debreu prices as used above. I just could not think of another symbol.

Just as in Chapter 3, for a long-lived asset j the Euler equations for agent 1 are:

$$q_j(s_t)p_{s_t} = \beta E\left\{p_{s_{t+1}}(q_j(s_{t+1}) + d_j(s_{t+1})|s_t\right\}$$

These equations are a system of S linear equations in S unknowns. The solution is

$$q_j \circ p = [I_S - \beta\Pi]^{-1}\beta\Pi(p \circ d_j).$$

For a short-lived asset j the Euler equations for agent 1 are:

$$q_j(s_t)p_{s_t} = \beta E\left\{p_{s_{t+1}}d_j(s_{t+1})|s_t\right\}$$

Knowing the Arrow-Debreu allocation therefore gives us the asset prices right away. It is pretty easy to use Negishi to show that Arrow-Debreu equilibria always exist. What's not so clear is under which conditions they are equivalent to GEI equilibria.

5.1.2 When are markets complete ?

Complete markets means that at each node $s^t \in \sigma$ the payoff matrix

$$W = \begin{pmatrix} q_1(s^t 1) + d_1(1) & \cdots & q_1(s^t S) + d_1(S) \\ \vdots & & \vdots \\ q_{J_1}(s^t 1) + d_{J_1}(1) & \cdots & q_{J_1}(s^t S) + d_{J_1}(S) \\ d_{J_1+1}(1) & \cdots & d_{J_1+1}(S) \\ \vdots & & \vdots \\ d_J(1) & \cdots & d_J(S) \end{pmatrix}$$

has full rank S.

For a given economy $\mathcal{E}$ with long-lived assets and with $J = S$, the rank of the payoff matrix is determined endogenously and there is no guarantee that markets are complete.

We now want to give conditions which ensure that there exits at least one sequential markets equilibrium with complete markets. The following theorem gives a sufficient condition.

Theorem 3 *Suppose that there are $J = S$ assets. Then for a full measure set of short-lived asset dividends there exists at least one efficient financial markets equilibrium.*

In order to prove this theorem, we need a lemma.

Lemma 5 *If equilibrium allocations are time-invariant functions of the shock alone, the assumption that the dividend matrix of the long-lived assets $[d_1 d_2 \cdots d_{J_1}]$ has full rank J_1 implies that in that equilibrium the payoff matrix for the long-lived assets*

$$\begin{pmatrix} q_1(1) + d_1(1) & \cdots & q_1(S) + d_1(S) \\ \vdots & & \vdots \\ q_{J_1}(1) + d_{J_1}(1) & \cdots & q_{J_1}(S) + d_{J_1}(S) \end{pmatrix}$$

has also full rank J_1.

Proof. If all endogenous variables in a financial markets equilibrium are time invariant, the stochastic Euler equations for agent 1, which are necessary conditions for optimality, for long-lived assets are

$$q_j(z) u_z^{1\prime}(c^1(z)) - \beta \sum_{s \in \mathcal{S}} \pi(s|z)(q_j(s) + d_j(s)) u_s^{1\prime}(c^1(s)) = 0 \quad \text{for } j \in \mathcal{J}^l, z \in \mathcal{S}.$$

Denoting by $u^{1\prime}$ the vector with the zth element $u_z^{1\prime}(c^1(z))$ and the component-wise product of S-vectors x and y by $x \circ y$ we can rewrite the equations as

$$q_j \circ u^{1\prime} - \beta\Pi \left[(q_j + d_j) \circ u^{1\prime}\right] = 0,$$

which in turn is equivalent to

$$q_j \circ u^{1\prime} = [I - \beta\Pi]^{-1}\, \beta\Pi(d_j \circ u^{1\prime}).$$

Adding $d^j \circ u^{1\prime}$ to both sides yields

$$\begin{aligned}(q_j + d_j) \circ u^{1\prime} &= [I - \beta\Pi]^{-1}\, \beta\Pi(d_j \circ u^{1\prime}) + d_j \circ u^{1\prime} \\ &= [I - \beta\Pi]^{-1}\, (d_j \circ u^{1\prime}).\end{aligned}$$

Now rank $[d_1 d_2 \cdots d_{J_1}] = J_1$ implies rank $[(d_1 \circ u^{1\prime})(d_2 \circ u^{1\prime}) \cdots (d_{J_1} \circ u^{1\prime})] = J_1$ and rank $[(q_1 + d_1)(q_2 + d_2) \cdots (q_{J_1} + d_{J_1})] = J_1$. $\square$

With this lemma, the proof of the theorem follows immediately from Kreps (1982):

Proof of Theorem 3.

Since for economies with complete markets Arrow-Debreu equilibria and financial market equilibria are equivalent it suffices to show that for any Arrow-Debreu equilibrium (with efficient and hence time-invariant allocations), the supporting asset prices are such that the payoff matrix $W(q, d)$ has full rank for a set of asset dividends of the short-lived assets having full Lebesgue measure when $J \geq S$. Lemma 5 implies that for $W(q, d)$ to have a rank deficiency one of the dividend vector of a short-lived asset must be linearly dependent on the remaining payoff vectors. So there must exist a nonzero vector $\alpha \in \mathbb{R}^S$ such that $W(q, d)\alpha = 0$ and, without loss of generality, $\alpha_S \neq 0$. Said equivalently, the system of equations $G(d^1, \alpha) = 0$ where $G : \mathbb{R}_{++}^{J_2 \times S} \times \mathbb{R}^S \to \mathbb{R}^{S+1}$ is a smooth function defined by

$$W(q, d)\alpha = 0 \tag{5.1}$$

$$\alpha_S - 1 = 0 \tag{5.2}$$

must have a solution. The function G depends on the $J_2 \times S$ exogenous parameters $d_j(y)$ for $j \in \mathcal{J}_2, s \in \mathcal{S}$, and the endogenous variables $\alpha_s, s \in \mathcal{S}$. A portion of the Jacobian of G appears as follows,

	d^S	α_S	
$G_{(5.1)}$	$\Lambda(\alpha_S)$		S
$G_{(5.2)}$	0	1	1
	S	1	

where $\Lambda(\alpha_S)$ is a diagonal matrix with all diagonal elements equal to α_S. Once again applying the transversality theorem shows that $\{\alpha \in \mathbb{R}^S : G(d^2, \alpha) = 0\} = \emptyset$ for almost all $d^2 \in \mathbb{R}^{J_2 \times S}$, and so the matrix $P(q, d)$ must have full rank S. □

Remarkably, the theorem implies that all equilibria must be efficient when there are S long-lived assets. Kreps (1982) states that given any equilibrium allocation, the payoff matrix supporting this allocation will have full rank generically in dividends. Theorem 3 generalizes the result of Kreps (1982) slightly, since it implies that, under the assumption that all dividend vectors are independent, we can focus on the dividends of the short-lived assets only.

5.2 Incomplete markets

We now consider the model of Duffie et al. (1994, Section 3). This model is a version of the Lucas (1978) asset pricing model with finitely many heterogeneous agents. There are H infinitely lived agents, $h \in \mathcal{H}$, and a single commodity in a pure exchange economy. The only assets are trees in unit net supply and agents are not allowed to take short positions in these trees. In equilibrium tree holdings therefore always lie in

$$\Theta = \{(\theta^h) \in \mathbb{R}_+^{JH} : \sum_{h \in \mathcal{H}} \theta^h = 1\}.$$

As we will see this will make our life much easier.

5.2.1 Existence of (Markov) Equilibrium

In order to define our notion of Markov equilibrium, we need to adapt some concepts from Duffie et al. (1994) to our model.

The State Space

We start of with defining a *very large* endogenous state space which consists of all current endogenous variables. Readers familiar with recursive methods will realize that this is not what one would usually identify as a natural state space. I will get back to this point below.

For now, let

$$z(\sigma) = (\theta_-(\sigma), c(\sigma), \theta(\sigma), q(\sigma) = ((\theta^h_-(\sigma))_{h\in\mathcal{H}}, (c^h(\sigma))_{h\in\mathcal{H}}, (\theta^h(\sigma))_{h\in\mathcal{H}}, q(\sigma))$$

be the endogenous state at node σ. These are beginning-of-period portfolio holdings of each agent, current consumption, new portfolio and prices. We define the endogenous state space as

$$\mathcal{Z} = \{z \in \mathbb{R}^{JH}_+ \times \mathbb{R}^H_+ \times \mathbb{R}^{JH}_+ \times \mathbb{R}^J_+ : \ \sum_{h\in\mathcal{H}} \theta^h_j = \sum_{h\in\mathcal{H}} \theta^h_{j-} = 1 \text{ for all } j \in \mathcal{J}\}.$$

By definition, in any financial markets equilibrium, all endogenous variables lie in $\mathcal{Z}$. We will make frequent use of the set of endogenous variables without beginning-of-period portfolios, and we denote this set by $\hat{\mathcal{Z}} = \mathbb{R}^H \times \Theta \times \mathbb{R}^J$.

We let the state space $\mathcal{Y}$ consist of all exogenous and endogenous variables that occur in the economy at some node σ, i.e. $\mathcal{Y} = \mathcal{S} \times \mathcal{Z}$, where $\mathcal{S}$ is the finite set of exogenous shocks and $\mathcal{Z}$ is the set of all possible endogenous variables.

Expectations Correspondence

For a set $\mathcal{X}$, let $\mathcal{X}^S$ be the Cartesian product of S copies of $\mathcal{X}$. Given a state $(s, z) \in \mathcal{Y}$, the 'expectations correspondence'

$$g : \mathcal{Y} \rightrightarrows \mathcal{Z}^S$$

describes all next period states that are consistent with market clearing and agents' first-order conditions. A vector of endogenous variables $(z^+_1, \ldots, z^+_S) \in g(s, z)$ if for all households $h \in \mathcal{H}$ the following conditions hold.

(a) For all $s = 1, \ldots, S$, the 'new' beginning-of-period portfolios coincide with the portfolios chosen today, i.e.

$$\theta^{h+}_{s-} = \theta^h$$

and the budget constraint holds, i.e.

$$c^{h+}_s = e^h(s) + \theta^{h+}_{s-} \cdot (d(s) - q^+_s) - \theta^{h+}_s \cdot q^+_s \geq 0$$

(b) Agents' first order conditions relate z to z^+, i.e. for each h, there exist multipliers $\lambda^h \in \mathbb{R}^J_+$ such that for all $j \in \mathcal{J}$ the following equations hold:

$$\lambda^h_j - q_j u'_h(c^h) + \beta E^s \left\{(d_j + q^+_j) u'_h(c^{h+})\right\} = 0 \quad (5.3)$$

$$\lambda^h_j \theta^h_j = 0 \quad (5.4)$$

Conditions (a) are just the budget constraint. The conditions (b) are part of the standard first-order conditions with respect to θ_j^h. Note that market clearing in asset markets is ensured by our definition of $\mathcal{Z}$; market clearing in commodity markets follows from Walras' law.

We need the following lemma to show existence.

Lemma 6 *The graph of g is a closed subset of $\mathcal{Y} \times \mathcal{Z}^S$.*

The lemma follows directly from the definition of g.

Definition of Generalized Markov Equilibrium

Markov equilibria are characterized by policy- and transition functions. Given a value of the state today, the policy function gives the new optimal policy and equilibrium prices. The transition function gives a probability distribution over states tomorrow. Unfortunately, if we take the state space to consist only of the shock and beginning-of-period portfolios (as one would usually do), these equilibria do not always exist.

Instead, we define a generalized Markov equilibrium to consist of a (non-empty valued) 'policy correspondence', P, and a 'transition function', F,

$$P : \mathcal{S} \times \Theta \rightrightarrows \hat{\mathcal{Z}} \text{ and } F : \text{ graph}(P) \rightarrow \mathcal{Z}^S$$

such that for all $(s, z) \in$ graph(P) and all $s' \in \mathcal{S}$

$$F(s, z) \in g(s, z) \text{ and } (s', F_{s'}(s, z)) \in \text{ graph}(P).$$

We say a generalized Markov equilibrium is simple if the associated policy correspondence is single valued. It is easy to verify that in that case the definition coincides with what one usually calls Markov equilibria.

A generalized Markov equilibrium is a financial markets equilibrium according to our previous definitions if we can show that first order conditions are necessary and sufficient. In order to show sufficiency of the first order conditions, we go back to lemma 2. What we need to show to apply the lemma to this model is that there are lower bounds to consumption and we need to assume that utility is bounded above. To get lower bounds for consumption, we assume that $u_h'(c) \rightarrow \infty$ as $c \rightarrow 0$ and that $e^h(s) > 0$ for all $s \in \mathcal{S}$. With this assumption it is clear that there exists a $\underline{c} > 0$ such that it is never optimal to consume less than this (because remaining in autarky and consuming endowments for the infinite future would give higher utility than consuming $\underline{c}$ now and infinite amounts for the rest of the economy.

Constructing Markov Equilibria

In constructing a Markov equilibrium (P, F), we closely follow the existence proof in Duffie et al. (1994). The basic idea of our approach is very similar to backward induction. We start with a compact set, $\mathcal{T} \subset \hat{\mathcal{Z}}$, that is large enough to ensure that for all finitely truncated economies the endogenous variables take equilibrium values in $\mathcal{T}$. The fact that such a set exists is established in Lemma 7 below. Assuming that next period's equilibrium variables can be described by some correspondence, $V^n : \mathcal{S} \times \Theta \rightrightarrows \mathcal{T}$, we define a new correspondence, V^{n+1}, by assigning to a given exogenous shock s and wealth distribution θ_- the set of all endogenous variables today that are consistent (in the sense that all agents' first-order conditions and market clearing hold) with some $((\theta_{1-}^+, \hat{z}_1), \ldots, (\theta_{S-}^+, \hat{z}_S))$ tomorrow for which $\hat{z}_{s'} \in V^n(s', \theta_{s'-})$. With this construction, $\bigcap_{n=1}^{\infty} V^n(s, \theta_-)$ is non-empty for all $s \in \mathcal{S}$, $\theta_- \in \Theta$.

Formally, given a compact set $\mathcal{T} \subset \hat{\mathcal{Z}}$, and a correspondence $V : \mathcal{S} \times \Theta \rightrightarrows \mathcal{T}$, define an operator, $G_{\mathcal{T}}$, which maps the correspondence $V : \mathcal{S} \times \Theta \rightrightarrows \mathcal{T}$ to a new correspondence $W : \mathcal{S} \times \Theta \rightrightarrows \mathcal{T}$ as follows:

$W = G_{\mathcal{T}}(V)$ if for all $\theta_- \in \Theta$ and all $s \in \mathcal{S}$,

$$\begin{aligned} W(s, \theta_-) = \{(c, \theta, q) \in \mathcal{T} : \; & \exists\ (z_1, \ldots, z_S) \in g(s, \theta_-, c, \theta, q) \text{ such that for all } s' \in \mathcal{S}, \\ & (c_{s'}, \theta_{s'}, q_{s'}) \in V(s', \theta_{s'-}) \\ & \text{where } z_{s'} = (\theta_{s'-}, c_{s'}, \theta_{s'}, q_{s'})\}. \end{aligned}$$

In order to construct a policy correspondence we first need a suitable set $\mathcal{T}$ that is large enough to contain all endogenous variables. We define a truncated economy with $T+1$ periods to be a finite horizon economy with an event tree, where all nodes of the form $\sigma = (s_0\ s_1\ \cdots\ s_T)$ are terminal nodes. Endowments and asset payoffs at nodes of the truncated tree, as well as agents' preferences over consumption at these nodes, are the same as in the infinite horizon economy.

Lemma 7 *For all $T \geq 1$ there exists a financial markets equilibrium for the truncated economy in which values of all prices, asset holdings, and consumption allocations lie in a compact set $\mathcal{T}^* \subset \hat{\mathcal{Z}}$.*

The proof of the lemma is left as an exercise and follows directly along the lines of the existence proof of Chapter 4.

Define V^0 by $V^0(s, \theta_-) = \mathcal{T}^*$ for all $\theta_- \in \Theta$ and all $s \in \mathcal{S}$. Given a correspondence

$V^n,\ n = 0, 1, \ldots,$ define recursively $V^{n+1} = G_{\mathcal{T}^*}(V^n)$. Finally let

$$V^*(s, \theta_-) := \bigcap_{n=1}^{\infty} V^n(s, \theta_-) \text{ for all } s \in \mathcal{S}, \theta_- \in \Theta. \tag{5.5}$$

We can now state our main theorem.

Theorem 4 *The correspondence V^* is non-empty valued and there exists a Markov equilibrium with policy correspondence V^*.*

We have thus proven that in the Lucas model with heterogenous agents equilibria always exist and that these can be characterized as an generalized Markov equilibrium. Unfortunately, for the general model one cannot go further.

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