This class

- Time $t = 0, 1, 2, ..., T \leq \infty$ and uncertainty
- Economic agents that maximize utility over consumption streams
- Assets that can be used to transfer wealth across time and states
- How can we price these assets

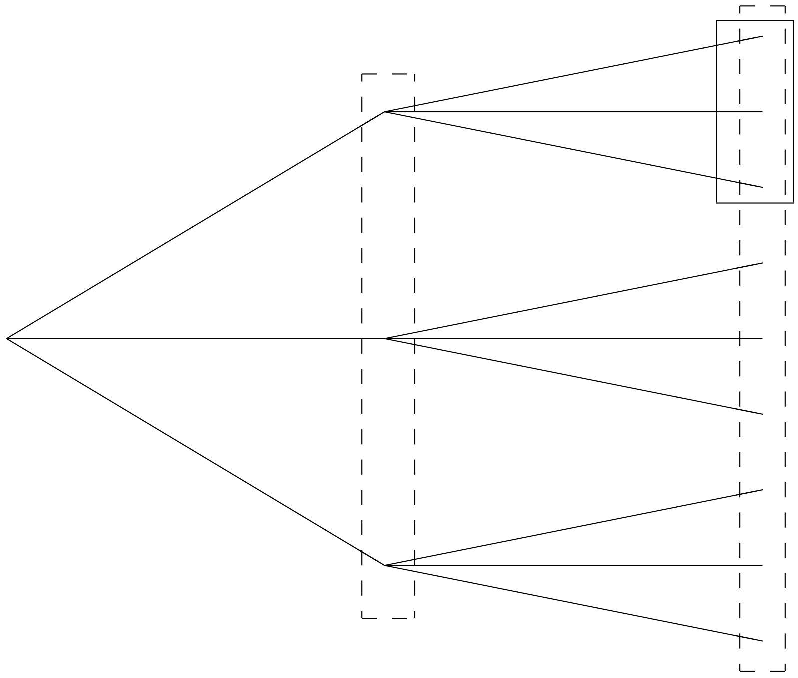
Structure

- Lecture 1: Uncertainty, assets, absence of arbitrage
- Lecture 2: The Lucas asset pricing model
- Lecture 3: General Equilibrium with Incomplete Asset Markets (GEI)
- Lecture 4: Infinite horizon models with heterogeneous agents

Event trees

- A set of nodes $\sigma \in \Sigma$
- Unique root node σ_0
- Each other node has a unique direct predecessor σ_-
- Each non-terminal node has a non-empty set of direct successor nodes $\Im(\sigma)$
- Collect all nodes with t predecessors in a set $\mathcal{N}_t$

Event trees



Stationary trees

- It often simplifies the notation hugely to assume that each (non-terminal) node has the same number of direct successors and that each terminal node is in $\mathcal{N}_T$.
- We associate with a node a history of shocks

$$\sigma \in \mathcal{N}_t \Leftrightarrow \sigma = (s_0 \ldots s_t) = s^t$$

$$s_t \in \mathcal{S} = \{1, ..., S\}$$

- Write s^{t+1} for a generic successor of s^t and s^{t-1} for the unique direct predecessor
- If s^t is followed by shock s we write $(s^t s)$.

Finance trees

- Underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ denotes the tribe (also called σ-algebra or σ-field depending on the context) of subsets of the set of possible states of the world Ω.
- At each date T a tribe $\mathcal{F}_t \subset \mathcal{F}$ denotes the set of events corresponding to the information available at date t. Filtration $\mathbb{F} = \{\mathcal{F}_0, \ldots, \mathcal{F}_T\}$ represents how information is revealed through time, $\mathcal{F}_t \subset \mathcal{F}_{t+1}$.
- We can then identify a node of the event tree by a date and a state of the world, ω_t, require that things that happen at t are $\mathcal{F}_t$ measurable
- Advantage: Ω does not need to be countable, relates to continuous time model

Infinite event trees

- A node is a finite history of shocks

$$\sigma = s^t = (s_0, ..., s_t)$$

- Countable many nodes in the event tree
- In contrast, if we consider sample paths, there must be a continuum, e.g. if $S = 10$ can associate each path with an element in $[0, 1)$
- I will choose the first interpretation. Does it make a difference?

General Equilibrium on an event tree

- Given a tree Σ with M nodes $\sigma \in \Sigma$
- Suppose there is one (perishable) commodity per node
- There are H agents with individual endowments $e^h \in \mathbb{R}_+^M$ and utility

$$u^h : \mathbb{R}_+^M \longrightarrow \mathbb{R}$$

- Agents want to trade to obtain consumption that gives higher utility than endowments

Assets

- At each node $\sigma \in \Sigma$ there are J assets $j \in \mathcal{J}$
- Each $j \in \mathcal{J}$ generates payoff at all direct successors $(a_j(\zeta))_{\zeta \in \Im(\sigma)} \in \mathbb{R}^{J+1}$.
- Assets pay off in both assets and the single commodity – $a_{j0}(\sigma)$ is the payoff in node σ commodity (and sometime write $d_j(\sigma)$ to refer to this payoff as a dividend) while $\tilde{a}_j = (a_{j1}, \ldots, a_{jJ})$ is the payoffs in node σ assets.
- The price of assets at node σ is denoted by $q(\sigma)$, a row vector.
- We collect payoffs of all assets at σ in a $J + 1 \times J$ matrix

$$A(\sigma) = (a_1(\sigma), \ldots, a_J(\sigma)).$$

Stocks and bonds

- A stock (or Lucas tree) is an asset that pays dividends and one unit of itself in the next period (does not pay in other assets).
- A one period asset (e.g. one period bond) pays only in commodities
- A bond of maturity T pays one unit of a bond of maturity $T - 1$ next period
- In this class, I will only consider stocks, in the notes everything is for general assets, but this just makes notation complicated !

Budget sets

- At each node an agent h faces the budget constraint

$$c(\sigma) - e^h(\sigma) \leq (1, q(\sigma))A(\sigma)\theta(\sigma_-) - q(\sigma)\theta(\sigma)$$

- In the case of only long-lived assets this gives

$$c(\sigma) - e^h(\sigma) \leq (q(\sigma) + d(\sigma)) \cdot \theta(\sigma_-) - q(\sigma) \cdot \theta(\sigma)$$

- We collect the set of all non-negative consumption processes and portfolio processes which satisfy these constraints at all nodes for an agent h in a budget set $\mathcal{B}^h(q)$.
- It will be useful to associate with a trading strategy $(\theta(\sigma))_{\sigma \in \Sigma}$ a so-called 'gain-process':

$$D^\theta(\sigma) = (q(\sigma) + d(\sigma))\theta(\sigma_-) - \theta(\sigma)q(\sigma)$$

Absence of arbitrage - Definition

Prices and payoffs $(q(\sigma), d(\sigma))_{\sigma \in \Sigma}$ preclude arbitrage if there is no trading strategy $(\theta(\sigma))_{\sigma \in \sigma}$ with $\theta(0-) = 0$ such that $D^{\theta}(\sigma) \geq 0$ for all $\sigma \in \Sigma$ and $D^{\theta}(\sigma) \neq 0$ for at least one $\sigma \in \Sigma$.

Absence of arbitrage - Characterization

Prices and payoffs $(q(\sigma), d(\sigma))_{\sigma\in\Sigma}$ preclude arbitrage if and only if there exists a strictly positive state-price process $(\alpha(\sigma))_{\sigma\in\Sigma}$ such that for all non-terminal $\sigma \in \Sigma$,

$$q(\sigma) = \frac{1}{\alpha(\sigma)} \sum_{\zeta\in\Im(\sigma)} \alpha(\zeta)(q(\zeta) + d(\zeta))$$

Substituting

- If T is finite, one can obtain an expression for asset prices as a linear function of all future dividends (i.e. commodity payoffs) by substituting out all future prices
- The price at node σ_0 of a stock with dividend process $(d_j(\sigma))_{\sigma\in\Sigma}$ is simply

$$q_j(\sigma_0) = \sum_{\sigma\in\Sigma} \alpha(\sigma) d_j(\sigma).$$

- For infinite T there might be bubbles and

$$q_j(\sigma_0) \geq \sum_{\sigma\in\Sigma} \alpha(\sigma) d_j(\sigma).$$

No time to discuss this here, see references in the notes.

Stochastic Discount Factor

- In finance, the state prices α are often rewritten as the 'stochastic discount factor'
- The return of a tree is

$$R_j(s^t) = \frac{q(s^t) + d(s_t)}{q(s^{t-1})}$$

- Absence of arbitrage implies that there are $m(s^t)$ such that

$$m(s^{t-1}) = E(R_j(s^t)m(s^t)|s^{t-1})$$

  The price of an asset is determined by the payoff's covariance with the stochastic discount factor

- With assumptions on preferences one can pin down m, e.g. CAPM

Outline of proof

- Define $\mathcal{M} \subset \mathbb{R}^M$ as $\mathcal{M} = \{(D^\theta(\sigma))_{\sigma\in\Sigma} : \theta$ is a trading strategy$\}$ the absence of arbitrage means that $\mathcal{M} \cap \mathbb{R}^M_+ = \{0\}$.
- If strictly positive α exist, the absence of arbitrage follows from simple calculation
- For the opposite direction, separate $\mathcal{M}$ and $\mathbb{R}^M_+$:
  Suppose M and K are closed convex cones in $\mathbb{R}^n$ that intersect precisely at zero. If K is not a linear subspace, then there is a nonzero linear functional F such that $F(x) < F(y)$ for each $x \in M$ and each nonzero $y \in K$.

Absence of arbitrage - Example

- Suppose there are 3 periods, two shocks each period (binomial event tree)
- Suppose there is a one period bond traded at each node and that there is a stock which pays dividends

$$d(1) = 2 \quad d(2) = 1 \quad d(11) = d(12) = 1 \quad d(21) = d(22) = 0$$

- How can we find possible no-arbitrage prices for period 0 ?

A geometric characterization

- Suppose there are J trees and M states in the event tree.
- No arbitrage prices at $t = 0$ are some subset of $\mathbb{R}^J$.
- Can characterize this set as the cone spanned by the dividends of the assets in a given state, across assets

Uniqueness of stochastic discount factor

- We will see later that stochastic discount factor is unique if markets are complete (i.e. sufficiently many assets)
- In this case, one can price all conceivable new assets (e.g. options) by the stochastic discount factor that one derives from existing assets
- Most of option pricing (e.g. Black Scholes) relies on this idea