Infinitely lived agents

- We consider a model with H heterogeneous agents who live forever.
- There is no production, only one commodity. Agents have endowments in the commodity which are time-invariant functions of the shock
- Agents maximize time-separable expected utility
- Lucas model with heterogenous agents

Assets

- Lucas trees: Infinitely lived assets that pay a dividend at all nodes $\sigma_t \in \sigma$. The dividend is a function of the shock alone. Each agent faces the following budget constraint

$$c^h(s^t) = \bar{e}^h(s_t) + \sum_{j \in \mathcal{J}} \theta_j^h(s^{t-1})(q(s^t) + d(s_t)) - \theta^h(s^t)q(s^t).$$

- One period assets: Contracts written contingent on next period's shock. Easiest example is a bond that pays one unit next period independently of the shock.

Some remarks on complete markets

- Condition for complete markets the same as in finite horizon
- When markets are complete, consumption time-invariant function of shock
- Prices of assets are given by marginal utilities of agents
- Incomplete markets make things incredibly difficult

Heterogeneous Lucas model

- H infinitely lived agents and a single commodity in a pure exchange economy.
- Endowments are $e^h(\sigma) > 0$ with $e^h(s^t) = \mathbf{e}^h(s_t)$
- h has von Neumann-Morgenstern utility over infinite consumption streams

$$U^h(c) = E_0 \sum_{t=0}^{\infty} \beta^t u_h(c_t)$$

- J infinitely lived assets in unit net supply. Each j pays shock dependent dividends $d_j(s)$, we denote its price at node s^t by $q_j(s^t)$. Agents trade these assets but are restricted to hold non-negative amounts of each asset. Portfolios are $\theta^h \geq 0$.
- Endogenous state could be initial portfolios or cash at hand (i.e. value of initial portfolio at current prices)

Euler equations

$z = (\theta_-, c, \theta, q)$, $\kappa \in \mathbb{R}_+^{HJ}$, equilibrium is determined by

$$g(\bar{s}, \bar{z}, \kappa, z(1), ..., z(S)) = 0, \text{ where}$$

$$\begin{aligned}
g_h^1 &= -\bar{q}u_h'(\bar{c}^h) + \beta E_{\bar{s}}\left[(q(s)+d(s))u_h'(c^h(s))\right] + \kappa^h \\
g_{hs}^2 &= \theta_-^h(s) - \bar{\theta}^h \\
g_{hs}^3 &= c^h(s) - \theta_-^h(s)\cdot(q(s)+d(s)) + \theta^h(s)\cdot q(s) - \mathbf{e}^h(s) \\
g_{hj}^4 &= \bar{\theta}_j^h \kappa_j^h \\
g_{js}^5 &= \textstyle\sum_{h\in\mathcal{H}} \theta_j^h(s) - 1
\end{aligned}$$

Conditions necessary and sufficient ?

- Given sequence of prices $q(s^t)$, need to show that first order conditions are necessary and sufficient for optimality
- Necessity of first order conditions is standard, but for sufficiency we need to make additional assumptions. Assume that $u^h(c)$ is bounded above and that $u_h(c) \to \infty$ as $c \to 0$.
- Also need that prices remain bounded, i.e. $\sup_\sigma q(\sigma) < \infty$

Recall Sufficiency-lemma

Assume that Bernoulli utility $u(.)$ is bounded above. Suppose asset prices are bounded, i.e. $\sup_\sigma q(\sigma) < \infty$. A process $(\bar{c}(\sigma), \bar{\theta}(\sigma))$, with $\sup_\sigma q \cdot \bar{\theta}(\sigma) < \infty$ and with $\sup_\sigma u'(c(\sigma)) < \infty$ solves an agent h's optimization problem if for all s^t the Euler equation holds, i.e.

$$-q(s^t)u'(\bar{c}(s^t)) + \sum_s \pi(s|s_t)(q(s^{t+1}) + d(s_{t+1}))u'(\bar{c}(s^{t+1})) = 0$$

Existence and characterization of equilibria

- Unfortunately, even with a single asset (even a bond) cannot use contraction mapping theorem to prove existence of recursive equilibrium
- In practice, it behaves as if it were a contraction ?
- Can interpret algorithm that are used to compute equilibria as approximating infinite horizon model with long finite horizon.

Existence of Markov equilibria

- In general, one cannot prove that recursive (or Markov) equilibria exist.
- However, one can show that competitive equilibria always exist. This can be done by looking at truncated economies and pass to the limit...
- For any $T < \infty$, consider a truncated economy where after T periods nobody has endowments. Use standard fixed point methods to prove existence for these 'truncated economies'.

Generalized Markov Equilibria

- Duffie, Geanakoplos, Mas-Colell and McLennan consider more general (much less useful) definition of Markov equilibria
- The state-space consist of all exogenous and endogenous variables that describe the state of the economy at some date-event:

$$\Omega = \mathcal{S} \times \mathcal{Z},$$

where $\mathcal{S}$ is the finite set of exogenous shocks and $\mathcal{Z}$, a subset of Euclidean space, is a comprehensive set of possible values for the endogenous variables at any date-event.

- An expectations correspondence,

$$G : \Omega \rightrightarrows \mathcal{Z}^S,$$

embodies all short run equilibrium conditions: (first order) conditions for individual optimization and market clearing conditions; to every pair of a realization of the shock and values of endogenous variables, it assigns values of the endogenous variables at all possible realization of the shock at the following date.

- An equilibrium set for an expectations correspondence, G, is a compact subset $\vec{\mathcal{Z}}^* \subset \mathcal{Z}^N$, such that

$$[(z_1, ..., z_S) \in \vec{\mathcal{Z}}^*] \quad \Rightarrow \quad [G(s, z_s) \cap \vec{\mathcal{Z}}^* \neq \emptyset, \text{ for all } s \in \mathcal{S}]$$

- Can write Z^* as the graph of an equilibrium correspondence, mapping shock and beginning-of-period portfolio holdings to new choices and prices

Existence of equilibrium set

- A T-horizon equilibrium for the correspondence G consists of a subset $\tilde{\Omega} \subset \Omega$ and $(s_t, z(\sigma_t)) : t = 1, \ldots, T$, $z(\sigma_t) \in G_{s_t}(s_{t-1}, z(\sigma_{t-1}))$, that satisfies $(s_t, z(\sigma_t)) \in \tilde{\Omega}$; the set $\tilde{\Omega}$ supports the equilibrium.
- If G is an expectations correspondence, such that
  1. there exists a compact subset $\mathcal{K} \subset \Omega$ that supports a T-horizon equilibrium for $T = 1, \ldots,$ and
  2. the graph of G is closed,

  then, an equilibrium set for G exists.

Proof

Define sets $\mathcal{T}_0 = \mathcal{K}$, and, recursively,

$$\mathcal{T}_n = \{(\bar{s}, \bar{z}_{\bar{s}}) \in \mathcal{K} : \text{ there exists } (z_1, ..., z_S) \in G(\bar{s}, \bar{z}_{\bar{s}}), \text{ with } (s, z_s) \in \mathcal{T}_{n-1}\}.$$

Since, by assumption, the expectations correspondence has a closed graph and since $\mathcal{K}$ is compact each $\mathcal{T}_n$ is compact.

We show by induction that for each $n > 1$, $\mathcal{T}_n \subset \mathcal{T}_{n-1}$. By definition $\mathcal{T}_1 \subset \mathcal{T}_0$. If $\mathcal{T}_n \subset \mathcal{T}_{n-1}$ and for some $(\bar{s}, \bar{z}) \in \mathcal{K}$ there exists $(z_1, ..., z_S) \in G(\bar{s}, \bar{z})$ with

$$(s, z_s) \in \mathcal{T}_n \subset \mathcal{T}_{n-1},$$

then, obviously, $(\bar{s}, \bar{z}) \in \mathcal{T}_n$ and $\mathcal{T}_{n+1} \subset \mathcal{T}_n$.

Since, by assumption, all $\mathcal{T}_n$ are non-empty, the set

$$\Omega^* = \bigcap_{n=0}^{\infty} \mathcal{T}_n$$

is non-empty. Define $\vec{\mathcal{Z}}^*$ as follows:

$$\vec{\mathcal{Z}}^* = \{z = (z_1, ..., z_S) : (s, z_s) \in \Omega^* \text{ for all } s \in \mathcal{S}\}$$

It follows from the construction that this is an equilibrium set for G $\square$

Why do we care?

- The existence of an equilibrium set implies existence of recursive ϵ-equilibrium
- The first order conditions do not hold exactly, but for small $\epsilon > 0$, we have

$$\| -u'(\bar{c}^h) + \beta \sum_{s'} \pi(s,s')(q(s') + d(s'))u'(c(s')) + \kappa \| < \epsilon$$

- This is what we compute in practice and we can show that it exists
- Obviously, this will not tell us why smooth methods should work here

Existence of approximate equilibria

- For the stochastic economy of overlapping generations, a Markov ϵ-equilibrium exists, for any $\epsilon > 0$.
- Given a (compact) equilibrium set, $\vec{\mathcal{Z}}^*$, for an expectations correspondence, G, there exists, for any $\delta > 0$, a finite collection of points, $\mathcal{F}^\delta$, such that

$$\sup_{\zeta \in \vec{\mathcal{Z}}^*} \inf_{\xi \in \mathcal{F}^\delta} \|\xi - \zeta\| < \delta,$$

and such that there exist a set of beginning-of-period portfolio holdings and stored commodities $\mathcal{T}$ and a function $w : \mathcal{S} \times \mathcal{T} \to \mathcal{Z}^N$, with

$$\mathcal{F}^\delta = \text{graph}(w);$$

the former follows from compactness of $\mathcal{Z}^*$ and the latter from the finiteness of $\mathcal{F}$.

- For each $(\bar{z}_1, \dots, \bar{z}_S) \in \mathcal{F}^\delta$, each $\bar{s} \in \mathcal{S}$, choose

$$\xi = (z_1, \dots, z_S) \in \mathcal{F}^\delta \text{ such that } \inf_{\zeta \in G(\bar{s}, \hat{z})} |\zeta - \xi| < \delta$$

for $\hat{z}$ with $|\hat{z} - \bar{z}_{\bar{s}}| < \delta$

Heterogeneous Lucas model and the data

- Heaton and Lucas (1996) show that with transitory shocks, incomplete markets do not change pricing implications significantly
- Constantinides Duffie show that with permanent shocks things become very different
- Empirical question: What can we say about income processes
- Cross-sectional returns ?
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