Introduction to Differential Information Economies

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1 Introduction

The main objective of equilirium concepts in economies with asymmetric information is to capture the idea of how contracts are formed (or written).[1] By now several notions have been introduced and since they are different they provide alternative equilibrium outcomes, (see, for example Glycopantis et. al. (2005).

The concepts which have been used are either non-cooperative, i.e. the Rational Expectations Equilibrium (REE) and the Warlasian Expectations Equilibrium (WEE)(or Radner Equilibrium), or cooperative, i.e. Core formulations and the Shapley value with asymmetric information.

All the above equilibrium outcomes can be characterized by means of a static optimization method. Of course such an optimization tehnique does not provide the process as to how the contract was reached as there is no dynamics involved. However we typically back up an equilibrium outcome with an interpetation as to how agents reached the equilibrium contract (outcome) and thus provide a relevant story.

We first recall the general idea of PBE. A PBE consists of a set of players' optimal behavioral strategies, and consistent with these, a set of beliefs which attach a probability distribution to the nodes of each information set. Consistency requires that the decision from an information set is optimal given the particular player's beliefs about the nodes of this set and the strategies from all other sets, and that beliefs are formed from updating, using the available information. If the optimal play of the game enters an information set then updating is of beliefs must Bayesian. Otherwise appropriate beliefs are assigned arbitrariry to the nodes of the set.

The extensive form implementation of an equilibrium concept enables us to achieve two objectives. First it helps to provide a dynamic interpretation or justification of static equilibrium concepts. This way we do not need to "tell a story". On the contrary, the extensive form game tree construction provides all the the dynamics as to how a contract is reached. Second for the appropriate concepts like the core and the Shapley value could it give a noncooperative foundation or justification and thus in a way recast and extend the well known "Nash program" to games with assymetric information.

The implementation analysis brings, also, out stongly the importance of the incentive compatibility of different solution concepts. In particular, solution concepts which are incentive compatible turn out to be implementable as a PBE of an extensive form game,

[1] This paper summarizes results obtained in Glycopantis et. al. given in the references.

constructed under reasonable rules, while the ones which are not incentive compatible are not thus implementable.

However we could implement solution concepts which are not incentive compatible as a PBE of an extensive form game by imposing incentive compatibility in an exogenous way which is rather ad hoc. In patricular we can introduce the idea of a court (or mechanism designer) whose role is to make sure that agents are penalized when they lie. The penalty threat makes the contract incentive compatible and therefore implementable. We believe that this is not appealling because agents have to rely on an exogenous person to make things work. This exogenous agent has no preferences or initial endoments, and the construction is rather artificial.

2 Differential information economy (DIE)

We assume that Ω and the number of goods, l, per state are finite. I is a set of n players and $\mathbb{R}^l_+$ will denote the set of positive real numbers.

A differential information exchange economy $\mathcal{E}$ is a set

$$\{((\Omega, \mathcal{F}), X_i, \mathcal{F}_i, u_i, e_i, q_i) : i = 1, \ldots, n\}$$

where

1. $\mathcal{F}$ is a σ-algebra generated by a partition of Ω;
2. $X_i : \Omega \rightarrow 2^{\mathbb{R}^l_+}$ is the set-valued function giving the *random consumption set* of Agent (Player) i, who is denoted by Pi;
3. $\mathcal{F}_i$ is a partition of Ω generating a sub-σ-algebra of $\mathcal{F}$, denoting the *private information* of Pi; $\mathcal{F}_i$ is a partition of Ω generating a sub-σ-algebra of $\mathcal{F}$, denoting the *private information*[2] of Pi;
4. $u_i : \Omega \times \mathbb{R}^l_+ \rightarrow \mathbb{R}$ is the *random utility* function of Pi; for each $\omega \in \Omega$, $u_i(\omega, .)$ is continuous, concave and monotone;
5. $e_i : \Omega \rightarrow \mathbb{R}^l_+$ is the *random initial endowment* of Pi, assumed to be $\mathcal{F}_i$-measurable, with $e_i(\omega) \in X_i(\omega)$ for all $\omega \in \Omega$;
6. q_i is an $\mathcal{F}$-measurable probability function on Ω giving the *prior* of Pi. It is assumed that on all elements of $\mathcal{F}_i$ the aggregate q_i is strictly positive. If a common prior is assumed on $\mathcal{F}$, it will be denoted by μ.

We assume that the players' information partitions are common knowledge. We will refer to a function with domain Ω, constant on elements of $\mathcal{F}_i$, as $\mathcal{F}_i$*-measurable*, although, strictly speaking, measurability is with respect to the σ-algebra generated by the partition.

Agents make contracts in the ex ante stage. In the interim stage, having received a signal concerning the event, i.e. the element of $\mathcal{F}_i$, containing the realized state of nature, they consider the incentive compatibility of the contract.

For any $x_i : \Omega \rightarrow \mathbb{R}^l_+$, the *ex ante expected utility* of Pi is given by

[2]Sometimes $\mathcal{F}_i$ will denote the σ-algebra generated by the partition, as will be clear from the context.

$$v_i(x_i) = \sum_{\Omega} u_i(\omega, x_i(\omega))q_i(\omega).$$

Let $\mathcal{G}$ be a partition of (or σ-algebra on) Ω, belonging to Pi. For $\omega \in \Omega$ denote by $E_i^{\mathcal{G}}(\omega)$ the element of $\mathcal{G}$ containing ω; in the particular case where $\mathcal{G} = \mathcal{F}_i$ denote this just by $E_i(\omega)$. Pi's conditional probability for the state of nature being ω', given that it is actually ω, is then

$$q_i\left(\omega' | E_i^{\mathcal{G}}(\omega)\right) = \begin{cases} 0 & : \quad \omega' \notin E_i^{\mathcal{G}}(\omega) \\ \frac{q_i(\omega')}{q_i\left(E_i^{\mathcal{G}}(\omega)\right)} & : \quad \omega' \in E_i^{\mathcal{G}}(\omega). \end{cases}$$

The *interim expected utility* function of Pi, $v_i(x|\mathcal{G})$, is given by

$$v_i(x|\mathcal{G})(\omega) = \sum_{\omega'} u_i(\omega', x_i(\omega'))q_i\left(\omega' | E_i^{\mathcal{G}}(\omega)\right),$$

which defines a $\mathcal{G}$-measurable random variable.

Denote by $L_1(q_i, \mathbb{R}^l)$ the space of all equivalence classes of $\mathcal{F}$-measurable functions $f_i : \Omega \to \mathbb{R}^l$; when a common prior μ is assumed $L_1(q_i, \mathbb{R}^l)$ will be replaced by $L_1(\mu, \mathbb{R}^l)$. L_{X_i} is the set of all $\mathcal{F}_i$-measurable selections from the random consumption set of Agent i, i.e.,

$$L_{X_i} = \{x_i \in L_1(q_i, \mathbb{R}^l) : \ x_i : \Omega \to \mathbb{R}^l \text{ is } \mathcal{F}_i\text{-measurable and } x_i(\omega) \in X_i(\omega) \ q_i\text{-}a.e.\}.$$

Let $L_X = \prod_{i=1}^{n} L_{X_i}$, $\bar{L}_{X_i} = \{x_i \in L_1(q_i, \mathbb{R}^l) : \ x_i(\omega) \in X_i(\omega) \ q_i\text{-}a.e.\}$ and $\bar{L}_X = \prod_{i=1}^{n} \bar{L}_{X_i}$. An element $x = (x_1, \ldots, x_n) \in \bar{L}_X$ will be called an *allocation* and $(y_i)_{i \in S} \in \prod_{i \in S} \bar{L}_{X_i}$ will be an allocation to S. In case there is only one good, we shall use the notation $L^1_{X_i}$, L^1_X etc. When a common prior is assumed $L_1(q_i, \mathbb{R}^l)$ will be replaced by $L_1(\mu, \mathbb{R}^l)$. Finally, the pooled information $\bigvee_{i \in S} \mathcal{F}_i$ of a coalition S will be denoted by $\mathcal{F}_S$. [3] We assume that $\mathcal{F}_I = \mathcal{F}$.

3 Cooperative equilibrium concepts: Core and Shapley value

First we define the ex ante concepts of the private core and weak fine core (see Yannelis (1991), Koutsougeras - Yannelis (1993), and Glycopantis et. al. (2001)).

Definition 3.1. An allocation $x \in L_X$ is said to be a *private core allocation* if

(i) $\sum_{i=1}^n x_i = \sum_{i=1}^n e_i$ and

(ii) there do not exist coalition S and allocation $(y_i)_{i \in S} \in \prod_{i \in S} L_{X_i}$ such that $\sum_{i \in S} y_i = \sum_{i \in S} e_i$ and $v_i(y_i) > v_i(x_i)$ for all $i \in S$.

If the feasibility condition (i) is replaced by (i)$'$ $\sum_{i=1}^n x_i \leq \sum_{i=1}^n e_i$ then free disposal is allowed.

Definition 3.2. An allocation $x = (x_1, \ldots, x_n) \in \bar{L}_X$ is said to be a *WFC allocation* if

[3] The "join" $\bigvee_{i \in S} \mathcal{F}_i$ denotes the smallest σ-algebra containing all $\mathcal{F}_i$, for $i \in S$.

(i) each $x_i(\omega)$ is F_I-measurable;

(ii) $\sum_{i=1}^n x_i(\omega) = \sum_{i=1}^n e_i(\omega)$, for all $\omega \in \Omega$;

(iii) there do not exist coalition S and allocation $(y_i)_{i \in S} \in \prod_{i \in S} \bar{L}_{X_i}$ such that $y_i(\cdot) - e_i(\cdot)$ is $\mathcal{F}_S$-measurable for all $i \in S$, $\sum_{i \in S} y_i = \sum_{i \in S} e_i$ and $v_i(y_i) > v_i(x_i)$ for all $i \in S$.

The weak fine core is also an ex ante concept. It captures the idea of an allocation which is ex ante "full information" Pareto optimal. The feasibility condition can also be relaxed here to (ii)$'$ $\sum_{i=1}^n x_i(\omega) \leq \sum_{i=1}^n e_i(\omega)$, for all $\omega \in \Omega$.

Finally we define the concept of weak fine value (WFV) (see Krasa - Yannelis (1996)). We must first define a transferable utility (TU) game in which each agent's utility is weighted by a non-negative factor λ_i, $(i = 1, ..., n)$, which allows for interpersonal comparisons. In a TU-game an outcome can be realized through transfers of payoffs among the agents. On the other hand a WFV allocation is realizable through a redistribution of payoffs and, no side payments are necessary. The WFV set is also non-empty.

Definition 3.3. A game with side payments $\Gamma = (I, V)$ consist of a finite set of agents $I = \{1, ..., n\}$ and a superadditive, real valued function V defined on 2^I such that $V(\emptyset) = 0$. Each $S \subseteq I$ is called a coalition and $V(S)$ is the 'worth' of the coalition S.

The Shapley value of the game Γ (Shapley (1953)) assigns to each Agent i a payoff, $Sh_i(V)$, given by the formula below. It measures the sum of the expected marginal contributions an agent can make to all the coalitions of which he/she can be a member.

$$Sh_i(V) = \sum_{\substack{S \subseteq I \\ S \supseteq \{i\}}} \frac{(\mid S \mid -1)!(\mid I \mid - \mid S \mid)!}{\mid I \mid!} [V(S) - V(S \backslash \{i\})]. \quad (1)$$

The Shapley value has the property that $\sum_{i \in I} Sh_i(V) = V(I)$, i.e. the implied allocation of payoffs is Pareto efficient.

We now define for each DIE, $\mathcal{E}$, with common prior μ, which is assumed for simplicity, and for each set of weights, $\lambda = \{\lambda_i \geq 0 : i = 1, \ldots, n\}$, the associated game with side payments (I, V_λ). We also refer to this as a *transferable utility* (TU) game.

Definition 3.4. Given $\{\mathcal{E}, \lambda\}$ an associated game $\Gamma_\lambda = (I, V_\lambda)$ is defined as follows: For every coalition $S \subset I$ let

$$V_\lambda(S) = \max_x \sum_{i \in S} \lambda_i \sum_{\omega \in \Omega} u_i(\omega, x_i(\omega))\mu(\omega) \quad (2)$$

subject to

(i) $\sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega)$, for all $\omega \in \Omega$, and

(ii) $x_i - e_i$ is $\bigvee_{i \in S} \mathcal{F}_i-$measurable.

We are now ready to define the WFV allocation.

Definition 3.5. An allocation $x = (x_1, \dots, x_n) \in \bar{L}_X$ is said to be a *WFV allocation* of the differential information economy, $\mathcal{E}$, if the following conditions hold

(i) Each net trade $x_i - e_i$ is $\bigvee_{i=1}^{n} \mathcal{F}_i$-measurable,

(ii) $\sum_{i=1}^n x_i = \sum_{i=1}^n e_i$ and

(iii) There exist $\lambda_i \geq 0$, for every $i = 1, ..., n$, which are not all equal to zero, with $\sum_{\omega \in \Omega} \lambda_i u_i(\omega, x_i(\omega))\mu(\omega) = Sh_i(V_\lambda)$ for all i, where $Sh_i(V_\lambda)$ is the Shapley value of Agent i derived from the game (I, V_λ), defined in (2) above.

Conditions (i) and (ii) are obvious and (iii) says that the expected utility of each agent multiplied by his/her weight, λ_i, must be equal to his/her Shapley value derived from the TU game (I, V_λ). For the actual utility that the agent will obtain the weight must not be taken into account. An agent could obtain the utility of a positive allocation even if λ_i were zero.

If in condition (ii) in Definitions 3.4 and (i) in 3.5 are replaced by $x_i - e_i$ is $\mathcal{F}_i$-measurable, for all i, then we obtain the definition of the *private value* allocation.

An immediate consequence of Definition 3.4 is that $Sh_i(V_\lambda) \geq \lambda_i \sum_{\omega \in \Omega} u_i(\omega, e_i(\omega))\mu(\omega)$ for every i, i.e. the value allocation is individually rational. This follows from the fact that the game (V_λ, I) is superadditive for all weights λ. Similarly, efficiency of the Shapley value implies that the weak-fine value allocation is weak-fine Pareto efficient.

For $n = 2$ the WFV belongs to the WFC. However for $n \geq 3$ *a value allocation may not be a core allocation, and therefore not a WEE equilibrium*(see for example Scafuri - Yannelis (1984)).

A table classifying core concepts under the assumption of no free disposal was discussed in Glycopantis et. al. (2005a). Of course there are alternative classifications as well.

4 Noncooperative equilibrium concepts: WEE and REE

We define a *price system* to be $\mathcal{F}$-measurable, non-zero function $p : \Omega \to \mathbb{R}^l_+$ and the *budget set* of Agent i is given by

$$B_i(p) = \{x_i : \ x_i : \Omega \to \mathbb{R}^l \text{ is } \mathcal{F}_i\text{-measurable } \ x_i(\omega) \in X_i(\omega) \\ \text{and } \sum_{\omega \in \Omega} p(\omega) x_i(\omega) \leq \sum_{\omega \in \Omega} p(\omega) e_i(\omega)\}.$$

We now define an ex ante equilibrium concept, which is due to Radner (1968).

Definition 4.1. A pair (p, x), where p is a price system and $x = (x_1, \dots, x_n) \in L_X$ is an allocation, is a *WEE* if

(i) for all i the consumption function maximizes v_i on $B_i(p)$

(ii) $\sum_{i=1}^n x_i \leq \sum_{i=1}^n e_i$ (free disposal), and

(iii) $\sum_{\omega \in \Omega} p(\omega) \sum_{i=1}^n x_i(\omega) = \sum_{\omega \in \Omega} p(\omega) \sum_{i=1}^n e_i(\omega)$.

It extends the Arrow - Debreu model, to allow for differential information. We allow for free disposal, because otherwise, a WEE with positive prices might not exist.

We consider also the case with $\sum_{i=1}^n x_i = \sum_{i=1}^n e_i$. When it exists, the WEE without free disposal is incentive compatible, as it is contained in the private core.

We note that a (free disposal) WEE is in the (free disposal) private core. On the other hand a WEE with free disposal may not be in the non-free disposal private core.

Next for the REE notion we need the following. Let $\sigma(p)$ be the smallest sub-σ-algebra of $\mathcal{F}$ for which a price system $p : \Omega \to \mathbb{R}^l_+$ is measurable and let $\mathcal{G}_i = \sigma(p) \vee \mathcal{F}_i$ denote the smallest σ-algebra containing both $\sigma(p)$ and $\mathcal{F}_i$. We shall also condition the expected utility of the agents on $\mathcal{G}$ which produces a random variable.

Definition 4.2. A pair (p, x), where p is a price system and $x = (x_1, \ldots, x_n) \in \bar{L}_X$ is an allocation, is a *REE* if

(i) for all i the consumption function $x_i(\omega)$ is $\mathcal{G}_i$-measurable;

(ii) for all i and for all ω the consumption function maximizes $v_i(x_i|\mathcal{G}_i)(\omega)$ subject to the budget constraint at state ω,

$$p(\omega)x_i(\omega) \le p(\omega)e_i(\omega);$$

(iii) $\sum_{i=1}^n x_i(\omega) = \sum_{i=1}^n e_i(\omega)$ for all $\omega \in \Omega$.

Conditioning on information obtained from prices as well makes the REE an interim concept. An REE is said to be *fully revealing* if $\mathcal{G}_i = \mathcal{F} = \bigvee_{i\in I} \mathcal{F}_i$ for all $i \in I$.

The definition of REE is taken from Radner (1978) and Allen (1981). An REE does not always exist, may not be fully Pareto optimal, and, as we shall see below, it may fail to be incentive compatible or implementable as a PBE.

5 Some comparisons of equilibrium concepts

We now make comparisons, using examples, between the various equilibrium notions. We shall then turn our attention to the incentive compatibility and possible implementation of such allocations. We indicate, by putting dates, where in Glycopantis et al. the various examples have been discussed.

The WEE, when they exist, are a subset of the corresponding private core ones. On the other hand, *the REE allocations need not be in the private core.* Finally, it is possible that without free disposal no WEE with positive prices exists but a REE does.

The measurability requirement of the private core allocations separates it from the WFC. We also note that no allocation which does not distribute the total resource could be in the WFC.

Example 5.1 (2001, 2003, 2005a) Consider the following three agents economy, $I = \{1, 2, 3\}$ with one commodity, i.e. $X_i = \mathbb{R}_+$ for each i, and three states of nature $\Omega = \{a, b, c\}$. The endowments and information partitions of the agents are given by

$$e_1 = (5, 5, 0), \qquad \mathcal{F}_1 = \{\{a, b\}, \{c\}\};$$

$$e_2 = (5, 0, 5), \qquad \mathcal{F}_2 = \{\{a, c\}, \{b\}\};$$

$$e_3 = (0, 0, 0), \qquad \mathcal{F}_3 = \{\{a\}, \{b\}, \{c\}\}.$$

$u_i(\omega, x_i(\omega)) = x_i^{\frac{1}{2}}$ and every player has the same prior distribution $\mu(\{\omega\}) = \frac{1}{3}$, for $\omega \in \Omega$. It was shown in Glycopantis et al. (2001) that, without free disposal, a private core allocation is given by

$$\begin{pmatrix} 4 & 4 & 1 \\ 4 & 1 & 4 \\ 2 & 0 & 0 \end{pmatrix}$$

.

The ith line refers to Player i and the columns from left to right to states a, b and c.

The private core is sensitive to information asymmetries. On the other hand in a WEE or a REE, Agent 3, irrespective of her private information, will receive zero quantities as he has no initial endowments.

Example 5.2 (2001, 2003, 2005a) We now consider Example 5.1 without Agent 3.

Throughout $\varepsilon, \delta \geq 0$.

A. REE

Now, a price function, $p(\omega)$, known to both agents, is defined on Ω. Apart from his own private $E_i \subseteq \mathcal{F}_i$, each agent also receives a price and the two signals are combined.

For any positive price vector and irrespective of whether free disposal is allowed or not the initial endowments is confirmed as an equilibrium allocation..

In general, with one good per state and monotonic utility functions, the measurability of the allocations implies that REE, fully revealing or not, simply confirms the initial endowments.

B. WEE

We denote the prices by $p(a) = p_1, p(b) = p_2, p(c) = p_3$. The measurability of allocations implies that we require consumptions $x_1(a) = x_2(b) = x$ and $x_1(c)$ for Agent 1, and $x_2(a) = x_2(c) = y$ and $x_2(b)$ for Agent 2. We can also write $x = 5 - \varepsilon, x_1(c) = \delta, y = 5 - \delta$ and $x_2(b) = \varepsilon$.

Without free disposal there is no WEE with prices in $\mathbb{R}^3_+$. For the case with free disposal, the prices are $p_1 = 0, p_2 = p_3 > 0$ and the corresponding allocation is

$$\begin{pmatrix} 4 & 4 & 1 \\ 4 & 1 & 4 \end{pmatrix}.$$

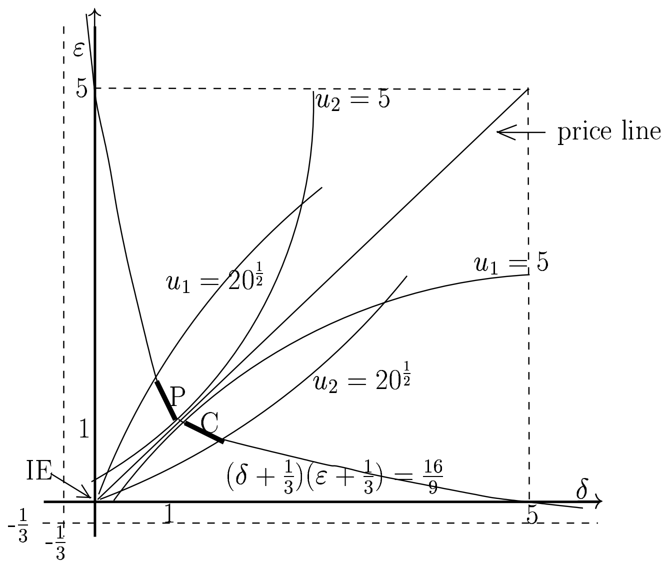
It corresponds to $\varepsilon, \delta = 1$ which means that in state a each of the agents throws away one unit of the good.

C. WFC

There are uncountably many such allocations, as for example

$$\begin{pmatrix} 5 & 2.5 & 2.5 \\ 5 & 2.5 & 2.5 \end{pmatrix}.$$

All WFC allocations will exhaust the resource in each state. From Example 5.1 we can see that a private core allocation is not necessarily a WFC allocation. For example the division (4, 4, 1), (4, 1, 4) and (2, 0, 0), to Agents 1, 2 and 3 respectively, is a private core



PC: private core; IE: Initial Endowments

Figure 1

but not a weak fine core allocation. The first two agents can pool their information and do better. They can realize the WFC allocation, (5, 2.5, 2.5), (5, 2.5, 2.5) and (0, 0, 0) which does not belong to the private core because of lack of measurability.

D. Private Core

Without free disposal the only allocation in the core is the initial endowments.

Free disposal can take the form:

$$\begin{pmatrix} 5-\varepsilon & 5-\varepsilon & \delta \\ 5-\delta & \varepsilon & 5-\delta \end{pmatrix}$$

for $\varepsilon, \delta > 0$.

The private core is the section of the curve $(\delta + \frac{1}{3})(\varepsilon + \frac{1}{3}) = \frac{16}{9}$ between the indifference curves corresponding to $\mathcal{U}_1 = 20^{\frac{1}{2}}$ and $\mathcal{U}_2 = 20^{\frac{1}{2}}$. Notice that the free disposal Radner equilibrium is in the private core, (see Figure 1).

E. WFV

First we note that the "join" $\mathcal{F}_1 \vee \mathcal{F}_2 = \{\{a\}\{b\}\{c\}\}$. So every allocation is $\mathcal{F}_1 \vee \mathcal{F}_2$-measurable and condition (i) of Definition 3.5 is satisfied. Condition (ii) is also immediately satisfied.

Detailed calculations show that there is a range of allocations which belong to WFV. In particular $x_1 = x_2 = (5, 2.5, 2.5)$ is a WFV allocation.

Example 5.3 (2005a, 2005b) We consider a two-agent economy, $I = \{1, 2\}$ with two commodities, i.e. $X_i = \mathbb{R}^2_+$ for each i, and three states of nature $\Omega = \{a, b, c\}$. The endowments, per state a, b, and c, respectively, and the information partitions of the agents are

$$e_1 = ((7,1),(7,1),(4,1)), \qquad \mathcal{F}_1 = \{\{a,b\},\{c\}\};$$
$$e_2 = ((1,10),(1,7),(1,7)), \qquad \mathcal{F}_2 = \{\{a\},\{b,c\}\}.$$

We shall denote $A_1 = \{a,b\},\ c_1 = \{c\},\ a_2 = \{a\},\ A_2 = \{b,c\}$.

We also have $u_i(\omega, x_{i1}(\omega), x_{i2}(\omega)) = x_{i1}^{\frac{1}{2}} x_{i2}^{\frac{1}{2}}$, and for all players $\mu(\{\omega\}) = \frac{1}{3}$, for $\omega \in \Omega$. We have that $u_1(7,1) = 2.65,\ u_1(4,1) = 2,\ u_2(1,10) = 3.16,\ u_2(1,7) = 2.65$ and the expected utilities of the initial allocations, multiplied by 3, are given by $\mathcal{U}_1 = 7.3$ and $\mathcal{U}_2 = 8.46$.

A. REE

First, we are looking for a fully revealing REE. Prices are normalized so that $p_1^1 = 1$ in each state. In effect we are analyzing an Edgeworth box economy per state.

state a: We find that $(p_1, p_2) = (1, \frac{8}{11})$; $x_{11}^* = \frac{85}{22}$, $x_{12}^* = \frac{85}{16}$, $x_{21}^* = \frac{91}{22}$, $x_{22}^* = \frac{91}{16}$; $u_1^* = 4.53, u_2^* = 4.85$.

state b: We find that $(p_1, p_2) = (1,1)$; $x_{11}^* = 4$, $x_{12}^* = 4$, $x_{21}^* = 4$, $x_{22}^* = 4$; $u_1^* = 4, u_2^* = 4$.

state c: We find that $(p_1, p_2) = (1, \frac{5}{8})$; $x_{11}^* = \frac{37}{16}$, $x_{12}^* = \frac{37}{10}$, $x_{21}^* = \frac{43}{16}$, $x_{22}^* = \frac{43}{10}$; $u_1^* = 2.93, u_2^* = 3.40$.

The normalized expected utilities of the equilibrium allocations are $\mathcal{U}_1 = 11.46$, $\mathcal{U}_2 = 12.25$. This completes the analysis of the fully revealing REE.

On the other hand straightforward calculations show that neither a partially revealing nor a non-revealing REE exists.

6 Incentive compatibility

An allocation is incentive compatible if no coalition of a agents, S, can misreport the realized state of nature to $I \setminus S$ and have a distinct possibility of making its members better off. If all members of $I \setminus S$ believe the statements of the members of S then each $i \in S$ expects to gain. For *coalitional Bayesian incentive compatibility* (CBIC) of an allocation we require that this is not possible.

For example, the private core allocation $x_1 = (4,4,1)$, $x_2 = (4,1,4)$ and $x_3 = (2,0,0)$, in Example 5.1, is incentive compatible. This follows from the fact that Agent 3 who would potentially cheat in state a has no incentive to do so. It has been shown in Koutsougeras - Yannelis (1993) that if the utility functions are monotone and continuous then private core allocations are *always* CBIC.

On the other hand the WFC allocations are not always incentive compatible, as the proposed redistribution $x_1 = x_2 = (5, 2.5, 2.5)$ in Example 5.2 shows. Indeed, if Agent 1 observes $\{a,b\}$, he has an incentive to report c and Agent 2 has an incentive to report b when he observes $\{a,c\}$.

CBIC coincides in the case of a two-agent economy with the concept of *Individually Bayesian Incentive Compatibility* (IBIC), which refers to the case when S is a singleton.

We consider here the concept of *Transfer Coalitionally Bayesian Incentive Compatible* (TCBIC) allocations. This allows for transfers between the members of a coalition, and is therefore a strengthening of the concept of Coalitionally Bayesian Incentive Compatibility (CBIC), (see Glycopantis et. al. (2001)).

Definition 6.1. An allocation $x = (x_1, \dots, x_n) \in \bar{L}_X$, with or without free disposal, is said to be TCBIC if it is not true that there exists a coalition S, states ω^* and $\omega^{'}$, with ω^* different from $\omega^{'}$ and $\omega^{'} \in \bigcap_{i \notin S} E_i(\omega^*)$ and a random, net-trade vector, $z = (z_i)_{i \in S}$ among the members of S,

$$(z_i)_{i \in S}, \sum_S z_i = 0$$

such that for all $i \in S$ there exists $\bar{E}_i(\omega^*) \subseteq Z_i(\omega^*) = E_i(\omega^*) \cap (\bigcap_{j \notin S} E_j(\omega^*))$, for which

$$\sum_{\omega \in \bar{E}_i(\omega^*)} u_i(\omega, e_i(\omega) + x_i(\omega^{'}) - e_i(\omega^{'}) + z_i) q_i\big(\omega | \bar{E}_i(\omega^*)\big) > \sum_{\omega \in \bar{E}_i(\omega^*)} u_i(\omega, x_i(\omega)) q_i\big(\omega | \bar{E}_i(\omega^*)\big). \tag{3}$$

Notice that $e_i(\omega) + x_i(\omega^{'}) - e_i(\omega^{'}) + z_i(\omega) \in X_i(\omega)$ is not necessarily measurable. The definition implies that no coalition can hope that by misreporting a state, every member will become better off if they are believed by the members of the complementary set.

Returning to Definition 6.1, one can define CBIC to correspond to $z_i = 0$ and then IBIC to the case when S is a singleton. Thus we have (not IBCI) $\Rightarrow$ (not CBIC) $\Rightarrow$ (not TCBIC). It follows that TCBIC $\Rightarrow$ CBIC $\Rightarrow$ IBIC.

In Koutsougeras - Yannelis (1993), Krasa - Yannelis (1996), it is shown that in an economy with monotone and continuous utility functions the private core and the private value are CBIC. Also, it can be easily seen that any no-free disposal Radner equilibrium belongs to the private core, therefore it is CBIC.

In the case of one good the $\mathcal{F}_i$-measurability of allocations characterizes TCBIC. This enables us to conclude that the redistribution

$$\begin{pmatrix} 5 & 2.5 & 2.5 \\ 5 & 2.5 & 2.5 \end{pmatrix}$$

is not CBIC because it is not $\mathcal{F}_i$-measurable.

On the other hand the no-trade allocation

$$\begin{pmatrix} 5 & 5 & 0 \\ 5 & 0 & 5 \end{pmatrix}$$

is incentive compatible. As we have seen this is a non-free disposal REE.

We note that in the case with *free disposal, private core and Radner equilibrium need not be incentive compatible.* In order to see this we notice that in Example 5.2 the (free disposal) Radner equilibrium is $x_1 = (4, 4, 1)$ and $x_2 = (4, 1, 4)$. The above allocation is not TBIC since if state a occurs agent 1 has an incentive to report state c and become better off.

Next we return to Example 3.5. We can see that the REE redistribution, which belongs also to the WFC, is not CBIC as follows. Suppose that P1 sees $\{a, b\}$ and P2 sees$\{a\}$ but misreports $\{b, c\}$. If P1 believes the lie then state b is believed. So P1 agrees to get the allocation (4, 4). P2 receives the allocation $e_2(a)+x_2(b)-e_2(b) = (1,10)+(4,4)-(1,7) = (4,7)$ with $u_2(4,7) = 5.29 > u_2(\frac{91}{22}, \frac{91}{16}) = 4.85$. Hence P2 has a possibility of gaining by misreporting and therefore the REE is not CBIC. On the other hand if P2 sees $\{b, c\}$ and P1 sees $\{c\}$, the latter cannot misreport $\{a, b\}$ and hope to gain if P2 believes it is b.

Suppose the true state of nature is $\bar{\omega}$. Any coalition can only see together that the state lies in $\bigcap_{i \in S} E_i(\bar{\omega})$. If they decide to lie they must first guess at what is the true state and they will do so at some $\omega^* \in \bigcap_{i \in S} E_i(\bar{\omega})$. Having decided on ω^* as a possible true state, they pick some $\omega^{'} \in \bigcap_{j \notin S} E_j(\omega^*)$ and assuming the system is not CBIC they hope, by each of them announcing $E_i(\omega^{'})$ to secure better payoffs.

This is all contingent on their being believed by $I \setminus S$, which depends on having been correct in guessing that $\omega^* = \bar{\omega}$. If $\omega^* \neq \bar{\omega}$, i.e they guess wrongly, then since $\bigcap_{j \notin S} E_j(\omega^*) \neq \bigcap_{j \notin S} E_j(\bar{\omega})$ the lie may be detected, since possibly $\omega^{'} \notin \bigcap_{j \notin S} E_j(\bar{\omega})$.

7 The possible implementation of equilibrium allocations

The definition of CBIC is about situations where a lie might be beneficial. However the extensive form forces us to consider the alternative of what happens if the lie is detected. It requires statements concerning earlier decisions by other players to lie or tell the truth and what payoffs will occur whenever a lie is detected. Only in this fuller description can players make a decision whether to risk a lie.

We recall that a PBE consists of a set of players' optimal behavioral strategies, and consistent with these, a set of beliefs which attach a probability distribution to the nodes of each information set. It is a variant of the idea of a sequential equilibrium of Kreps and Wilson (1982).

In employing game trees in the analysis we adopt the definition of IBIC. It is still difficult to apply the more appopriate CBIC idea whch implies IBIC. The game-theoretic equilibrium concept employed will be that of PBE. A play of the game will be a directed path from the initial to a terminal node.

In terms of the game trees, a core allocation will be IBIC if there is a profile of optimal behavioral strategies along which no player misreports the state of nature he has observed. This allows for the possibility that players have an incentive to lie from information sets which are not visited by an optimal play.

We examine whether cooperative or Warlasian, noncooperative, static equilibrium allocations, can be supported as the outcome of a dynamic, noncoperative solution concept. The fundamental issue is to connect the idea of implementation, in the form of a PBE of an extensive form game, to the static CBIC property. We also examine the role that a third party could play in supporting an equilibrium.

In view of the widespread use of REE as an equilibrium concept, a separate section is dedicated to the discussion of its possible implementability.

A general conclusion is that only static equilibrium allocations with the CBIC property can be supported, under reasonable rules, as PBE outcomes. From this point of view, private core has a distinct advantages as an equilibrium concept.

7.1 Non-implementation of WEE, WFC and WFV allocations

We consider Example 5.2. We show here is that lack of IBIC implies that two agents do not sign a proposed contract because they have an incentive to cheat. Therefore PBE leads to no-trade.

We shall investigate the possible implementation of the allocation

$$\begin{pmatrix} 4 & 4 & 1 \\ 4 & 1 & 4 \end{pmatrix}$$

contained in a proposed contract between P1 and P2. As we have seen, with free disposal this is a WEE allocation.

This allocation is not IBIC because, as we explained in the Section 6, if Agent 1 observes $A_1 = \{a, b\}$, he has an incentive to report c and Agent 2 has an incentive to report b when he observes $A_2 = \{a, c\}$.

We construct a game tree and employ reasonable rules for calculating payoffs. In fact we look at the contract

$$\begin{pmatrix} 5 & 4 & 1 \\ 5 & 1 & 4 \end{pmatrix}.$$

In the analysis below we assume that the players move sequentially. Nature chooses a state with equal probabilities. P1 then acts first and cannot distinguish between a and b. When P2 is to act he cannot distinguish between a and c but he knows exactly what P1 has chosen before him.

The *rules* for calculating the payoffs in terms of quantities, i.e. the terms of the contract, are:

(i) If the declarations by the two players are incompatible, that is (c_1, b_2) then no-trade takes place and the players retain their initial endowments.
That is the case when either state c, or state b occurs and Agent 1 reports state c and Agent 2 state b. In state a both agents can lie and the lie cannot be detected by either of them. They are in the events A_1 and A_2 respectively, they get 5 units of the initial endowments and again they are not willing to cooperate. Therefore whenever the declarations are incompatible, no trade takes place and the players retain their initial endowments.
(ii) If the declarations are (A_1, A_2) then even if one of the players is lying, this cannot be detected by his opponent who believes that state a has occurred and both players have received endowment 5. Hence no-trade takes place.
(iii) If the declarations are (A_1, b_2) then a lie can be beneficial and undetected. P1 is trapped and must hand over one unit of his endowment to P2. Obviously if his initial endowment is zero then he has nothing to give.
(iv) If the declarations are (c_1, A_2) then again a lie can be beneficial and undetected. P2 is now trapped and must hand over one unit of his endowment to P1. Obviously if his initial endowment is zero then he has nothing to give.

For the calculations of payoffs the revelation of the actual state of nature is not required. We could specify that a player does not lie if he cannot get a higher payoff by doing so.

We assume that each player, given his beliefs, chooses optimally from his information sets.

In Figure 2 we indicate, through heavy lines, plays of the game, obtained through backward induction, which are the outcome of the choices by nature and the optimal behavioral strategies by the players. The interrupted lines signify that nature simply chooses among three alternatives, with equal probabilities. The fractions next to the nodes of the information sets are obtained, whenever possible through Bayesian updating.

These probabilities are calculated as follows. From left to right, we denote the nodes in I_1 by j_1 and j_2, in I_2 by n_1 and n_2 and in $I_2^{'}$ by n_3 and n_4. Given the choices by nature, the strategies of the players described above and using the Bayesian formula for updating beliefs we can calculate, for example, the conditional probabilities

$$Pr(n_1/A_1) = \frac{Pr(A_1/n_1) \times Pr(n_1)}{Pr(A_1/n_1) \times Pr(n_1) + Pr(A_1/n_2) \times Pr(n_2)} = \frac{1 \times 0}{1 \times 0 + 1 \times \frac{1}{3} \times \frac{1}{2}} = 0 \quad (4)$$

and

$$Pr(n_3/c_1) = \frac{Pr(c_1/n_3) \times Pr(n_3)}{Pr(c_1/n_3) \times Pr(n_3) + Pr(c_1/n_4) \times Pr(n_4)} = \frac{1 \times \frac{1}{3}}{1 \times \frac{1}{3} + 1 \times \frac{1}{2} \times \frac{1}{3}} = \frac{2}{3}. \quad (5)$$

For all choices by nature, at least one of the players tells a lie on the optimal play. The players by lying avoid the possibility of having to make a payment to their opponent and stay with their initial endowments. The PBE obtained above confirms the initial endowments. The decisions to lie imply that the players will not sign the contract $(5, 4, 1)$ and $(5, 1, 4)$.

For all choices by nature, at least one of the players tells a lie on the optimal play. The players by lying avoid the possibility of having to make a payment and the PBE confirms the initial endowments. The decisions to lie imply that the players will not sign the contract $(5, 4, 1)$ and $(5, 1, 4)$. A similar conclusion would have been reached if we investigated directly the allocation (4, 4, 1) and (4, 1, 4).

Finally suppose we were to modify (iii) and (iv) of the *rules* i.e.:

(iii) If the declarations are (A_1, b_2) then a lie can be beneficial and undetected, and P1 is trapped and must hand over half of his endowment to P2. Obviously if his endowment is zero then he has nothing to give.
(iv) If the declarations are (c_1, A_2) then again a lie can be beneficial and undetected. P2 is now trapped and must hand over half of his endowment to P1. Obviously if his endowment is zero then he has nothing to give.

The new rules would imply the following changes in the payoffs in Figure 2, from left to right. The second vector would now be (2.5, 7.5), the third vector (7.5, 2.5), the sixth vector (2.5, 2.5) and the eleventh vector (2.5, 2.5). The analysis in Glycopantis et. al. shows that the weak fine core allocation in which both agents receive (5, 2.5, 2.5) cannot be implemented as a PBE. Again this allocation is not IBIC. The same allocation belongs, for equal weights to the agents, also to the WFV.

Finally we note that, the PBE implements the initial endowments allocation

$$\begin{pmatrix} 5 & 5 & 0 \\ 5 & 0 & 5 \end{pmatrix}$$

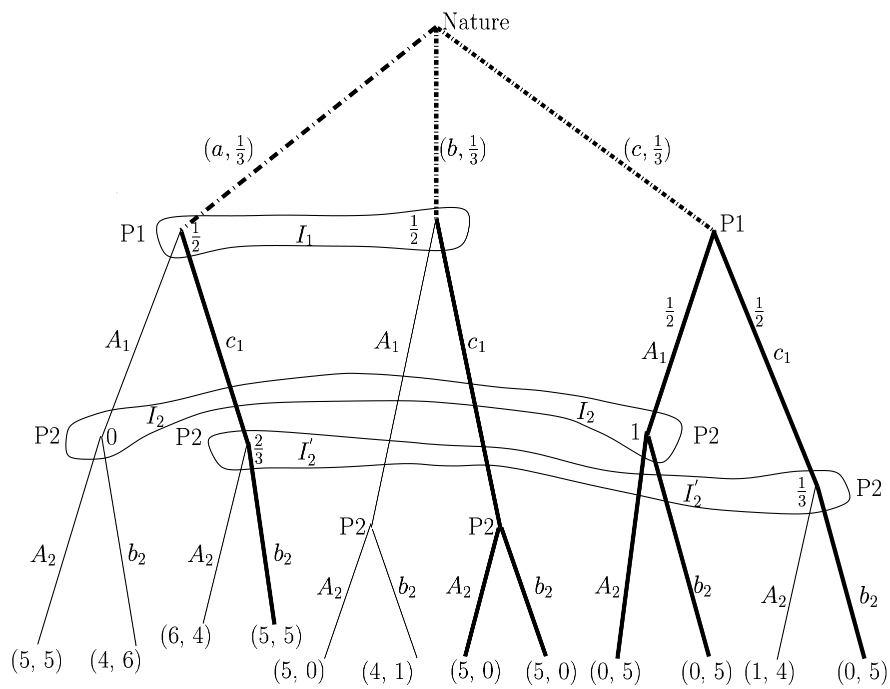


Figure 2

which in the case of non-free disposal, coincides with the REE. However as we shall see below a REE is not in general implementable.

7.2 Implementation of Radner equilibria and of WFC allocations through the courts

We shall show that the Radner (private core) allocation,

$$\begin{pmatrix} 4 & 4 & 1 \\ 4 & 1 & 4 \end{pmatrix}$$

can be implemented as a PBE through an exogenous third party. This might be a court which imposes penalties when an agent lies.

Nature chooses with equal probabilities and P1 acts first and cannot distinguish between a and b. When P2 is to act we assume that not only he cannot distinguish between a and c but also he does not know what P1 has chosen before him.

The *rules* are:

(i) If a player lies about his observation, then he is penalized by 1 unit of the good. If both players lie then they are both penalized. For example if the declarations are (c_1, b_2) and state a occurs both are penalized. If they choose (c_1, A_2) and state a occurs then the first player is penalized. If a player lies and the other agent has a positive endowment then the court keeps the quantity subtracted for itself. However, if the other agent has no endowment, then the court transfers to him the one unit subtracted from the one who lied.

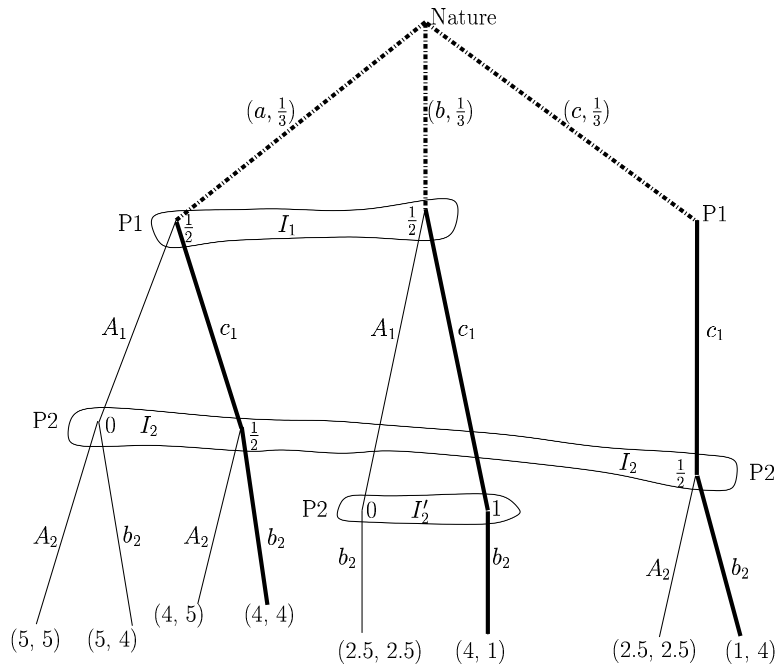


Figure 3

(ii) If the declarations of the two agents are consistent, that is (A_1, A_2) and state a occurs, (A_1, b_2) and state b occurs, (c_1, A_2) and state c occurs, then they divide equally the total endowments in the economy.

We obtain through backward induction the equilibrium strategies by assuming that each player chooses optimally, given his stated beliefs.

Figure 3 indicates, through heavy lines, optimal plays of the game. The fractions next to the nodes of the information sets are obtained through Bayesian updating.

Finally, suppose that the penalties are changed as follows. The court is extremely severe when an agent lies while the other agent has no endowment. It takes all the endowment from the one who is lying and transfers it to the other player.

Now P2 will play A_2 from I_2 and P1 will play A_1 from I_1. Therefore invoking an exogenous agent implies that the PBE will now implement the WFC allocation

$$\begin{pmatrix} 5 & 2.5 & 2.5 \\ 5 & 2.5 & 2.5 \end{pmatrix}.$$

7.3 Implementation of private core allocations

Here we draw upon the discussion in Glycopantis et. al. (2001, 2003). In the case we consider now there is no court and therefore the agents in order to decide must listen to the choices of the other agents before them. P3 is one of the agents and we investigate his role in the implementation of private core allocations. Again we define $A_1 = \{a, b\}$ and $A_2 = \{a, c\}$.

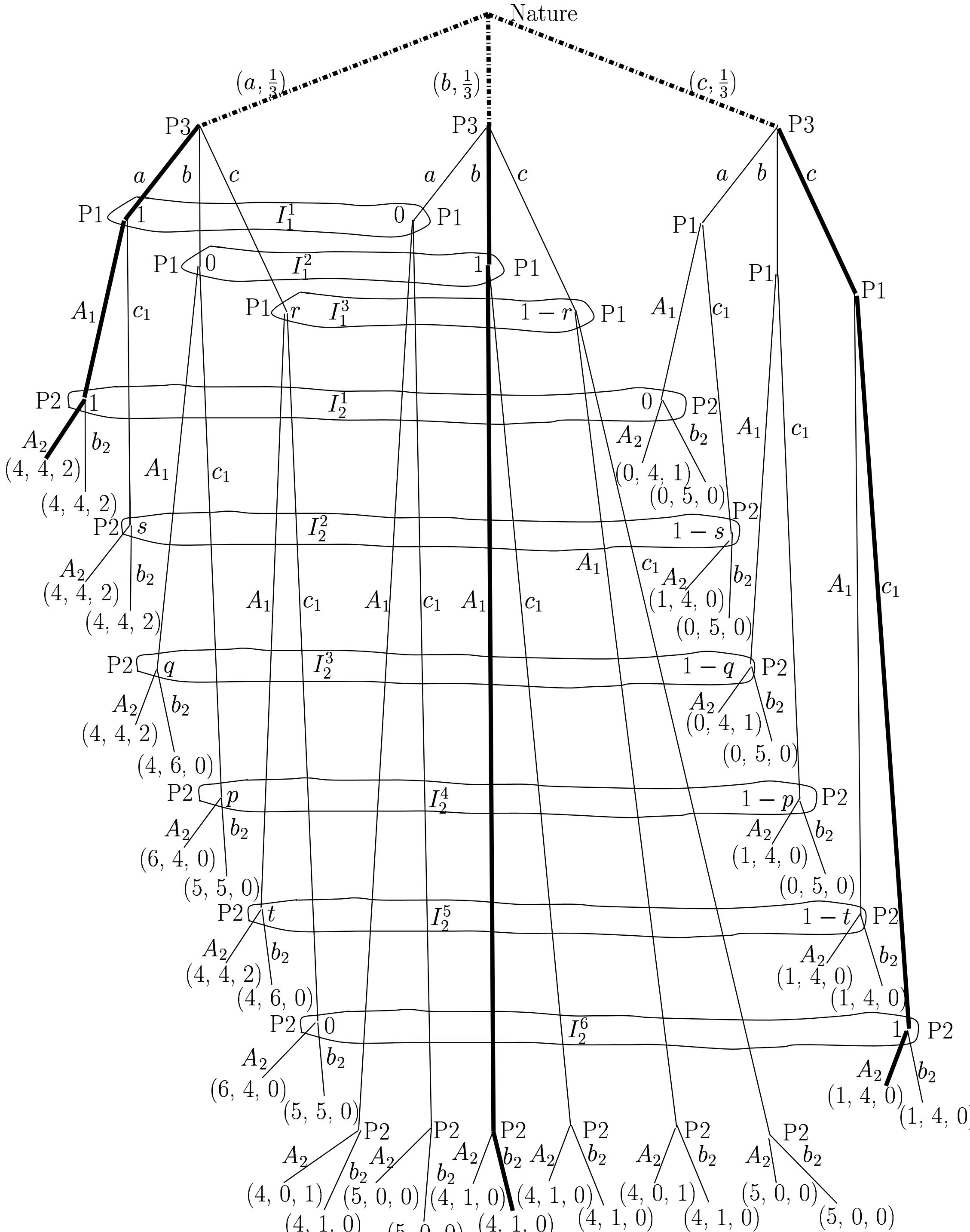

Nature
(a, 1/3)
(b, 1/3)
(c, 1/3)
P3
P3
P3
a
b
c
P1
P1
P1
P1
P1
P1
P1
P1
I_1^1
I_1^2
I_1^3
1
0
0
1
r
1 − r
A_1
c_1
P2
I_2^1
I_2^2
I_2^3
I_2^4
I_2^5
I_2^6
1
0
s
1 − s
q
1 − q
p
1 − p
t
1 − t
0
1
A_2
b_2
(4, 4, 2)
(4, 6, 0)
(6, 4, 0)
(5, 5, 0)
(0, 4, 1)
(0, 5, 0)
(1, 4, 0)
(4, 0, 1)
(4, 1, 0)
(5, 0, 0)

Figure 4

Private core without free disposal seems to be the most satisfactory concept. The third agent, who has superior information, acting as an intermediary, implements the contract and gets rewarded in state a.

We shall consider the private core allocation

$$\begin{pmatrix} 4 & 4 & 1 \\ 4 & 1 & 4 \\ 2 & 0 & 0 \end{pmatrix}$$

of Example 5.1.

We know that such core allocations are CBIC and we shall show now how they can be supported as PBE of a noncooperative game.

P1 cannot distinguish between states a and b and P2 between a and c. P3 sees on the screen the correct state and moves first. He can either announce exactly what he saw or he can lie. Obviously he can lie in two ways. When P1 comes to decide he has his information from the screen and also he knows what P3 has played. When it is the turn of P2 to decide he has his information from the screen and he also knows what Nin-implementatiionP3 and P1 played before him. Both P1 and P2 can either tell the truth about the information they received from the screen or they can lie.

The *rules* of calculating payoffs, i.e. the terms of the contract, are as follows:
If P3 tells the truth we implement the redistribution in the matrix above which is proposed for this particular choice of nature.
If P3 lies then we look into the strategies of P1 and P2 and decide as follows:
(i) If the declaration of P1 and P2 are incompatible we go to the initial endowments and each player keeps his.
(ii) If the declarations are compatible we expect the players to honour their commitments for the state in the overlap, using the endowments of the true state, provided these are positive. If a player's endowment is zero then no transfer from that agent takes place as he has nothing to give.

In Figure 4 we indicate through heavy lines the equilibrium paths. The directed paths (a, a, A_1, A_2) with payoffs (4, 4, 2), (b, b, A_1, b_2) with payoffs (4, 1, 0) and (c, c, c_1, A_2) with payoffs (1, 4, 0) occur, each, with probability $\frac{1}{3}$. It is clear that nobody lies on the optimal paths and that the proposed reallocation is incentive compatible and hence it will be realized.

The beliefs, which can be calculated through Bayesian updating are indicated by the probabilities attached to the nodes of the information sets, with arbitrary r, s, q, p and t between 0 and 1, (see Glycopantis et. al. (2003)).

We also note that the same payoffs, i.e. (4, 4, 2), (4, 1, 0) and (1, 4, 0), can be confirmed as a PBE for all possible orders of the players.

Further we can show that the PBE in Figure 4 can also be obtained as a sequential equilibrium. Now, it is also required that the optimal behavioral strategies, and the beliefs consistent with these, are the limit of a sequence consisting of completely mixed behavioral strategies, and the implied beliefs. Throughout the sequence it is only required that beliefs are consistent with the strategies. The latter are not expected to be optimal.

We are looking for a sequence of positive probabilities attached to all the choices from each information set and beliefs consistent with these such that their limits are the results given in Figure 4.

First we specify the positive probabilities, i.e. the completely mixed strategies, with which the players choose the available actions. The sequence is obtained through $\{n = 1, 2, ...\}$.

In the first instance we consider the singletons from left to right belonging to P3. At the first one the positive probabilities attached to the various actions are given by $(a, 1-\frac{2}{n};\ b, \frac{1}{n};\ c, \frac{1}{n})$, at the second one by $(a, \frac{1}{n};\ b, 1-\frac{2}{n};\ c, \frac{1}{n})$ and at the third one by $(a, \frac{1}{n};\ b, \frac{1}{n};\ c, 1-\frac{2}{n})$.

Then we come to the probabilities with which P1 chooses his actions from the various information sets belonging to him. From I_1^1 and I_1^2 the choices and the probabilities attached to these are $(A_1, 1-\frac{1}{n};\ c_1, \frac{1}{n})$, and from I_1^3, as well as from all the singletons, they are $(A_1, \frac{1}{n};\ c_1, 1-\frac{1}{n})$.

With respect to P2 choices and probabilities are given as follows. From I_2^1 and I_2^6 they are $(A_2, 1-\frac{1}{n};\ b_2, \frac{1}{n})$ and from I_2^2, I_2^3, I_2^4 and I_2^5 they are $(A_2, \frac{1}{n};\ b_2, 1-\frac{1}{n})$. With respect to the singletons belonging to P2 we have for all of them $(A_2, \frac{1}{n};\ b_2, 1-\frac{1}{n})$.

Beliefs are indicated by the probabilities attached to the nodes of the information sets. Glycopantis et. al. (2003) explain in detail how a particular PBE can be confirmed as a sequential one.

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