STRESS TRANSPORT MODELLING – 1

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Introduction – 1

- Both linear and non-linear EVM's use *algebraic* relations for the stresses.
- This means the stresses respond instantly to changes in the mean strain, and leads to a direct link between the stress anisotropy and mean strains. If mean velocity gradients vanish at some point in the flow, even a NLEVM will return isotropic turbulence at that position.
- Physically, the individual stresses get generated, convected, diffused and dissipated at different rates. We now, therefore, explore a modelling strategy that takes account of this.
- One can construct an exact transport equation for each stress component $\overline{u_i u_j}$, by manipulating the equations for the fluctuating velocities u_i and u_j. The result does contain unknown products – which have to be modelled – but the modelling is now at a more fundamental level than the approach of obtaining an effective turbulent viscosity.

Introduction – 2

- An important aspect of stress transport modelling is that the generation terms are represented exactly.
- In this first lecture we will:
  - Examine the stress transport equations, and identify some of the physical features built into them.
  - Describe simple models that can be used to close the set of equations.
- In a second lecture we will look at more methods of devising more advanced stress transport models.

Stress Transport Equations – 1

- To obtain a transport equation for $\overline{u_i u_j}$, the equation for u_j is multiplied by u_i and averaged:

$$\overline{u_i \left[\frac{\partial u_j}{\partial t} + U_k \frac{\partial u_j}{\partial x_k} = \cdots \right]}$$

- Adding this to the equation for u_i multiplied by u_j (also averaged), results in an equation which begins

$$\overline{u_i \frac{\partial u_j}{\partial t}} + \overline{u_j \frac{\partial u_i}{\partial t}} + U_k \left[\overline{u_i \frac{\partial u_j}{\partial x_k}} + \overline{u_j \frac{\partial u_i}{\partial x_k}} \right] = \cdots$$

or

$$\frac{\partial \overline{u_i u_j}}{\partial t} + U_k \frac{\partial \overline{u_i u_j}}{\partial x_j} \equiv \frac{D \overline{u_i u_j}}{Dt} = \cdots$$

Stress Transport Equations – 2

- After some manipulation, the stress transport equation becomes

$$\frac{D\overline{u_i u_j}}{Dt} = -\left(\overline{u_i u_k}\frac{\partial U_j}{\partial x_k} + \overline{u_j u_k}\frac{\partial U_i}{\partial x_k}\right) - 2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}$$

$$+\overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}$$

$$-\frac{\partial}{\partial x_k}\left[\overline{u_i u_j u_k} + \overline{p u_i}/\rho\delta_{jk} + \overline{p u_j}/\rho\delta_{ik} - \nu\frac{\partial\overline{u_i u_j}}{\partial x_k}\right]$$

which is often written in shorthand notation

$$\frac{D\overline{u_i u_j}}{Dt} = P_{ij} - \varepsilon_{ij} + \phi_{ij} + d_{ij}$$

Stress Transport Equations – 3

- There are similarities with the turbulent kinetic energy equation:
  - P_{ij} is the generation rate of the turbulent stress by mean shear.
  - ε_{ij} is the viscous dissipation rate of the stress component.
  - d_{ij} is the diffusion rate of the turbulent stress by turbulent and viscous action.
- For practical computations, models are needed for ϕ_{ij}, ε_{ij} and d_{ij}. However, the important generation terms do not require modelling.
- One difference between the turbulent kinetic energy and stress equations is that the k transport equation has no equivalent of the process ϕ_{ij}.
- If we sum the equations for the normal stresses (set $i = j$) we produce an equation for (twice) the turbulence energy ($D\overline{u_i u_i}/Dt \equiv 2\, Dk/Dt$).

Stress Transport Equations – 4

- For ϕ_{ij} we get:

$$\phi_{ii} = \overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i}\right)} = 0 \qquad \text{(by continuity)}$$

- This process – called the “pressure-strain” or “pressure-scrambling” term – thus makes no *direct* contribution to the level of turbulence energy.

- However, ϕ_{ij} does act to redistribute turbulence energy from one stress component to another.

- As a vast generalization, if the normal stress in one direction is less than in the other directions, it will receive energy through ϕ_{ij}.

- The process ϕ_{ij} is thus often thought of as tending to return the turbulence towards an isotropic state.

The Production Tensor

- The production term are exact, in that they only contain Reynolds stresses and mean strains, and therefore do not require further modelling:

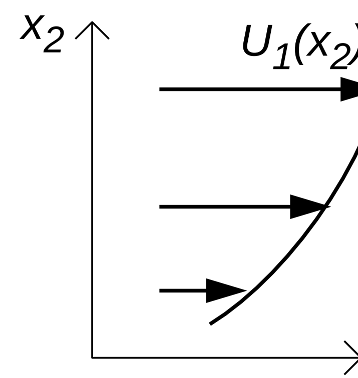
$$P_{ij} = -\left(\overline{u_i u_k} \frac{\partial U_j}{\partial x_k} + \overline{u_j u_k} \frac{\partial U_i}{\partial x_k} \right)$$

- Before considering the modelling of other terms, we first examine how these generation rates behave in a few example flows.

Simple Shear Flow

- In a simple shear flow, with $U_1(x_2)$, $U_2 = U_3 = 0$,

$$P_{11} = -2\,\overline{u_1 u_2}\,\frac{\partial U_1}{\partial x_2}$$
$$P_{22} = P_{33} = 0$$
$$P_{12} = -\overline{u_2^2}\,\frac{\partial U_1}{\partial x_2}$$



- This shows why:
  - $\overline{u_1 u_2}$ is (usually) of the opposite sign to $\partial U_1/\partial x_2$: its generation term is of the opposite sign.
  - The streamwise normal stress is typically larger than the others: only $\overline{u_1^2}$ is generated directly by P_{ij}, and the pressure-strain ϕ_{ij} then acts to 'redistribute' some of this energy into $\overline{u_2^2}$ and $\overline{u_3^2}$.
  - Free flows spread more rapidly than near-wall flows: $\overline{u_2^2}/k$ is larger in free flows than in near-wall flows, leading to higher levels of shear stress generation.

Generation Rates on Curved Surfaces – 1

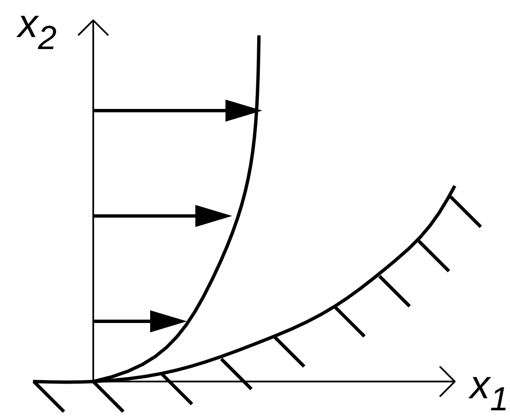
- It is known that even weak surface curvature can have a strong effect on turbulence levels. The reasons for this can be seen from the generation terms.

- In addition to the primary shear $\partial U_1/\partial x_2$, there is now a shear associated with curvature $\partial U_2/\partial x_1$. This results in

$$P_{12} = -\overline{u_2^2}\,\frac{\partial U_1}{\partial x_2} - \overline{u_1^2}\,\frac{\partial U_2}{\partial x_1}$$

and

$$P_{22} = -\overline{u_1 u_2}\,\frac{\partial U_2}{\partial x_1}$$



- For the concave wall shown, $\partial U_1/\partial x_2 > 0$, and $\partial U_2/\partial x_1 > 0$.

- Both terms in P_{12} are thus of the same sign. As noted earlier, $\overline{u_1^2}$ is significantly larger than $\overline{u_2^2}$ (particularly in the viscosity affected sublayer).

Generation Rates on Curved Surfaces – 2

- Thus, although $\partial U_2/\partial x_1$ is much smaller than $\partial U_1/\partial x_2$, the difference in magnitude of the associated normal stresses means that the second term in P_{12} is not neligible when compared to the first.
- In addition, the generation term in the $\overline{u_2^2}$ equation is positive (since $\overline{u_1 u_2}$ is negative), helping to further increase P_{12}.
- The shear stress production is thus augmented partly by the secondary strain term in P_{12} and partly because the curvature enhances $\overline{u_2^2}$.
- Since stress transport models represent P_{ij} exactly, they should (at least qualitatively) capture this effect.
- In contrast, a linear eddy-viscosity model takes

$$-\overline{u_1 u_2} = \nu_t \left(\frac{\partial U_1}{\partial x_2} + \frac{\partial U_2}{\partial x_1} \right)$$

and since the secondary strain is much smaller than the primary, its effect in this case is negligible.

Modelling the Stress Transport Equations

- Although generation terms are treated exactly, models are needed for the dissipation (ε_{ij}), diffusion (d_{ij}) and pressure-strain correlation (ϕ_{ij}).
- There are a number of properties which the model equations should ideally have:
  - The correct tensorial form: they should have the same symmetries and contraction properties that the exact processes do.
  - Coordinate invariance: they should be independent of the frame of reference – including accelerating frames.
  - Realizability: models should not predict physically impossible values such as negative normal stresses.
  - Consistency with physical wall/surface limits.
  - Geometry independence: models should not be dependent on details of the specific geometry being studied.
- Models described in this lecture satisfy the first two conditions.

Dissipation, ε_{ij}

- Dissipative processes arise from the smallest scale eddies:

$$\varepsilon_{ij} = 2\nu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}$$

- Turbulence energy gets cascaded from larger eddies to smaller ones.
- If there is a wide enough range of scales, one can argue that the smallest eddies will be (almost) isotropic – <u>local isotropy</u>.
- Hence, at high Reynolds numbers, the dissipative terms are often assumed to be isotropic:

$$\varepsilon_{ij} = 2/3\varepsilon\delta_{ij}$$

  – an equal effect on all normal stresses, and none on the shear stress.
- The turbulence energy dissipation rate ε is obtained from its own transport equation, similar to that used in a k-ε scheme.

Diffusion, d_{ij}

- From the exact expression, the diffusion can be seen to be due to triple moments, pressure-velocity correlations and viscous effects:

$$d_{ij} = -\frac{\partial}{\partial x_k}\left(\overline{u_i u_j u_k} + \overline{p u_j}/\rho\delta_{ik} + \overline{p u_i}/\rho\delta_{jk} - \nu\frac{\partial\overline{u_i u_j}}{\partial x_k}\right)$$

- Pressure-diffusion is usually negligible, except very close to a wall or surface.
- The generalized gradient diffuion hypothesis (Daly & Harlow 1970) is often employed:

$$\overline{u_k\phi} \propto -\frac{k}{\varepsilon}\overline{u_k u_l}\frac{\partial\phi}{\partial x_l}$$

for some quantity ϕ.
- Applying this to the triple moments gives

$$d_{ij} = \frac{\partial}{\partial x_k}\left(c_s\frac{k}{\varepsilon}\overline{u_k u_l}\frac{\partial\overline{u_i u_j}}{\partial x_l}\right)$$

Pressure-Strain, ϕ_{ij} – 1

- As already seen, ϕ_{ij} is redistributive: it has no direct influence on levels of turbulence energy.

$$\phi_{ij} = \overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}$$

- One can obtain and solve a Poisson equation for p, and thus arrive at an expression for ϕ_{ij}:

$$\phi_{ij} = -\frac{1}{4\pi}\int_V \left(\frac{\partial^3 \overline{u_l' u_k' u_i}}{\partial r_l \partial r_k \partial r_j} + \frac{\partial^3 \overline{u_l' u_k' u_j}}{\partial r_l \partial r_k \partial r_i}\right)\frac{dV}{|r|}$$

$$-\frac{1}{2\pi}\int_V \frac{\partial U_k'}{\partial r_l}\left(\frac{\partial^2 \overline{u_l' u_i}}{\partial r_k \partial r_j} + \frac{\partial^2 \overline{u_l' u_j}}{\partial r_k \partial r_i}\right)\frac{dV}{|r|}$$

- In buoyant flows there is a third part to ϕ_{ij}, involving the fluctuating buoyant forces.

Pressure-Strain, ϕ_{ij} – 2

- One thus expects to have contributions to ϕ_{ij} from turbulence-turbulence interactions and from mean strains (and from buoyancy terms in buoyant flows). Hence ϕ_{ij} is often modelled as

$$\phi_{ij} = \phi_{ij1} + \phi_{ij2}(+\phi_{ij3})$$

- In a simple shear flow ϕ_{ij} acts to reduce the anisotropy of the stresses – by redistributing energy from the streamwise component into the other two normal stresses.
- If one generates an initially anisotropic turbulence, it will return towards isotropy once the external strains have been removed. This must be due to ϕ_{ij1}, the “slow pressure-strain”, or “return to isotropy”, term.

Pressure-Strain Modelling – 1

- A simple linear model for the turbulence-turbulence interactions is thus (Rotta 1951):

$$\phi_{ij1} = -c_1 \varepsilon a_{ij}$$

  where $a_{ij} = \overline{u_i u_j}/k - 2/3\delta_{ij}$.

- This redistributes energy to reduce the anisotropy of the stresses.
- ϕ_{ij2} has a similar effect of tending to reduce the anisotropy (at least in simple shear flow).
- A simple (and widely used) model for the “rapid” pressure strain term is

$$\phi_{ij2} = -c_2(P_{ij} - 1/3P_{kk}\delta_{ij})$$

  (Naot et al 1970, Launder et al 1975).

- This acts to redistribute the generation rates to reduce anisotropy. (“Isotropization of Production” or IP model)

Pressure-Strain Modelling – 2

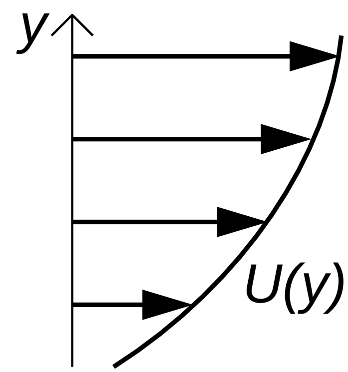
- In a simple shear, where only P_{11} and P_{12} are non-zero, this gives

$$\phi_{112} = -c_1 \varepsilon a_{11} - 2/3\, c_2 P_{11}$$

$$\phi_{222} = -c_1 \varepsilon a_{22} + 1/3\, c_2 P_{11}$$

$$\phi_{332} = -c_1 \varepsilon a_{33} + 1/3\, c_2 P_{11}$$

$$\phi_{122} = -c_1 \varepsilon a_{12} - c_2 P_{12}$$



- ϕ_{ij2} acts as a sink for $\overline{u_1^2}$ and $\overline{u_1 u_2}$, but a source for $\overline{u_2^2}$ and $\overline{u_3^2}$.
- Since $\overline{u_1^2}$ will be larger than $\overline{u_2^2}$ and $\overline{u_3^2}$, ϕ_{ij1} will also act to reduce $\overline{u_1^2}$ and increase the other two normal stresses. It will also act as a sink for $\overline{u_1 u_2}$.

Accounting for Wall-Reflection Effects – 1

- In a simple shear flow with velocity gradient dU_1/dx_2, the model described so far gives identical values for $\overline{u_2^2}$ and $\overline{u_3^2}$.
- This is *not* what is found experimentally, even in free flows: in near-wall flows even larger anisotropy is expected.
- Typical stress anisotropy levels:

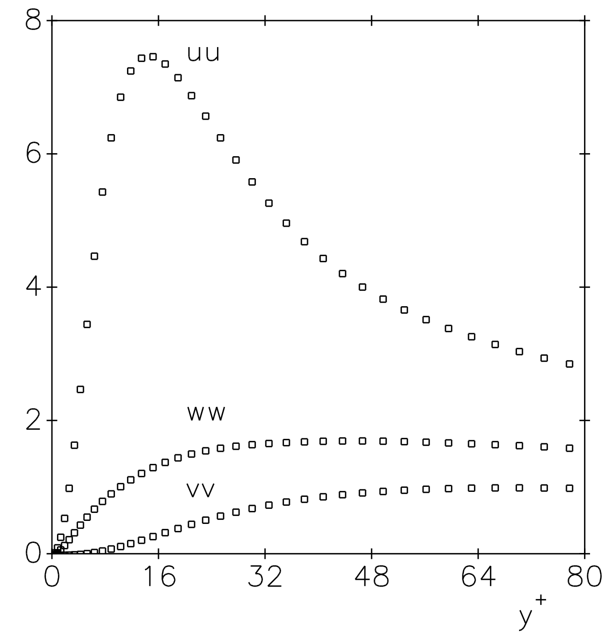
	$\overline{u_1^2}/k$	$\overline{u_2^2}/k$	$\overline{u_3^2}/k$	$\overline{u_1u_2}/k$
Free shear	0.95	0.47	0.55	0.32
Near-wall flow	1.20	0.25	0.55	0.25

- Equal values for $\overline{u_2^2}$ and $\overline{u_3^2}$ would imply significant errors near a wall.

Accounting for Wall-Reflection Effects – 2

- One of the effects of the wall (or surface) is to impede the energy transfer into the stress component normal to the wall.

- Pressure fluctuations get reflected from walls or surfaces, and this leads to a damping of the velocity fluctuations normal to the surface.



- This also leads to a reduction of the shear stress.

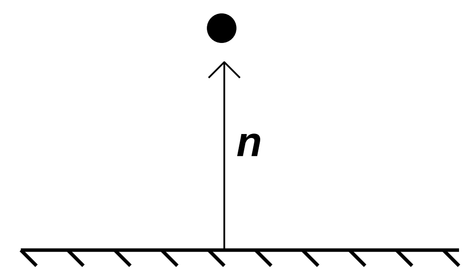
- Additional terms are often included in ϕ_{ij} to account for these effects.

Wall-Reflection Modelling – 1

- The wall damping effect is often modelled by employing the turbulence lengthscale to control the strength of the wall corrections.

- The quantity f_y defined by

$$f_y = \frac{k^{3/2}/\varepsilon}{2.5 x_p n_p}$$



where n is the unit vector normal to the wall, is typically close to unity near the wall, but decreases as one moves away from the wall.

- Note, however, that this can be difficult to apply in computing complicated geometries.

Wall-Reflection Modelling – 2

- A widely-used wall correction to ϕ_{ij1} (Shir 1973) can be written:

$$\phi^w_{ij1} = c_{1w}\frac{\varepsilon}{k}\left(\overline{u_k u_l}\, n_l n_k \delta_{ij} - 3/2\,\overline{u_i u_k}\, n_j n_k - 3/2\,\overline{u_j u_k}\, n_i n_k\right) f_y$$

  where c_{1w} is typically taken as 0.5.

- For a single wall with $n_1 = n_3 = 0$, $n_2 = 1$ this gives:

$$\phi^w_{111} = \phi^w_{331} = c_{1w}\frac{\varepsilon}{k}\overline{u_2^2} f_y$$

$$\phi^w_{221} = -2\, c_{1w}\frac{\varepsilon}{k}\overline{u_2^2} f_y$$

$$\phi^w_{121} = -3/2\, c_{1w}\frac{\varepsilon}{k}\overline{u_1 u_2} f_y$$

- The model thus acts to impede the transfer of energy into $\overline{u_2^2}$ as the wall is approached.

Wall-Reflection Modelling – 3

- Corrections are usually also applied to ϕ_{ij2}. The Gibson-Launder (1978) model for ϕ^w_{ij2} can be written:

$$\phi^w_{ij2} = c_{2w} \left(\phi_{kl2}\, n_k n_l \delta_{ij} - 3/2\, \phi_{ik2}\, n_j n_k - 3/2\, \phi_{jk2}\, n_i n_k \right) f_y$$

- This acts to oppose the ϕ_{ij2} redistribution process. For $n = (0, 1, 0)$,

$$\phi^w_{112} = \phi^w_{332} = c_{2w}\, \phi_{222}\, f_y$$

$$\phi^w_{222} = -2\, c_{2w}\, \phi_{222}\, f_y$$

$$\phi^w_{122} = -3/2\, c_{2w}\, \phi_{122}\, f_y$$

- But in an impinging flow (where, because of the different strain rates, energy is generated in the component normal to the wall), this has the effect of *increasing* the stress component normal to the wall!

Wall-Reflection Modelling – 4

- An alternative model for ϕ^w_{ij2}, proposed by Craft & Launder (1992), has the desired effect of damping the wall-normal stress component in both wall-parallel and impinging flows:

$$\phi^w_{ij2} = -0.08 \frac{\partial U_l}{\partial x_k} \overline{u_l u_k} (\delta_{ij} n_t n_t - 3 n_i n_j) f_y$$

$$- 0.1 k a_{lm} \left(\frac{\partial U_k}{\partial x_m} n_l n_k \delta_{ij} - 3/2 \, \frac{\partial U_i}{\partial x_m} n_l n_j - 3/2 \, \frac{\partial U_j}{\partial x_m} n_l n_i \right) f_y$$

$$+ 0.4 k \frac{\partial U_k}{\partial x_l} n_l n_k \left(n_i n_j - 1/3 \, n_t n_t \delta_{ij} \right) f_y$$

Summary

- In this first lecture on sress transport modeling, we have covered:
  - Where the stress transport equations come from.
  - The physical processes that appear in the equations.
  - Some simple modelling ideas to close the equations.

- In the next lecture we will look at routes to devise more advanced models: in particular the idea of realizability and the development of two-component-limit models.

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