Some pathological phenomena of finite difference approximations for parabolic blow-up problems

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Abstract

The blow-up problems for parabolic PDEs sometimes help our understanding of singular behaviours of fluid flows. They accordingly received attention from both mathematicians and physicists for several decades. My talk is concerned with numerical computations of blow-up phenomena.

Attention is focused on $u_t = u_{xx} + f(u)$ $(0 < t, 0 < x < 1)$ with the Dirichlet boundary condition $u = 0$ on $x = 0$ and $x = 1$. It is known that a solution with a large initial data blows up in finite time, if an appropriate growth condition on f is imposed. I consider questions such as : how a finite difference scheme can reproduce the blow-up phenomena? For instance, I would like to know whether the one-point blow-up, which is proved to occur in certain $f(u)$, is truly reproduced by the finite difference solutions.

With $f(u) = u^2$, T. Nakagawa in 1976 considered a finite difference scheme and he mathematically proved that his discrete solutions blew up in finite time if a certain largeness condition on the initial data was assumed and that the discrete blow-up time converged to the 'real' blow-up time if the mesh sizes tended to zero. His work was followed by several mathematical papers, but some qualitative problems were left unanswered.

A new step forward was carried out by Y.-G. Chen in 1986, who considered the equation with $f(u) = u^{1+\alpha}$ and proved, among others, that some discrete solutions could blow up at more than one point, while a one-point blow-up occurred in the continuous problem. Quite general blow-up conditions were established for semi-discretized equations by T. K. Ushijima in 2000.

The purpose of my talk is to provide clues to those problems which were left unanswered or untouched in these papers. In particular, I will show, by examples, what is called the blow-up set cannot be numerically captured by a seemingly nice scheme.

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