Pricing Mortgage-Backed Securities

– Continuous Time Stanton's Model [a]

Min DAI

Department of Mathematics

National University of Singapore

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[a] Joint with Yue-Kuen Kwok and Hong You

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- Historical Work

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- Numerical Schemes

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- Numerical Schemes
- Applications to Other Derivatives

Introduction

What is mortgage?

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What is mortgage?

- a loan in order to purchase a house
- collateralized by the house
- similar to an amortizing bond
- A mortgagor has the **right** to prepay his loan at any time

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- attractive yields with little or no credit risk, trade in a liquid secondary market
- dramatic growth since its inception in 70s
  - total notional amount of MBS and collateralized mortgage obligations outstanding as of 30 June 2002 $3.9 trillion
  - total notional amount of US treasury debt $3.5 trillion

The Pricing of MBS

- Black-Scholes Option Pricing

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- The key point is how to deal with the prepayment behavior

Historical Work

structural models

reduced-form models

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- optimal prepayment policy (like the optimal exercise policy of American options)

reduced-form models

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reduced-form models

- prepayment policy given according to historic data
- a simple linear parabolic PDE model

Structural Model

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- assuming interest rate follows CIR model in the risk-neutral world

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- mortgagors prepay whenever $V = \varphi(t)$
- basic structure model

$$\begin{cases} V_t + \frac{1}{2}\sigma^2 r V_{rr} + \kappa(\mu - r)V_r - rV + C(t) \geq 0 \\ V \leq \varphi(t) \\ either\ must\ take\ equality\ at\ any\ point\ (r,t) \in [0,\infty) \times [0,T] \\ with\ boundary\ and\ initial\ conditions \end{cases}$$

back

Structural Model

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- suboptimal prepayment: some mortgagors prepay due to exogenous reasons, e.g. migration, divorce or purchase of a better house
- exogenous prepayment governed by Poisson process dJ, with density $\lambda(r, t)$

$$\begin{cases} P(dJ = 0) = 1 - \lambda\delta t, & \text{no suboptimal prepayment occurs} \\ P(dJ = 1) = \lambda\delta t, & \text{suboptimal prepayment occurs} \end{cases}$$

Structural Model

Dunn and McConnell model

$$\begin{cases} V_t + \frac{1}{2}\sigma^2 r V_{rr} + \kappa(\mu - r)V_r - rV + C(t) + \lambda(r,t)[\varphi(t) - V] \geq 0 \\ V(r,t) \leq \varphi(t) \\ either\ must\ take\ equality\ at\ any\ point\ (r,t) \in [0,\infty) \times [0,T] \\ with\ boundary\ and\ initial\ conditions \end{cases}$$

compare with most basic structural model

back

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drawbacks

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- predicted rate of prepayment does not match observation

major reason

- mortgagors may not prepay even when it is optimal

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- do not explain the true underlying process
- may not perform well out-of-sample

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when

$$V(r,t) > \varphi(r,t)$$

- in summary, probability of total prepayment

$$P(\text{prepayment}) = \begin{cases} P_e = \lambda\delta t & V \leq \varphi \\ P_r = (\lambda+\rho)\delta t & \text{otherwise} \end{cases}$$

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- repeat these steps until $t_0 = 0$

Stanton's Model

Other features

- mortgage liability vs asset

Stanton’s Model

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- heterogenous transaction cost

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But Stanton’s Model is only a numerical algorithm.

New Model

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New Model

- continuous time version of Stanton's model
- inherit all features of Stanton's model
- **random** exogenous and endogenous prepayment modelled by two independent Poisson processes

$$dJ = \begin{cases} 0, & \text{exogenous prepayment does not occur} \\ 1, & \text{exogenous prepayment occurs} \end{cases}$$

$$dK = \begin{cases} 0, & \text{mortgagor makes no endogenous prepayment decision} \\ 1, & \text{mortgagor makes endogenous prepayment decision} \end{cases}$$

and

$$\mathrm{E}(dJ) = P(dJ = 1) = \lambda dt$$

$$\mathrm{E}(dK) = P(dK = 1) = \rho dt$$

New Model

With interest rate governed by CIR model,

$$dM = [M_t + \frac{1}{2}\sigma^2 r M_{rr} + \kappa(\mu - r)M_r + C(t)]dt + \sigma\sqrt{r}M_r dz$$
$$+ [\varphi(t) - M]dJ + \min(\varphi(t) - M, 0)dK$$

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compare with Dunn and McConnell model

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- Stanton's numerical scheme is shown to be consistent with the continuous time model

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- nonlinear parabolic partial differential equation
- does not impose $\varphi(r, t)$ as the upper bound
- Stanton's numerical scheme is shown to be consistent with the continuous time model
- provides a financial explanation for penalty approximation for variational inequality problem (see Forsyth and Vetzal, 2002)

Numerical Scheme

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  - Crank-Nicolson scheme with generalized Newton iteration
  - hybrid scheme using Crank-Nicolson and Adam-Bashforth

Numerical Scheme 1

Crank-Nicolson scheme with generalized Newton iteration

$$\begin{aligned} M_t + \frac{1}{2}\sigma^2 r M_{rr} &+ \kappa(\mu - r)M_r - rM + C \\ &+ \lambda(\varphi - M) + \rho \min(\varphi - M, 0) = 0 \end{aligned}$$

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- all terms are expanded at point $(y_i, t_{n+\frac{1}{2}})$ to obtain

$$AM^n = BM^{n+1} + \eta^{n+\frac{1}{2}} + \rho \min\{\varphi^{n+\frac{1}{2}} - \frac{1}{2}(M^{n+1} + M^n), 0\}$$

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$$f(x^{(k+1)}) = f(x^{(k)}) + f'(x^{(k)})(x^{(k+1)} - x^{(k)})$$

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- use generalized Newton method

$$f(x^{(k+1)}) = f(x^{(k)}) + f'(x^{(k)})(x^{(k+1)} - x^{(k)})$$

- perform iteration at each step

Numerical Scheme 2

hybrid scheme using Crank-Nicolson and Adam-Bashforth

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- nonlinear term expanded explicitly (Adam-Bashforth) to obtain

$$AM^n = BM^{n+1} + \eta^{n+\frac{1}{2}} + \rho \min\{\varphi^{n+\frac{1}{2}} - \frac{1}{2}(3M^{n+1} - M^{n+2}), 0\}$$

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- no iteration required

Numerical Result 1

ε_1		Stanton's Scheme		CN with Newton Iteration	
N_t	N_y	ε_1	ratio	ε_1	ratio
50	50	127.513522		2.413700	
100	100	64.293427	2.0	0.588450	4.1
200	200	32.275461	2.0	0.146250	4.0
400	400	16.169679	2.0	0.036505	4.0
800	800	8.092824	2.0	0.009120	4.0
1600	1600	4.048412	2.0	0.002277	4.0
3200	3200	2.024710	2.0	0.000566	4.0
6400	6400	1.012484	2.0	0.000139	4.1
12800	12800	0.506278	2.0	0.000032	4.4

Stanton vs Crank-Nicolson with Newton Iteration

Numerical Result 2

ε_1		Stanton's Scheme		Hybrid Scheme	
N_t	N_y	ε_1	ratio	ε_1	ratio
50	50	57.449132		0.670530	
100	100	27.242231	2.1	0.158820	4.2
200	200	13.468115	2.0	0.038450	4.1
400	400	6.705991	2.0	0.009536	4.0
800	800	3.346585	2.0	0.002374	4.0
1600	1600	1.671741	2.0	0.000591	4.0
3200	3200	0.835489	2.0	0.000147	4.0
6400	6400	0.417650	2.0	0.000036	4.1
12800	12800	0.208801	2.0	0.000008	4.3

Stanton vs hybrid scheme

Numerical Result

	order	iteration
Stanton's scheme	1	no
CN with Newton Iteration	2	yes
Hybrid scheme	2	no

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	order	iteration
Stanton's scheme	1	no
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Conclusion: Hybrid scheme is most suitable to solve this model

Applications

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- model other derivatives: convertible bonds, callable warrants, ...

Recap

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Thank you