Some branching process models and the experience of SARS in Singapore

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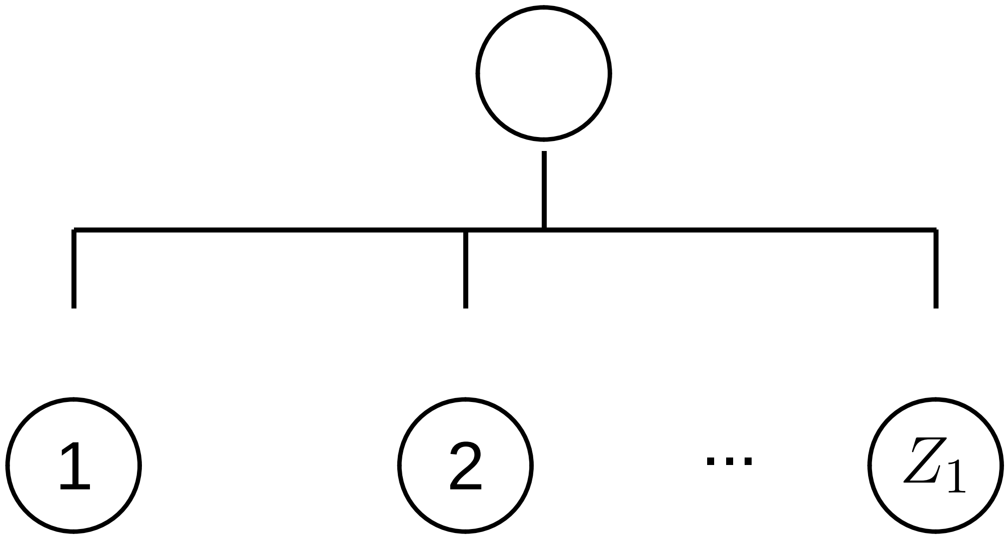
Branching process

Galton-Watson branching process $\{Z_0, Z_1, ...\}$

Z_n is population size of nth generation

$$Z_n = U_1 + U_2 + ... + U_{Z_{n-1}}$$

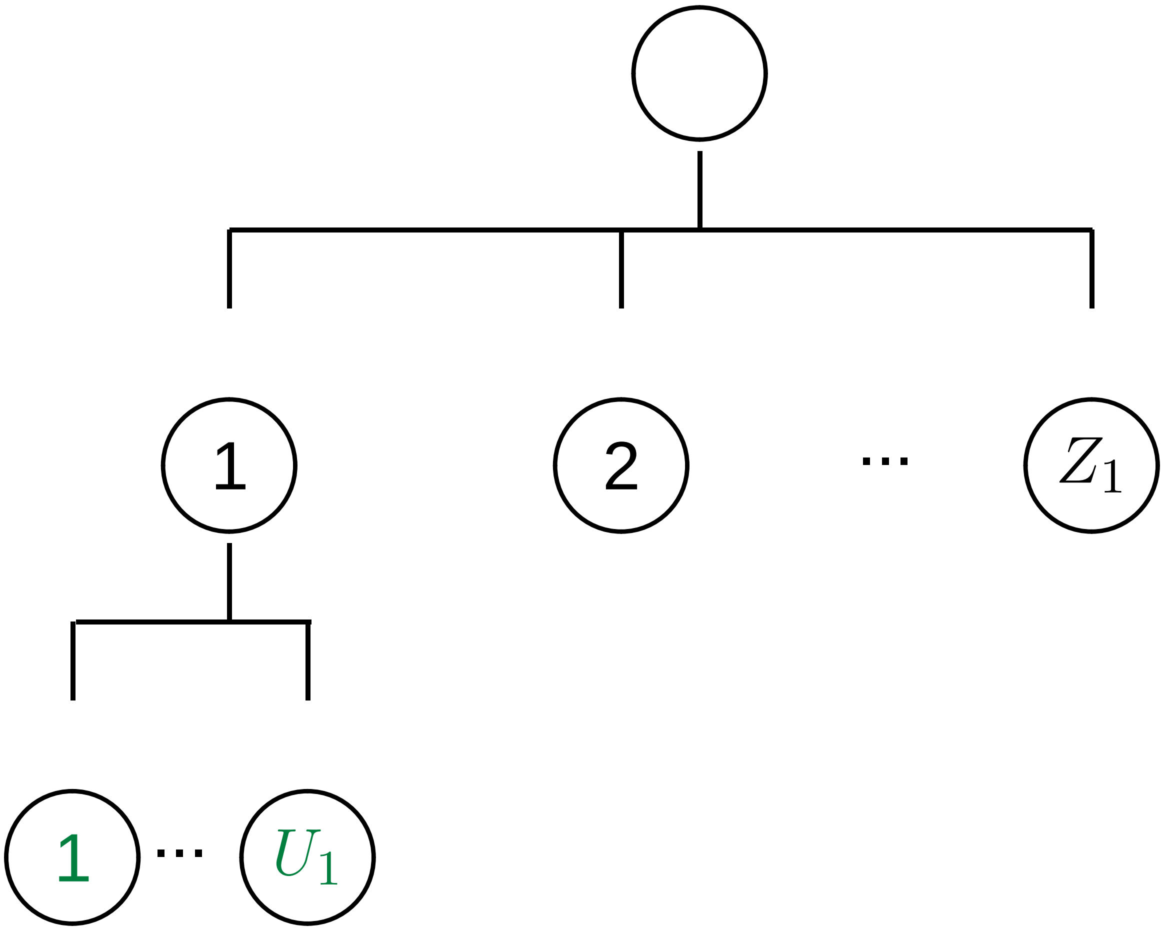
U_i= no. of offspring from ith person in previous generation.

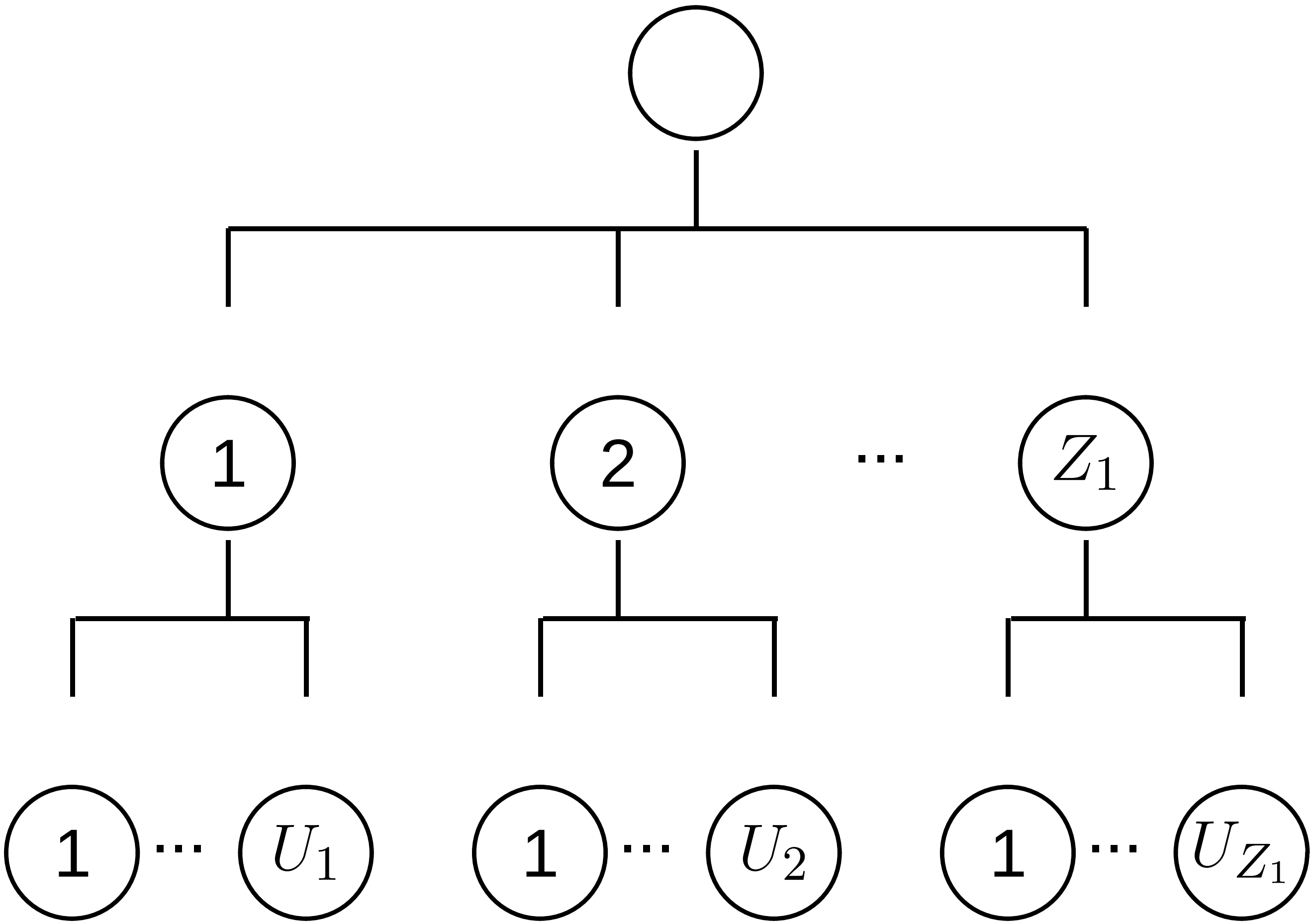


Z_0=1 parent

Z_1 offspring

in 1st gen


1
2
...
Z_1
1
...
U_1



$Z_2 = U_1 + U_2 + \ldots + U_{Z_1}$ offspring in 2nd gen

The spread of SARS

'Parents' are those already infected.

For an individual parent,
no. of 'offspring'= no. of people infected by the parent.

Z_0 is the initial number of infectives.

Offspring distribution

Assume $\{U_i\}$ i.i.d. some common distribution.

- $E(U_1) = m$
- $var(U_1) = \sigma^2$
- probability generating function, $f(s) = E\left(s^{U_1}\right)$

Some classical results

- for $Z_0 = 1$, probability of extinction q, where $f(q) = q$

$$q^k = P(Z_n = 0 \text{ for some } n \geq 1 | Z_0 = k)$$

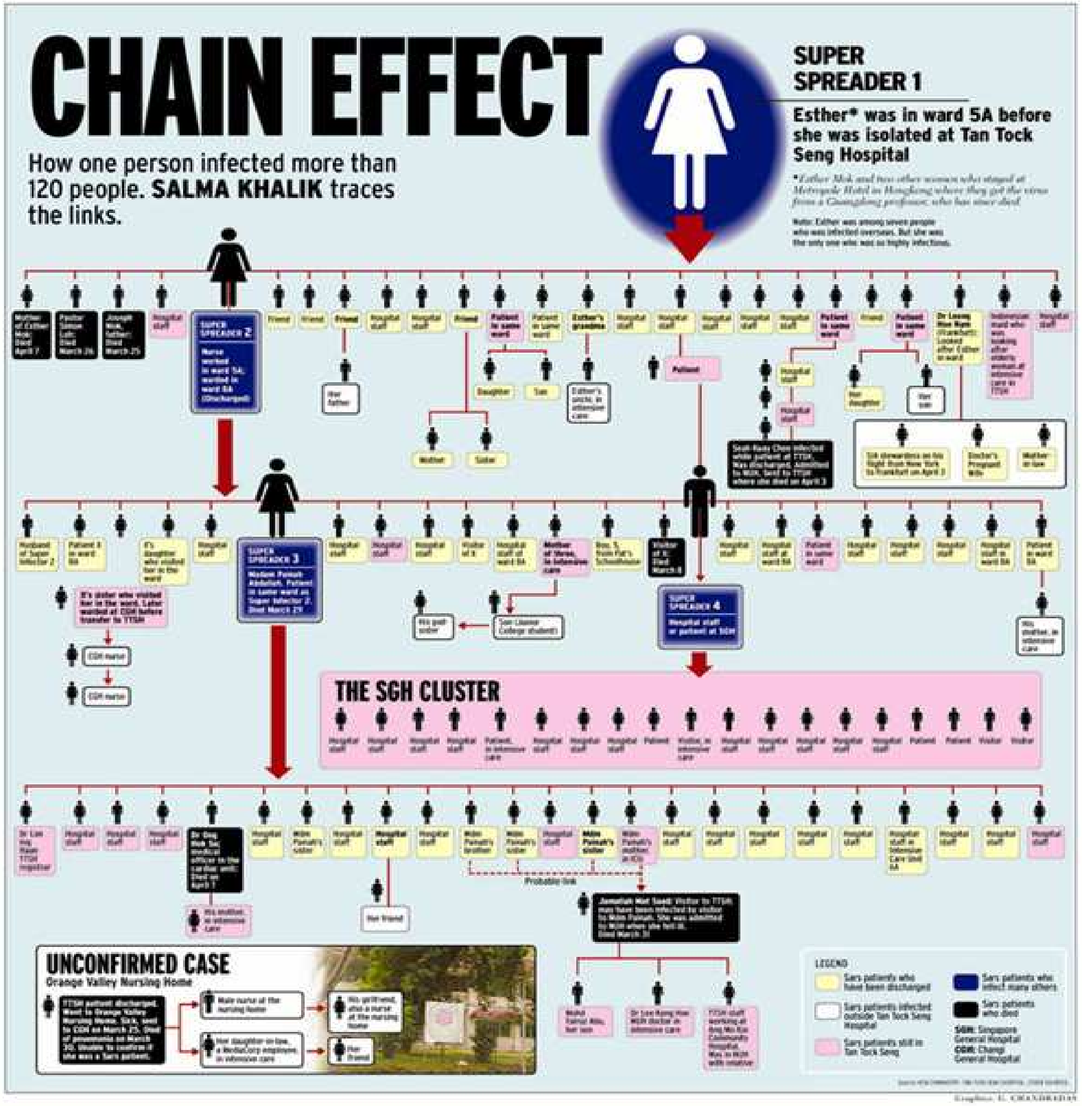
- $q < 1 \Leftrightarrow m > 1$

  $m \leq 1 \Rightarrow$ extinction with probability 1
  $m > 1 \Rightarrow$ explosion with positive probability

Total no. of offspring, $T = Z_1 + ... + Z_n + ...$

- $E(Z_n) = m^n Z_0$
- If extinction is certain,
  $E(T) = \frac{m}{1-m} Z_0, \quad var(T) = \frac{\sigma^2}{(1-m)^3} Z_0$

Example data



Straits Times

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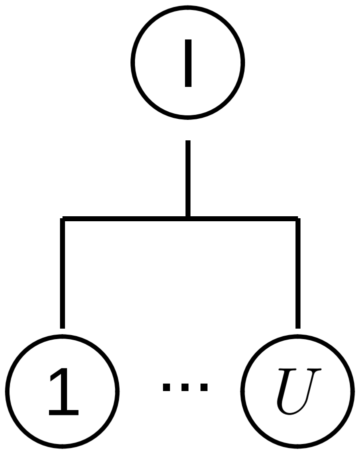
$$\hat{m} = 0.975$$

$$\hat{T} = 275$$

2-type model

Use 2-type branching process model to describe the spread of SARS in Singapore

I. an infected person is roaming freely



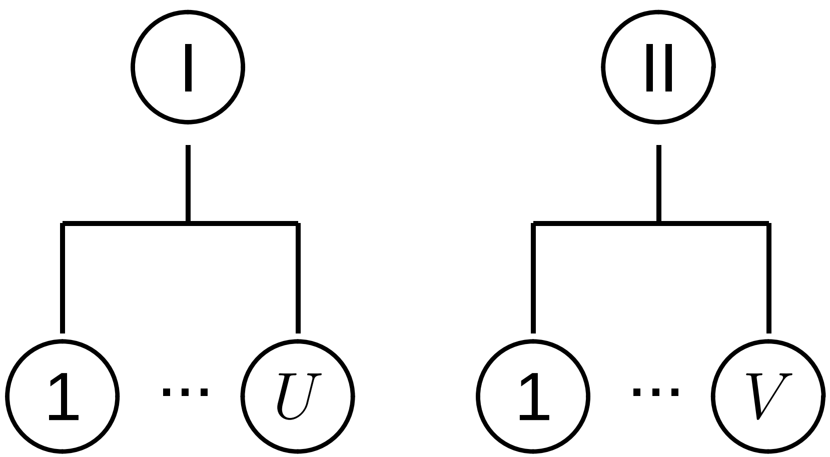
Type I offspring

2-type model

Use 2-type branching process model to describe the spread of SARS in Singapore

I. an infected person is roaming freely

II. an infected person is under medical control, eg.in hospital or under quarantine



2 offspring rv, U and V

Type I offspring

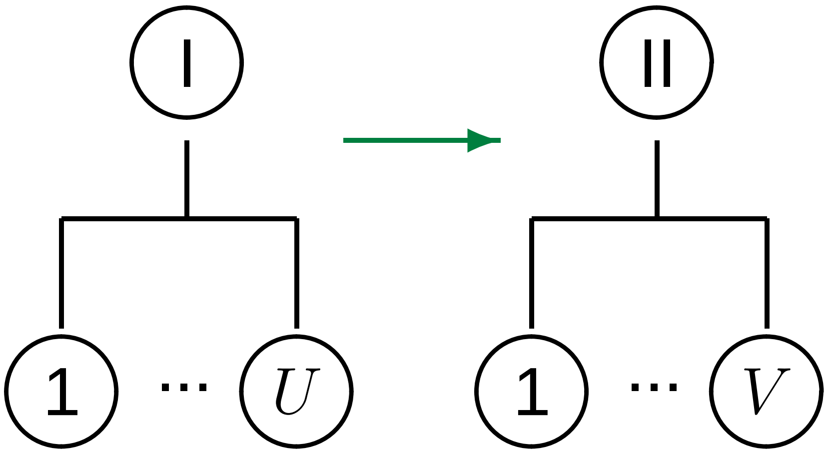
Type II offspring

2-type model

Use 2-type branching process model to describe the spread of SARS in Singapore

I. an infected person is roaming freely

II. an infected person is under medical control, eg.in hospital or under quarantine



Type I offspring Type II offspring

For nth gen,

X_n = no. of type I SARS cases

Y_n = no. of type II SARS cases

Assume $Y_0 = 0$.

Branching process:

$$\begin{aligned} X_n &= U_1 + ... + U_{X_{n-1}} \\ Y_n &= V_1 + ... + V_{X_{n-1}} + V_{X_{n-1}+1} + ... + V_{X_{n-1}+Y_{n-1}} \end{aligned}$$

Let

- $E(U_1)$ = $m_1 \quad var(U_1) = \sigma_1^2$
- $E(V_1)$ = $m_2 \quad var(V_1) = \sigma_2^2$

Can show that

- $E(X_n) = m_1^n X_0, \quad E(Y_n) = m_2 \frac{m_1^n - m_2^n}{m_1 - m_2} X_0$

Excluding the index cases, let

T_X be total no. of type I cases

$$T_X = X_1 + X_2 + ...$$

T_Y be total no. of type II cases

$$T_Y = Y_1 + Y_2 + ...$$

Then

- $E(T_X) = \frac{m_1 X_0}{1-m_1}$
- $E(T_Y) = \frac{m_2 X_0}{(1-m_1)(1-m_2)}$

For total no. of SARS cases,

- $E(T_X + T_Y + X_0) = \frac{X_0}{(1-m_1)(1-m_2)}$

  $var(T_X + T_Y) = \frac{\sigma_1^2 X_0}{(1-m_1)^3(1-m_2)^2} + \frac{\sigma_2^2 X_0}{(1-m_1)(1-m_2)^3}$

Effective anti-SARS measures $\Rightarrow \quad m_1 \approx 0, \sigma_1^2 \approx 0$

$\Rightarrow E(T_X + T_Y + X_0) \approx \frac{X_0}{(1-m_2)}$

$var(T_X + T_Y) \approx \frac{\sigma_2^2 X_0}{(1-m_2)^3}$

Similar effects for m_2, σ_2^2.

3-type model

Break type II into two, giving a 3-type model

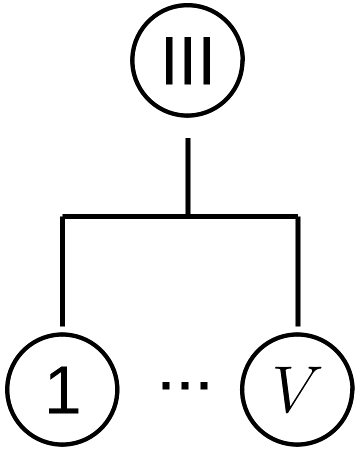
I. community at large

II. home quarantine

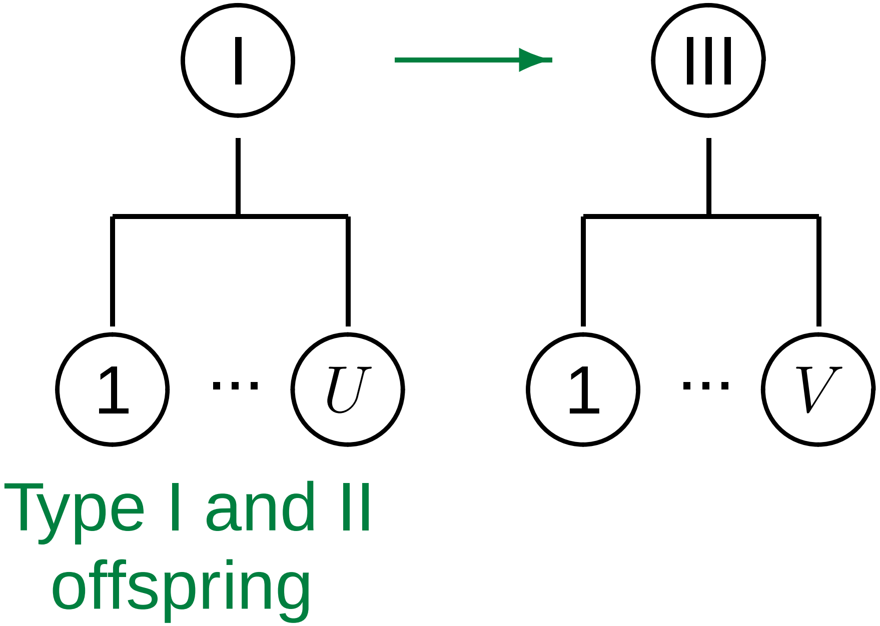
III. hospital

Initial infective period, IIP:

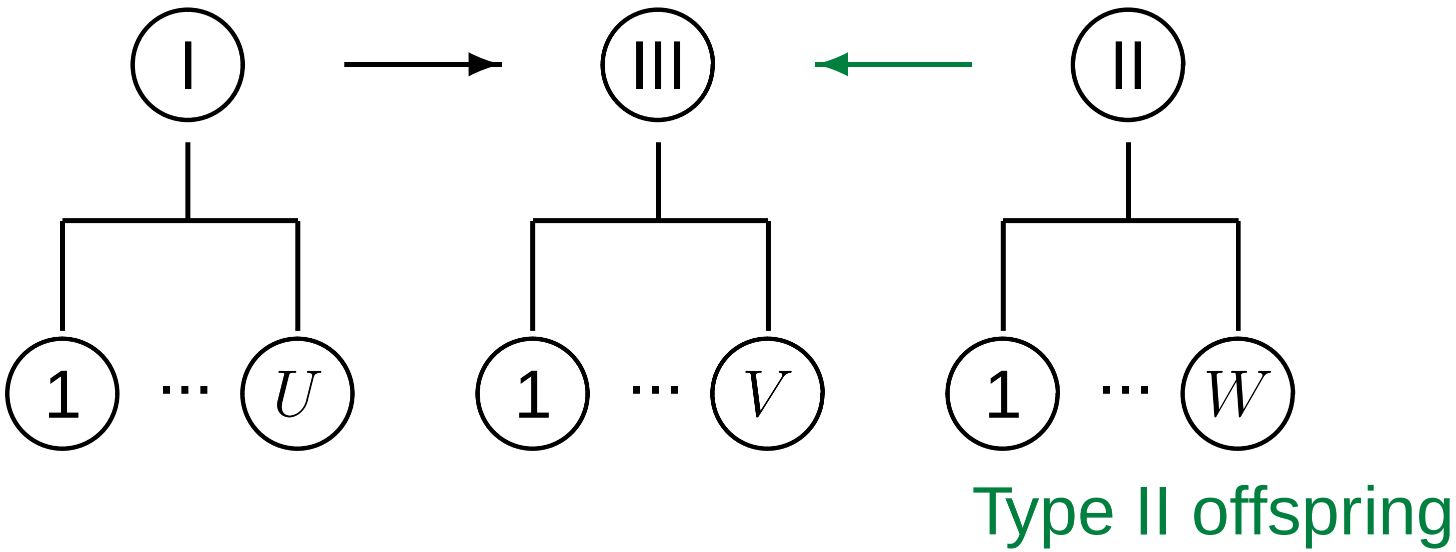
period between becoming infective and being identified as SARS case



Type III offspring



$U = U_I + U_{II}$. Short IIP $\Rightarrow U_I = 0$, small U_{II}


I
III
II
1
$\cdots$
U
1
$\cdots$
V
1
$\cdots$
W
Type II offspring

For this model (with $Z_0 = 1$), can show that

$$
\begin{aligned}
E(T) &= \frac{1}{(1-m_I)(1-m_3)}\left\{1+\frac{m_{II}}{1-m_2}\right\} \\
var(T) &= \frac{var\left\{\frac{1-m_2+m_{II}}{(1-m_1)(1-m_2)}U_I+\frac{1}{1-m_2}U_{II}\right\}}{(1-m_I)(1-m_3)^2}+ \\
&\quad \frac{\left\{\frac{m_{II}}{(1-m_2)^3}\sigma_2^2+\left(1+\frac{m_{II}}{(1-m_2)(1-m_3)}\right)\sigma_3^2\right\}}{(1-m_I)(1-m_3)^2}
\end{aligned}
$$

m_I, m_{II}, m_2, m_3 offspring distribution means

σ_2^2, σ_3 offspring distribution variances.

Some remarks

- Some branching models to describe SARS spread
- Easy interpretation
- Effect of offspring distribution mean
- Effect of anti-SARS measures