Amenable Groups and Randomness

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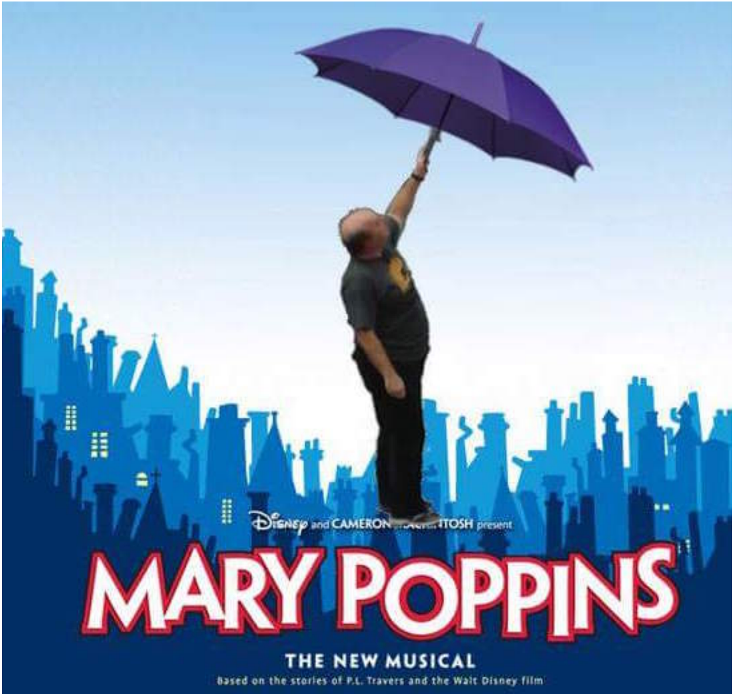
MARY POPPINS
THE NEW MUSICAL
Based on the stories of P.L. Travers and the Walt Disney film

The Setting

- A – a countable alphabet.
- G – a countable computable group.
- A^G – the set of functions from G to A.

Place the product topology on A^G.

Question:
Can we transfer the theory of algorithmic randomness, particularly prefix-free complexity to A^G?

The uninteresting part

- Fix an isomorphism between G and $\mathbb{N}$.
- This gives a homeomorphism between A^G and $A^{\mathbb{N}}$.
- Transfer the theory via this isomorphism.
- Use this to define 1-randomness.

Uninteresting because this ignores the group structure of G. But where exactly does the groups structure come in?

Group Actions as Dynamical Systems

Here is a typical example of a dynamical system.

- Let X be a set and let $T : X \to X$ be an automorphism of X.
- We can regard this system as $\mathbb{Z}$ acting on X with the action defined by
$$a(n, x) = T^n(x).$$

In general any group action can be thought of as a dynamical system.

We will use $\cdot$ to denote the left-shift action of G on A^G. If $g \in G$ and $x \in A^G$, then $g \cdot x$ is defined by

$$(g \cdot x)(h) = x(g^{-1}h).$$

Approximation Sequence

Call $(F_i)_{i\in\mathbb{N}}$ an *approximation sequence* to G if

- Each F_i is a finite subset of G.
- Each element of G is contained in all but finitely many F_i.

Example

- An approximation sequence to $\mathbb{Z}$ is given by $F_i = [-i, \ldots, i]$.
- If G is finitely generated, we can take

$$F_n = \{g \in G \colon g = s_1 \ldots s_n \text{ where each } s_i \text{ is a generator or its inverse or } 1_G\}.$$

If $x \in A^G$ and $(F_i)_{i\in\mathbb{N}}$ is an approximation sequence to G, then we will think of the "initial segments" of x as being $x\restriction_{F_0}, x\restriction_{F_1}, \ldots$.

Initial segment complexity for elements of A^G

- If $x \restriction_{F_n}$ is an initial segment of x, then what is its prefix free complexity?
- If $\sigma \in A^{F_n}$, then we can regard σ as a finite subset of $G \times A$. $K(\sigma)$ can be defined to be the complexity of this finite subset.
- Note that given a description of σ we can uniformly compute the domain of σ as well as the values of $\sigma(g)$ for each g in the domain.

Dimension

We will use initial segment complexity to look at analogues of effective Hausdorff dimension and effective packing dimension for elements of A^G.

Definition

- $\mathrm{ldim}(x) := \liminf_{n\to\infty} \frac{K(x \upharpoonright_{F_n})}{|F_n|}$
- $\mathrm{udim}(x) := \limsup_{n\to\infty} \frac{K(x \upharpoonright_{F_n})}{|F_n|}$

Note that this definition is dependent on the approximation sequence picked, but we will see that in certain cases we can remove this dependence.

Invariance of dimension under group action

- Given $x \upharpoonright_{F_n}$ and g, how can we determine $(g \cdot x) \upharpoonright_{F_n}$?
- Because $(g.x)(h) = x(g^{-1}h)$, if $g^{-1}h \in F_n$ (equivalently $h \in gF_n$), then we can determine $(g.x)(h)$.

Hence

$$K((g \cdot x) \upharpoonright_{F_n}) \leq K(x \upharpoonright_{F_n}) + K(g) + \lceil \log(A) \rceil |F_n \setminus gF_n|$$

$$\liminf_{n\to\infty} \frac{K((g \cdot x) \upharpoonright_{F_n})}{|F_n|} \leq \liminf_{n\to\infty} \frac{K(x \upharpoonright_{F_n})}{|F_n|} + \lceil \log(A) \rceil \liminf_{n\to\infty} \frac{|F_n \setminus gF_n|}{|F_n|}$$

If this last term tends to 0 then $\mathrm{ldim}(g \cdot x) \leq \mathrm{ldim}(x)$.

Følner sequences

Definition

An approximation sequence $(F_i)_{i\in\mathbb{N}}$ to G is called a *Følner sequence* if for all $g \in G$,

$$\lim_n \frac{|gF_n \Delta F_n|}{|F_n|} = 0.$$

If we define dimension using Følner sequences, then for all $g \in G$ and all $x \in A^G$

- $\mathrm{ldim}(g \cdot x) = \mathrm{ldim}(x)$
- $\mathrm{udim}(g \cdot x) = \mathrm{udim}(x)$

When does a group have a Følner sequence?

Amenable Groups

Theorem

A countable group G has a Følner sequence if and only if it is amenable.

Definition

A group G is *amenable* if there exists a finitely additive measure μ on the powerset of G such that $\mu(G) = 1$ and for all $g \in G$ and $E \subseteq G$, $\mu(gE) = \mu(E)$.

Amenable groups and paradoxical groups

Definition

A group G is *paradoxical* if there exists $A_1, \ldots, A_m, B_1, \ldots B_m$ disjoint subsets of G and $g_1, \ldots g_m, h_1, \ldots h_m$ such that

- $G = \bigcup_{i=1}^{m} g_i A_i$, and
- $G = \bigcup_{i=1}^{n} h_i B_i$.

Theorem (Tarski)

A group G is paradoxical if and only if it is not amenable.

Examples of Amenable groups

- All abelian groups are amenable.
- All finitely generated groups of polynomial growth are amenable.
- Subgroups of amenable groups are amenable.
- If N is a normal subgroup of G and each of N, G/N are amenable then so is G

Topological Entropy

- Let X be a closed subset of A^G that is also closed under the left shift action i.e. $g \in G$ and $x \in X$ implies $g \cdot x \in X$.
- As the mappings $x \mapsto g \cdot x$ are continuous, we can consider X and the left shift as a topological dynamical system.
- Let $X \restriction_{F_n} = \{x \restriction_{F_n} : x \in X\}$

Definition

The *topological entropy* of X is denoted $\text{ent}_T(X)$ and defined to be

$$\lim_{n \to \infty} \frac{\log |X \restriction_{F_n}|}{|F_n|}.$$

Topological Entropy and Dimension

Lemma

If $x \in X$ and $X \upharpoonright_{F_n}$ is uniformly recursive then

$$\mathrm{udim}(x) \leq \mathrm{ent}_T(X).$$

$$K(x \upharpoonright_{F_n}) \leq K(n) + \log |X \upharpoonright_{F_n}|$$

and so

$$\lim_{n\to\infty} \frac{K(x \upharpoonright_{F_n})}{|F_n|} \leq \lim_{n\to\infty} \frac{K(n)}{|F_n|} + \lim_{n\to\infty} \frac{|X \upharpoonright_{F_n}|}{|F_n|}$$

Topological Entropy and Dimension

Theorem

Let G be a computable amenable group and let X be a computable subshift of A^G. If for all $x \in X$, $\mathrm{ldim}(x) \leq s$, then $\mathrm{ent}_T(X) \leq s$.

This implies that

$$\mathrm{ent}_T(X) = \inf_{Z \in 2^{\mathbb{N}}} \sup\{\mathrm{ldim}^Z(x) \colon x \in X\}.$$

Hence $\mathrm{ent}_T(X)$ is equal to the Hausdorff dimension of X (by Lutz, Mayordomo and Hitchcock).

- Case G is $\mathbb{N}$ is due to Furstenberg.
- Case G is $\mathbb{N}^d$ or $\mathbb{Z}^d$ is due to Simpson (2014).
- Case G is an amenable group is new. (Dimension must be defined using an appropriate Følner sequence.)

Ornstein and Weiss's Work

- Used to prove for a subclass of amenable groups.
- After Lindenstrauss adapted their techniques to give a new proof for all groups.
- Need to restrict to bi-invariant Følner sequences that are tempered

$$\left|\bigcup_{i\leq n} F_i^{-1}F_{n+1}\right| \leq b\left|F_{n+1}\right|$$

Adding measures

Our next objective is to add a measure to space A^G and consider it as a measure-preserving dynamical system. We will consider two main theorems:

- Birkhoff ergodic theorem.
- Shannon-McMillian-Brieman theorem.

If μ is a probability measure on A^G then we say that $x \in A^G$ is μ 1-random if it is not contained in any Martin-Löf test.

Ergodic group actions

Let $(X, \mathcal{X})$ be a measurable space and μ a probability measure on this space. A group action $a : G \times X \to X$ is *measure preserving* if

- For each $g \in G$, $x \mapsto a(g, x)$ is measurable.
- For each $g \in G$ and $E \in \mathcal{X}$, $\mu(a(g, E)) = \mu E$.

A measure preserving group action is *ergodic* if for all $E \in \mathcal{X}$ and all $g \in G$ $a(g, E) \subseteq E$ implies that $\mu E = 0$ or $\mu E = 1$.

Question

If $a : G \times X$ is an ergodic action for $(X, \mathcal{X})$, and $E \in \mathcal{X}$ is it true that for μ almost all $x \in X$,

$$\lim_{n \to \infty} \frac{|\{g \in F_n : a(g, x) \in E\}|}{|F_n|} = \mu E?$$

i.e. Does Birkhoff's ergodic theorem hold.

Lindenstrass Theorem

- (Lindenstrass 1999) Birkhoff's ergodic theorem holds if G is an amenable group.
- Provided we use tempered bi-invariant Følner sequences.
- (Moriakov 2017) Has effectivised this proof and shown that it holds for all 1-random points.
- Lindenstrass also generalised the Shannon-McMillian-Brieman Theorem to amenable groups.
- Before looking at the Shannon-McMillian-Brieman Theorem, lets review some concepts of entropy.

Entropy

Definition

Let P be a discrete probability measure on the countable set $\{c_1, c_2, \ldots\}$. The *Shannon entropy* of P is defined by

$$H(P) = \sum_{i=1}^{\infty} -P(c_i) \log P(c_i).$$

"The expected length of an optimal prefix-free code."

Kolmogorov-Sinai entropy

Let's return to the space A^G and the left shift action of G on A^G. Let μ be a measure on A^G such that left shift action is ergodic.

- By ergodicity, there are h_u and h_l such that for μ almost all $x \in A^G$,
$$\text{ldim}(x) = h_l \quad \text{and} \quad \text{udim}(x) = h_u.$$
- If $\sigma \in A^{F_n}$, denote by $[\![\sigma]\!]$ the set $\{x \in A^G : x \restriction_{F_n} = \sigma\}$.
- Define $H_n = \sum_{\sigma \in A^{F_n}} \mu[\![\sigma]\!] . \log \mu[\![\sigma]\!]$.
- The Kolmogorov-Sinai entropy of μ is defined to be
$$h(\mu) = \lim_{n\to\infty} \frac{H_n}{n}.$$
- Note that $h_l \leq h(\mu) \leq h_u$.

Shannon-McMillian-Brieman

Theorem (Lindenstrass (1999))

Let μ be an ergodic measure for the left-shift action on A^G. Let h be the Kolmogorov-Sinai entropy of $(A^G, \cdot, \mu)$. Let (F_n) be a tempered Følner sequence for G. Then for μ-almost all $x \in A^G$.

$$\lim_n \frac{-\log \mu [\![x \restriction_{F_n}]\!]}{|F_n|} = h.$$

This is a simplified version of Lindenstrass's result.

A restatement

Theorem (Shannon-McMillian-Brieman)

Let G be a computable group and let μ be a computable ergodic measure for the left-shift action on A^G. If dimension is defined using a tempered Følner sequence, then

- *For almost all x, $\text{ldim}(x) = \text{udim}(x)$,*
- *This value is independent of the tempered Følner sequence used.*

If x is 1-random then

$$\text{ldim}(x) = \liminf_{n\to\infty} \frac{\mu[\![x \upharpoonright_{F_n}]\!]}{|F_n|} \quad \text{and} \quad \text{udim}(x) = \limsup_{n\to\infty} \frac{\mu[\![x \upharpoonright_{F_n}]\!]}{|F_n|}.$$

Effective version

Theorem (Shannon-McMillian-Brieman effective version)

Let G be a computable group and let μ be a computable ergodic measure for the left-shift action on A^G. If dimension is defined using a tempered Følner sequence, then

- *If x is μ 1-random*, $\mathrm{ldim}(x) = \mathrm{udim}(x)$,
- *This value is independent of the tempered Følner sequence used.*

- The case that G is $\mathbb{N}$ was proved by V'yugin (1998), Hoyrup (2013).
- The new proof is simpler

Entropy as an Isomorphism Invariant

- Let A be an alphabet of size n. Call the system (A^G, μ) where μ is the product of uniform measures on A, the *full n shift over G*.
- Kolmogorov-Sinai entropy originates in the proof that the full 2 shift over $\mathbb{Z}$ is not isomorphic to the full 3 shift over $\mathbb{Z}$.
- In fact there is no factor map from the full 2 shift over $\mathbb{Z}$ to the full 3 shift over $\mathbb{Z}$.
- Reason: factor maps must be decreasing in entropy.

Theorem (Ornstein-Weiss)

If G is infinite and amenable then the Kolmogorov-Sinai entropy classifies Bernoulli shifts over G up to isomorphism

Entropy for Non-amenable groups

- Bowen - Entropy for Free groups and then generalised to Sofic groups.
- Seward - Rokhlin entropy.

Future directions analyse these entropies from the perspective of algorithmic randomness.