# Limitwise monotonic functions and classifications of structures

Alexander Melnikov

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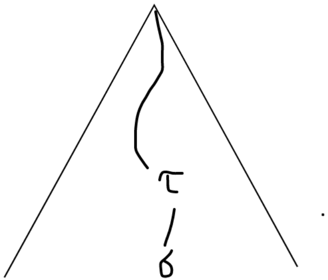
Happy Birthday, the Father-Node of Logic in New Zealand!



# Introduction

**Idea:** Approach classification problems in computable algebra from the perspective of pure recursion theory (neither via definability nor via algebra).

**The main tools:** Limitwise monotonic approximations, priority arguments, and various tricks separating algebra from combinatorics.

**Definition**

A function $f : \omega \to \omega \cup \{\infty\}$ is **limitwise monotonic** if there exists a (total) recursive $g : \omega \times \omega \to \omega$ such that

$$f(x) = \sup_y g(x, y),$$

for all $x$.

If we forbid $\infty$ then it gives a special subclass of $\Delta^0_2$ functions.

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Why do we care?

Limitwise monotonic functions show up in computable:

1. equivalence structures;
2. linear orders ($\eta$-presentations, shuffle sums, initial segments etc.);
3. abelian groups;
4. models of $\aleph_1$-categorical structures
5. many other things that “grow”.

See a survey of Downey, Kach, Turetsky; see also my paper with Kalimullin and Khoussainov.

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This class is useless and trivial.

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## Part 1: The problem of Khisamiev-Ash-Knight-Oates is hard

**Countable abelian p-groups** can be viewed as layers of equivalence structures (multisets) living on a tree.

1. A group G is **reduced** if the tree is well-founded.
2. Iterate the Ulm derivative $G \to G'$ to form (essentially) equivalence structures $G/G'$.
3. We have the sequence $G_\alpha = G^{(\alpha)}/G^{(\alpha+1)}$ that terminates at $u(G)$, the **Ulm type** of the group.
4. The sequence of **Ulm factors** $G_\alpha = G^{(\alpha)}/G^{(\alpha+1)}$ fully describes the group (this fact is non-trivial).

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**Theorem (Khisamiev; Ash-Knight-Oates)**

For a reduced abelian $p$-group $G$ of finite Ulm type $m$, TFAE:

1. $G$ has a computable copy;
2. $G_0, G_1, \ldots, G_m$ have $\Delta^0_1, \Delta^0_3, \ldots, \Delta^0_{2m+1}$-copies, respectively.

Recall each $G_i$ is (essentially) a limitwise monotonic function.

**Problem**

What happens when the Ulm type of G is $\omega$?

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(Essentially:) The case of Ulm type $\omega$ is hard.

We proved: Given a computable $G$, calculating the index of its $n^{th}$ $0^{(2n)}$-monotonic function requires $0^{(2n+3)}$.

If such a sequence is played by God, we must analyse **an iterated** $0'''$ **in its full generality** to either build a copy of $G$ or construct a counter-example.

Our proof is the first known example of an iterated $0'''$.

Have you noticed? **This was all about equivalence structures.**

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Friedberg enumerations of structures

Suppose $\mathcal{K}$ is a class of (computable) algebraic structures.

**Definition**

A computable enumeration of structures in $\mathcal{K}$ is *Friedberg* if it is 1-1 up to isomorphism.

Very few classes admit a Friedberg enumeration.

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Is there a Friedberg enumeration of the class of computable equivalence structures?

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**$\Pi^0_4$-complete problem.**

Compare this to c.e. sets where $W_e = W_j$ is $\Pi^0_2$.

There were earlier attempts by Goncharov and Knight, and by Miller R., Lange, and Steiner.

**Theorem (Downey, M., Ng)**

There **exists** a Friedberg enumeration of computable eq. structures.

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We know that **reduced** abelian $p$-groups of a fixed finite Ulm type (observed by Goncharov and Knight).

Remarkably, if we drop "reduced" than such an enumeration exists:

**Theorem (with Ng)**

1. For each $m < \omega$, there exists a Friedberg enumeration of all computable abelian $p$-groups of Ulm type $\leq m$.
2. There exists a Friedberg enumeration of all computable abelian $p$-groups of finite Ulm type.

This are the first non-trivial and natural algebraic classes with a Friedberg enumeration. The proof is rather technical and **uses a Friedberg enumeration of computable eq. structures.**

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# A problem of Mal′cev

A structure is **computably categorical** if it has a unique computable copy, up to computable isomorphism.

**Problem (Maltsev, in the 1960-s)**

Describe computably categorical abelian groups.

We have nice satisfactory classifications for:

- $p$-groups (Smith, indep. Goncharov)
- torsion-free (Nurtazin)
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It is not hard to show:

**Fact**

There exist c.c. but not relatively c.c. torsion abelian groups.

Thus, there should not be any **algebraic description** of c.c. torsion groups.

We decided to look at the **index set**

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Using known techniques it can be pushed down to $\Pi^0_5$.

**Theorem (M. and Ng)**

The index set

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is $\Pi^0_4$-complete.

- $\Pi^0_4$-harness of the index set is the easy(er) part.
- The proof relies on several subtle **algebraic reductions.**
- We use that a certain diagonalization attempt on **equivalence structures** must fail.
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From computable groups to Polish groups

## Definition

A computable Polish group is a computable Polish (metric) space equipped with computable group operations.

We consider Polish groups up to topological isomorphism.

Suppose $K$ is a natural class of Polish groups (e.g., connected compact groups).

**Can we classify members of $K$?**

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## Theorem (M.)

1. The index sets of **profinite** and of **connected compact** Polish groups are arithmetical.
2. The topological isomorphism problems for **profinite abelian groups** and for **connected compact abelian** groups are $\Sigma^1_1$-complete.

We can list all partial computable Polish groups: $G_0, G_1, G_2, \ldots$

- $\{i : G_i$ is a connected topological group$\}$ is Arithmetical.
- $\{(i, j) : G_i \cong G_j$ and $G_i, G_j$ are connected$\}$ is $\Sigma^1_1$-complete.

The result is uniform. It follows connected and profinite (abelian) groups are **unclassifiable**.

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- Computable Polish space theory.
- Computable (discrete) abelian group theory (e.g., the old result of Dobrica on bases, **the result of Downey and Montalban** mentioned by Julia, etc.).
- Abstract harmonic analysis.

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A profinite group is *recursive* if it is the limit of a computable surjective inverse system of finite groups.

($\widehat{G}$ stands for the Pontryagin dual of $G$.)

### Theorem (M.)

Let $G$ be a countable torsion abelian group. Then

- $G$ is computable iff $\widehat{G}$ is a recursive profinite group;
- $G$ is computably categorical iff $\widehat{G}$ is computably categorical (as a recursive profinite group).

### Corollary (follows from M. and Ng)

The index set of c.c. recursive profinite groups is $\Pi^0_4$-complete.

eq. structures $\to$ (discrete) abelian groups $\to$ Polish groups.

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**Definition (Smith and La Roche, after Nerode)**

A profinite group is *recursive* if it is the limit of a computable surjective inverse system of finite groups.

($\widehat{G}$ stands for the Pontryagin dual of $G$.)

**Theorem (M.)**

Let $G$ be a countable torsion abelian group. Then

- $G$ is computable iff $\widehat{G}$ is a recursive profinite group;
- $G$ is computably categorical iff $\widehat{G}$ is computably categorical (as a recursive profinite group).

**Corollary (follows from M. and Ng)**

The index set of c.c. recursive profinite groups is $\Pi^0_4$-complete.

eq. structures $\rightarrow$ (discrete) abelian groups $\rightarrow$ Polish groups.

# Spasibo

Thanks!