Effective fractal dimension theory: exploring the extreme cases (II)

Elvira Mayordomo

Universidad de Zaragoza, Iowa State University

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Left out Monday

- Open problems:
  - Partially complete problems
  - BPP sources
- References

Today

0. Introduction of effective dimension
1. Resource-bounded Hausdorff dimension for Complexity Classes
2. **Compression and dimension for low resource bounds. Very effective construction of a normal sequence**
3. Looking back at fractal geometry, other metric spaces

Compression characterizations of effective measure

- **Constructive Hausdorff dimension can be entirely defined using Kolmogorov complexity**

Theorem

For every $A \subseteq \{0,1\}^\infty$, $\mathrm{cdim}(A) = \inf_{x \in A} \frac{\mathrm{K}(x{\upharpoonright}n)}{n}$.

For a finite string w, $\mathrm{K}(w)$ is the length of the shortest description from which w can be computably recoverered

Space-bounded Kolmogorov complexity characterizations

$$\mathrm{KS}^f(w) = \min\{|p| \mid U(p) = w \text{ in space } f(|w|)\}$$

Theorem

For every $A \subseteq \{0,1\}^\infty$,
$\dim_{\mathrm{pspace}}(A) = \inf_{q \textit{ polynomial}} \inf_{x \in A} \frac{\mathrm{KS}^q(x\restriction n)}{n}$.

What about time?

- Time-bounded Kolmogorov complexity is hard to work with due to invertibility issues
- p-dimension (predictability) can be characterized in terms of a class of polynomial time reversible compressors: *compressors that do not start from scratch*
- I will leave out the technical definition, but for instance a compressor C for which $C(w)$ and $C(wu)$ have a common prefix of length at least $|C(w)| - O(\log(|w|))$ does not start from scratch

Theorem

p-dimension is exactly the best compression rate achievable through polynomial-time compressors that do not start from scratch

Pushdown dimension

We consider BPD the set of pushdown machines that work with a bounded number of λ-transitions per input symbol

Theorem

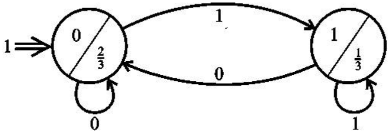
BPD-dimension is exactly the best compression rate achievable through BPD-compressors

Still open for general PD-computation

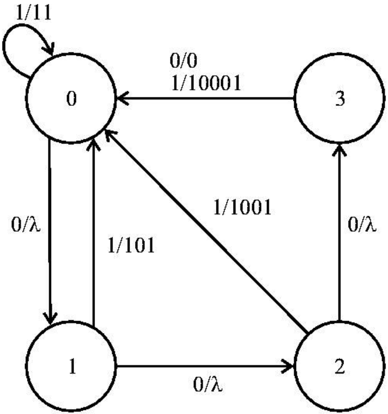
Finite state dimension

We considered the case of Finite-State computation

$\dim_{\mathrm{FS}}(A) = \inf\{s \mid$ there is a Finite State s-gale that succeeds on $A\}$



Finite state compression



The input can be recovered given the output and the final state

Finite state dimension characterization

Theorem

Finite-state dimension is exactly the best compression rate achievable through finite-state compressors, that is,

$$\dim_{\mathrm{FS}}(x) = \inf_{C\ \mathrm{FS}-comp} \liminf_{n} \frac{|C(x \upharpoonright n|}{n}$$

FS-compressors and Lempel-Ziv algorithm

- **Lempel-Ziv algorithm** subsumes FS-compressors:

$$\rho_{LZ}(x) = \liminf \liminf_{n} \frac{|LZ(x \upharpoonright n|}{n}$$

Theorem

For every $x \in \{0,1\}^\infty$

$$\rho_{LZ}(x) \leq \dim_{\mathrm{FS}}(x)$$

- Lempel-Ziv algorthm is universal for FS dimension/compression
- It is known that there are sequences for which $\rho_{LZ}(x) < \dim_{\mathrm{FS}}(x)$

Comparison among different levels

Theorem

PD-compression is incomparable with the Lempel-Ziv compression algorithm:

- *There are sequences for which PD-compression is better than LZ.*
- *There are sequences for which Lempel-Ziv compression is better than PD.*

- There is a FS-random sequence that is not PD-random (note: FS-random is equivalent to FS-dimension 1)
- There is a sequence such that $\dim_{\mathrm{PD}}(x) < \dim_{\mathrm{FS}}(x) < 1$

Open question

- Is PD-dimension 1 different from FS-dimension 1?
- Can PD-dimension be characterized in terms of compression?

Open questions

- There are characterizations of effective fractal dimension in terms of Kolmogorov complexity/compressibility at the most and least restricted computation levels
- They happen for completely different reasons
- Understanding what happens at intermediate levels can have useful applications for learning/compression
- Understanding what happens with FS-dimension/randomness may be useful for number theory

References for compression and dimension

- E. Mayordomo. A Kolmogorov complexity characterization of constructive Hausdorff dimension. Information Processing Letters, 84(1):1-3, 2002.
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References for compression and dimension

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- D. Doty and J. Nichols. Pushdown dimension. Theoretical Computer Science, 381(1-3):105-123, 2007.
- W. Merkle, J. Reimann. On selection functions that do not preserve normality. Theory of Computing Systems, 39(5):685697, 2006.

Very effective construction of an absolutely normal sequence

- At the lowest resource-bounded level, FS, dimension meets number theory
- Sequences with FS-dimension 1 are exactly Borel normal sequences
- FS-dimension is not closed under base change
- Can we use a constructive probabilistic method to construct an absolutely normal sequence?

Normal numbers

Borel, 1909:

- A real number α is **normal** in base b $(b \geq 2)$ if, for every finite sequence w of base-b digits,

$$\lim_n \frac{\mathrm{N}_\alpha(w, n)}{n} = \frac{1}{b^{|w|}}$$

  the asymptotic, empirical frequency of w in the base-b expansion of α is $b^{-|w|}$.

- α is **absolutely normal** if it is normal in every base $b \geq 2$.

Theorem (Cassels 1959, Schmidt 1960)

There exist a number that is normal in base 2 but not in base 3.

Examples of normal numbers

- Champernowne's sequence

$$0.123456\ldots$$

- Copeland-Erdös sequence

$$0.235711\ldots$$

Pretty far from natural

Absolutely Normal numbers

Theorem (Borel)

Almost every real number (i.e., every real number outside a set of Lebesgue measure 0) is absolutely normal.

- Computer analyses of the expansions of π, e, $\sqrt{2}$, $\ln 2$, and other irrational numbers that arise in common mathematical practice **suggest** that these numbers are absolutely normal.
- No such "natural" example of a real number has been proven to be normal in *any* base, let alone absolutely normal.
- The **conjectures that every algebraic irrational is absolutely normal** and that π is absolutely normal are especially well known open problems.
- In cryptographic applications constants such as π and $\sqrt{2}$ are used and expected to be "somehow random" (nothing up my sleeve numbers)

Computing absolutely normal numbers

- We are interested in the **complexity of** explicitly **computing** an absolutely normal real number
- Sierpinski and Lebesgue gave explicit constructions of absolutely normal numbers in 1917 (intricate limiting processes, no complexity or insight into the nature of the numbers constructed)
- Turing (1936, unpublished) gave a constructive proof that almost all real numbers are absolutely normal and then *derived* constructions of absolutely normal numbers from this proof.

Turing vision

- We believed that Schmidt (1960) was the first to construct absolutely normal numbers
- But the most surprising part was Turing's idea of effective measure and its application as an effective probabilistic method
- As analysed by Figueira, Becher and Picci (2007) Turing's unpublished note shows is that the set of non-normal numbers has computable measure 0
- The formalization of effective measure and randomness did not come until the sixties: Martin-Löf (paper 1966), Von-Mises, Solomonoff (1960), Kolmogorov, ...
- We now know that normality is a type of randomness

Computing absolutely normal numbers

- (Becher, Heiber, and Slaman 2013, simultaneous work from other authors) Algorithm that computes an absolutely normal number in polynomial time.
- Specifically, they compute the binary expansion of an absolutely normal number x, with the nth bit of x appearing after $O(n^2 \mathrm{polylog}(n))$ steps.
- Here we present a new algorithm that computes an absolutely normal in **nearly linear time**. Our algorithm computes the binary expansion of an absolutely normal number x, with the nth bit of x appearing after $O(n \mathrm{polylog}(n))$ steps.

Note: The term "nearly linear time" was introduced by Gurevich and Shelah (1989). While linear time computability is very model-dependent, nearly linear time is very robust.

Gales and martingales in base b

- $\Sigma_b = \{0, \ldots, b-1\}$ the base b alphabet
- Σ_b^* are finite sequences, Σ_b^∞ infinite sequences
- For $s \in [0, \infty)$, an s-gale is a function $\Sigma_b^* \to [0..\infty)$ such that for $w \in \Sigma_b^*$

$$d(w) = \frac{d(w0) + d(w1)}{b^s}$$

- A **martingale** is a function $d : \Sigma_b^* \to [0..\infty)$ with the fairness property, for every finite sequence w,

$$d(w) = \frac{\sum_{i \in \Sigma_b} d(wi)}{b}$$

- The success set of an s-gale d is

$$S^\infty[d] = \left\{ x \in \Sigma_b^\infty \,\middle|\, \limsup_n d(x \upharpoonright n) = \infty \right\}$$

- **Notice that if d is an s-gale then $d'(w) = b^{(1-s)|w|} d(w)$ is a martingale**

Finite-state randomness

Definition

x is **FS random** is no finite automata computable martingale succeeds on x

Notice that if $\dim_{\mathrm{FS}}(x) < 1$ then x is not FS-random

Normality and Finite-state randomness

- If x is the base b representation of a non-normal number, w is a finite string that is "unbalanced" in x, for instace i.o. w appears more often than it should, a finite automata can bet a bit more than its fair share and make infinite money ...
- Clearly FS random sequences are representations of base b normal numbers
- Even better **FS-random** $=$ **normal** –Schnorr and Stimm (1972)

Finite-State dimension

Schnorr and Stimm (1972) implicitly defined **finite-state martingales** and proved that every sequence $S \in \Sigma_b^\infty$ obeys this dichotomy:

1. If S is b-normal, then no finite-state base-b martingale succeeds on S. (In fact, every finite-state base-b martingale decays exponentially on S.)
2. If S is not b-normal, then some finite-state base-b martingale **succeeds exponentially** on S.

Using dimension terminology

1. If S is b-normal, then S is FS-random.
2. If S is not b-normal, then $\dim_{\mathrm{FS}}(S) < 1$.

Therefore **FS-dimension 1** $=$ **normal**

Remember ...

- **Objective** Compute a (provably) absolutely normal number $x \in (0,1)$ **fast**.
- Absolutely normal number means that is normal in every base
- We need to construct a single real number that is b-normal for every base b
- We will use Lempel-Ziv algorithm that is universal for FS-compressors **in a single base**

Lempel-Ziv martingales

Feder (1991) implicitly defined the **base-b Lempel-Ziv martingale** $d_{\mathrm{LZ}(b)}$ and proved that it is at least as successful **on every sequence** as every finite-state martingale.

$\therefore$ if $S \in \Sigma_b^\infty$ is not normal, then $\dim_{d_{\mathrm{LZ}(b)}}(S) < 1$.

$\therefore$ $x \in (0,1)$ is absolutely normal if none of the martingales $d_{\mathrm{LZ}(b)}$ succeed **exponentially** on the base-b expansion of x.

Moreover, $d_{\mathrm{LZ}(b)}$ has a fast and beautiful theory.

Celebrated Lempel-Ziv compression algorithm and martingale can be both computed very efficiently (time very close to linear)

Lempel-Ziv martingales

How $d_{\mathrm{LZ}(b)}$ works:
Parse $w \in \Sigma_b^*$ into distinct **phrases**, using a growing tree whose leaves are all of the previous phrases.
At each step, bet on the next digit in proportion to the number of leaves below each of the b options.

Base change notation

- For a real x, $\mathrm{seq}_b(x) \in \Sigma_b^\infty$ is the base-b representation of x
- For a sequence $S \in \Sigma_b^\infty$, $\mathrm{real}_b(S) \in [0,1]$ is the real number represented by S

How to construct an absolutely normal number

- For each base b, we need to construct x such that b-Lempel-Ziv martingale does not succeed on x
- We need to construct a single real number that is b-normal for every base b
- It suffices to translate b-Lempel-Ziv martingale into base 2 (very efficiently)
- We need a martingale $d : \Sigma_2^* \to [0, \infty)$ that succeeds on base 2 representations of the numbers for which b-Lempel-Ziv martingale succeeds
- For this translation to be possible (**and efficient**) the martingale must be quite well behaving ...

How to construct an absolutely normal number

1. Transform b-Lempel-Ziv martingale into a better behaving and still efficient martingale that still succeeds on not b-normal sequences
2. Efficiently change base for the resulting martingale
3. Efficiently combine all resulting martingales into one
4. Diagonalize resulting martingale

Savings Accounts, strong success

- The value of Lempel-Ziv martingale $d_{\mathrm{LZ}(b)}$ on a certain infinite string S can fluctuate a lot
- This makes base change more complicated (and time consuming)
- We use the notion of "savings account" here, we are looking at an alternative martingale that **keeps money aside for the bad times to come**

The strong success set of an s-supergale d is

$$S^{\infty}_{\mathrm{str}}[d] = \left\{ x \in \{0,1\}^{\infty} \,\middle|\, \lim_{n} d(x \upharpoonright n) = \infty \right\}$$

Savings Accounts, strong success

- We construct a new martingale d'_b that is a conservative version of $d_{\mathrm{LZ}(b)}$
- d'_b strongly succeeds at least on non-b-normal sequences

$$\{S \,|\dim_{LZ}(S) < 1\} \subseteq S^{\infty}_{\mathrm{str}}[d'_b]$$

- d'_b can be computed in nearly linear time
- If $S \notin S^{\infty}_{\mathrm{str}}[d'_b]$ then S is b-normal

Base Change

- We want an absolutely normal real number x, that is, the base b representation $seq_b(x)$ is not in $S^\infty[d'_b]$
- For this we convert d'_b into a base-2 martingale $d_b^{(2)}$ succeeding on the **base-2 representations of the reals with base-b representation in $S^\infty_{\mathrm{str}}[d'_b]$**
- Again, $d_b^{(2)}$ succeeds on $\mathrm{seq}_2(\mathrm{real}_b(S^\infty_{\mathrm{str}}[d'_b])$

$$\mathrm{real}_b(S^\infty_{\mathrm{str}}[d'_b]) \subseteq \mathrm{real}_2(S^\infty_{\mathrm{str}}[d_b^{(2)}])$$

- We use Carathéodory construction to define measures
- Computing in nearly linear time is also delicate
- In fact our computation $\widehat{d_b^{(2)}}$ approximates slowly $d_b^{(2)}$

$$|\widehat{d_b^{(2)}}(y) - d_b^{(2)}(y)| \leq \frac{1}{|y|^3}$$

Absolutely Normal Numbers

- From previous steps we have a family of martingales $(d_b^{(2)})_b$ so that $d_b^{(2)}$ **succeeds on base-2 representations of non-b-normal sequences**
- For each b we have a nearly linear time computation $\widehat{d_b^{(2)}}$
- We want to construct $S \notin S^\infty[d_b^{(2)}]$ for every b
- Nearly linear time makes it painful to construct a martingale d for the union of $S^\infty[d_b^{(2)}]$
- Then we diagonalize over d to construct S

Martingale diagonalization

- For a martingale d, how to construct x such that d martingale does not succeed on x (with time similar to the computation time for d)?
- Recursive construction, if we have the prefix $x \upharpoonright n$ choose the next symbol i such that

$$d(x \upharpoonright ni)$$

  is the minimum over all possible symbols
- By the fairness condition of a martingale

$$d(w) = \frac{\sum_{i \in \Sigma_b} d(wi)}{b}$$

  d does not succeed on the resulting x
- Time is $n \cdot t(n)$ if d is computable in time $t(n)$

Time bounds ...

- All the steps were performed in nearly linear time on a common **time bound independent of base** b
- Many technical details were simplified in this presentation ... please read paper

Base invariance

- Normality corresponds exactly to the lowest level of algorithmic randomness, Finite-State randomness
- Finite-State randomness and Finite-State dimension are not closed under base change
- p-dimension and p-randomness are closed under base change
- What about intermediate levels, PD, LZ, nearly linear time?

Conclusions

- Lots of remaining questions,
  - can we substitute “suspected” absolute normal numbers by proven absolutely normal numbers in Cryptography?
  - “biased-normality”? (based on FS-dimension)
  - Tight complexity for the operation of base change
  - The algorithm of Becher, Heiber, and Slaman’s has nearly quadratic time but (apparently) a much lower discrepancy. Can we improve our discrepancy while maintaining nearly linear time?

References for construction of absolutely normal numbers

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- V. Becher, P.A. Heiber, and T. Slaman, A polynomial-time algorithm for computing absolutely normal numbers, Information and Computation 232: 1–9, 2013.
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Next lecture

0. Introduction of effective dimension
1. Resource-bounded Hausdorff dimension for Complexity Classes
2. Compression and dimension for low resource bounds. Very effective construction of a normal sequence
3. **Looking back at fractal geometry, other metric spaces**