Computability and model-theoretic aspects of families of sets and its generalizations

Kalimullin I.Sh.

Kazan Federal University
e-mail:ikalimul@gmail.com

Aspects of Computation, Singapore, 2017
in celebration of the research work of Professor Rod Downey

Wehner's family

Theorem (Wehner, 1999).The family

$$\mathcal{W} = \{\{n\} \oplus F : F \text{ is finite } \& \ F \neq W_n\}$$

is (uniformly) c.e. in a degree $\mathbf{x}$ if and only if $\mathbf{x} > \mathbf{0}$.

Wehner's family

Theorem (Wehner, 1999).The family

$$\mathcal{W} = \{\{n\} \oplus F : F \text{ is finite } \& \ F \neq W_n\}$$

is (uniformly) c.e. in a degree $\mathbf{x}$ if and only if $\mathbf{x} > \mathbf{0}$.

Corollary. There is a countable algebraic structure $\mathcal{A}$ s.t. $\mathcal{A}$ has an $\mathbf{x}$-computable structure if and only if $\mathbf{x} > \mathbf{0}$.

Jump inversions

Theorem (Goncharov, Harizanov, Knight,McCoy, Miller, Solomon, 2005). There is a (strong) jump inversion in the class of structures, i.e. functor F s.t.

$$\mathcal{A} \text{ has an } \mathbf{x}'\text{-comp. copy} \iff F(\mathcal{A}) \text{ has an } \mathbf{x}\text{-comp. copy.}$$

Jump inversions

Theorem (Goncharov, Harizanov, Knight,McCoy, Miller, Solomon, 2005). There is a (strong) jump inversion in the class of structures, i.e. functor F s.t.

$$\mathcal{A} \text{ has an } \mathbf{x}'\text{-comp. copy} \iff F(\mathcal{A}) \text{ has an } \mathbf{x}\text{-comp. copy.}$$

Corollary. For every $n \in \omega$ there is a countable algebraic structure $\mathcal{A}$ s.t. $\mathcal{A}$ has an $\mathbf{x}$-computable structure if and only if $\mathbf{x}^{(n)} > \mathbf{0}^{(n)}$.

Jump inversions

Theorem (Goncharov, Harizanov, Knight,McCoy, Miller, Solomon, 2005). There is a (strong) jump inversion in the class of structures, i.e. functor F s.t.

$$\mathcal{A} \text{ has an } \mathbf{x}'\text{-comp. copy } \iff F(\mathcal{A}) \text{ has an } \mathbf{x}\text{-comp. copy.}$$

Corollary. For every $n \in \omega$ there is a countable algebraic structure $\mathcal{A}$ s.t. $\mathcal{A}$ has an $\mathbf{x}$-computable structure if and only if $\mathbf{x}^{(n)} > \mathbf{0}^{(n)}$.

Proof. Let $\mathcal{B}$ has an $\mathbf{x}$-computable copy iff $\mathbf{x} > \mathbf{0}^{(n)}$.

Jump inversions

Theorem (Goncharov, Harizanov, Knight,McCoy, Miller, Solomon, 2005). There is a (strong) jump inversion in the class of structures, i.e. functor F s.t.

$$\mathcal{A} \text{ has an } \mathbf{x}'\text{-comp. copy} \iff F(\mathcal{A}) \text{ has an } \mathbf{x}\text{-comp. copy.}$$

Corollary. For every $n \in \omega$ there is a countable algebraic structure $\mathcal{A}$ s.t. $\mathcal{A}$ has an $\mathbf{x}$-computable structure if and only if $\mathbf{x}^{(n)} > \mathbf{0}^{(n)}$.

Proof. Let $\mathcal{B}$ has an $\mathbf{x}$-computable copy iff $\mathbf{x} > \mathbf{0}^{(n)}$. Then $\mathcal{A} = F^n(\mathcal{B})$.

Iterated jump inversions

Theorem (Goncharov, Harizanov, Knight,McCoy, Miller, Solomon, 2005). There are iterated jump inversions for successive computable ordinals α, i.e. functors $F^{(\alpha)}$ s.t.

$$\mathcal{A} \text{ has an } \mathbf{x}^{(\alpha)}\text{-comp. copy} \iff F^{(\alpha)}(\mathcal{A}) \text{ has an } \mathbf{x}\text{-comp. copy.}$$

Iterated jump inversions

Theorem (Goncharov, Harizanov, Knight,McCoy, Miller, Solomon, 2005). There are iterated jump inversions for successive computable ordinals α, i.e. functors $F^{(\alpha)}$ s.t.

$$\mathcal{A} \text{ has an } \mathbf{x}^{(\alpha)}\text{-comp. copy} \iff F^{(\alpha)}(\mathcal{A}) \text{ has an } \mathbf{x}\text{-comp. copy.}$$

Corollary. For every successive $\alpha \in \omega_1^{CK}$ there is a countable algebraic structure $\mathcal{A}$ s.t. $\mathcal{A}$ has an $\mathbf{x}$-computable structure if and only if $\mathbf{x}^{(\alpha)} > \mathbf{0}^{(\alpha)}$.

Iterated jump inversions

Theorem (Goncharov, Harizanov, Knight,McCoy, Miller, Solomon, 2005). There are iterated jump inversions for successive computable ordinals α, i.e. functors $F^{(\alpha)}$ s.t.

$$\mathcal{A} \text{ has an } \mathbf{x}^{(\alpha)}\text{-comp. copy} \iff F^{(\alpha)}(\mathcal{A}) \text{ has an } \mathbf{x}\text{-comp. copy.}$$

Corollary. For every successive $\alpha \in \omega_1^{CK}$ there is a countable algebraic structure $\mathcal{A}$ s.t. $\mathcal{A}$ has an $\mathbf{x}$-computable structure if and only if $\mathbf{x}^{(\alpha)} > \mathbf{0}^{(\alpha)}$.

Proof. Let $\mathcal{B}$ has an $\mathbf{x}$-computable copy iff $\mathbf{x} > \mathbf{0}^{(\alpha)}$.

Iterated jump inversions

Theorem (Goncharov, Harizanov, Knight,McCoy, Miller, Solomon, 2005). There are iterated jump inversions for successive computable ordinals α, i.e. functors $F^{(\alpha)}$ s.t.

$$\mathcal{A} \text{ has an } \mathbf{x}^{(\alpha)}\text{-comp. copy} \iff F^{(\alpha)}(\mathcal{A}) \text{ has an } \mathbf{x}\text{-comp. copy.}$$

Corollary. For every successive $\alpha \in \omega_1^{CK}$ there is a countable algebraic structure $\mathcal{A}$ s.t. $\mathcal{A}$ has an $\mathbf{x}$-computable structure if and only if $\mathbf{x}^{(\alpha)} > \mathbf{0}^{(\alpha)}$.

Proof. Let $\mathcal{B}$ has an $\mathbf{x}$-computable copy iff $\mathbf{x} > \mathbf{0}^{(\alpha)}$. Then $\mathcal{A} = F^{(\alpha)}(\mathcal{B})$.

Jump inversion for families

Question. Are there jump inversion functors for families? Are there families which are uniformly $\boldsymbol{x}$-c.e. if and only if $\mathbf{x}^{(\alpha)} > \mathbf{0}^{(\alpha)}$?

Jump inversion for families

Question. Are there jump inversion functors for families? Are there families which are uniformly $\boldsymbol{x}$-c.e. if and only if $\mathbf{x}^{(\alpha)} > \mathbf{0}^{(\alpha)}$?

Answer: Only for $\alpha = \mathsf{0}$ and $\alpha = \mathsf{1}$ (K., Faizrahmanov, 2015).

The family enumerable in non-low$_1$ degrees

Theorem. (Andrews, Cai,K,Lempp, Miller, Montalban, 2016). Let $\emptyset' \equiv_T \delta \in \omega^\omega$ such that the set

$$C = \{\sigma \in \omega^{<\omega} \mid \sigma \not\subseteq \delta\}$$

is c.e. Then the family

$$\mathcal{V} = \{\{n\} \oplus (C \cup F) \mid F \text{ is finite and } F \neq W_n^\delta\}$$

is X-c.e. iff $\mathbf{x} \not\leq \mathbf{0}'$.

The family enumerable in non-low$_1$ degrees

Theorem (K., Faizrahmanov, 2015). Let $\emptyset' \equiv_T \delta \in \omega^\omega$ such that the set

$$C = \{\sigma \in \omega^{<\omega} \mid \sigma \not\subset \delta\}$$

is c.e. Then the family

$$\mathcal{V} = \{\{n\} \oplus (C \cup F) \mid F \text{ is cofinite and } \overline{F} \neq W_n^\delta\}$$

is X-c.e. iff $\mathbf{x}' > \mathbf{0}'$.

The case $\alpha = 2$

Theorem. (Faizrahmanov, K.) There is no family $\mathcal{F}$ which is $\mathbf{x}$-c.e. iff $\mathbf{x}'' > \mathbf{0}''$.

The case $\alpha = 2$

Theorem. (Faizrahmanov, K.) There is no family $\mathcal{F}$ which is $\mathbf{x}$-c.e. iff $\mathbf{x}'' > \mathbf{0}''$.

Proof. Let X be a low$_3$ c.e. set which is not low$_2$.

The case $\alpha = 2$

Theorem. (Faizrahmanov, K.) There is no family $\mathcal{F}$ which is $\mathbf{x}$-c.e. iff $\mathbf{x}'' > \mathbf{0}''$.

Proof. Let X be a low$_3$ c.e. set which is not low$_2$.
Then the index set $\{e : \Phi_e^X \text{ is non-low}_2\}$ is $(\Pi_5^X = \Pi_5)$-complete.

The case $\alpha = 2$

Theorem. (Faizrahmanov, K.) There is no family $\mathcal{F}$ which is $\mathbf{x}$-c.e. iff $\mathbf{x}'' > \mathbf{0}''$.

Proof. Let X be a low$_3$ c.e. set which is not low$_2$.
Then the index set $\{e : \Phi_e^X$ is non-low$_2\}$ is $(\Pi_5^X = \Pi_5)$-complete.
But if $\mathcal{F}$ is X-c.e. then the index set $\{e : \mathcal{F}$ is Φ_e^X-c.e.$\}$ is $\Sigma_5^X = \Sigma_5$.

Generalizations of families for jump inversions

Definition. A $\mathbf{0}$-family is any subset of ω.

Generalizations of families for jump inversions

Definition. A 0-family is any subset of ω.
An n-family, $0 < n < \omega$, is a countable set of m-families, $m < n$.

Generalizations of families for jump inversions

Definition. A 0-family is any subset of ω.
An n-family, $0 < n < \omega$, is a countable set of m-families, $m < n$.
An n-family $\mathcal{U}$ is $\mathbf{x}$-c.e. if the m-families $\mathcal{V} \in \mathcal{U}$, $m < n$, are uniformly $\mathbf{x}$-c.e.

Generalizations of families for jump inversions

Definition. A 0-family is any subset of ω.
An n-family, $0 < n < \omega$, is a countable set of m-families, $m < n$.
An n-family $\mathcal{U}$ is $\mathbf{x}$-c.e. if the m-families $\mathcal{V} \in \mathcal{U}$, $m < n$, are uniformly $\mathbf{x}$-c.e.

Generalizations of families for jump inversions

Definition. A 0-family is any subset of ω.
An n-family, $0 < n < \omega$, is a countable set of m-families, $m < n$.
An n-family $\mathcal{U}$ is $\mathbf{x}$-c.e. if the m-families $\mathcal{V} \in \mathcal{U}$, $m < n$, are uniformly $\mathbf{x}$-c.e.

Observation. Every n-family $\mathcal{U}$ can be coded into a structure $\mathcal{G}_{\mathcal{U}}$ such that $\mathcal{U}$ is $\mathbf{x}$-c.e. iff $\mathcal{G}_{\mathcal{U}}$ has a $\mathbf{x}$-computable copy.

Jump inversion for n-families

An n-family $\mathcal{U}$ is $\mathbf{x}'$-c.e. iff the $(n+1)$-family

$$\mathcal{E}(\mathcal{U}) = \begin{cases} \{\omega\} \cup \{\{x\} : x \in A\}, & \text{if } n = 0 \text{ and } \mathcal{U} = A \subseteq \omega, \\ \{\mathcal{E}(\mathcal{V}) : \mathcal{V} \in \mathcal{U}\}, & \text{if } n > 0, \end{cases}$$

is $\mathbf{x}$-c.e.

Double jump inversion for n-families

An n-family $\mathcal{U}$ is $\mathbf{x}''$-c.e. iff the $(n+1)$-family

$$\mathcal{D}(\mathcal{U}) = \begin{cases} \{\text{all finite sets}\} \cup \{\overline{\{x\}} : x \in A\}, & \text{if } n = 0 \text{ and } \mathcal{U} = A \subseteq \omega, \\ \{\mathcal{D}(\mathcal{V}) : \mathcal{V} \in \mathcal{U}\}, & \text{if } n > 0, \end{cases}$$

is $\mathbf{x}$-c.e.

Double jump inversion for n-families

An n-family $\mathcal{U}$ is $\mathbf{x}''$-c.e. iff the $(n+1)$-family

$$\mathcal{D}(\mathcal{U}) = \begin{cases} \{\text{all finite sets}\} \cup \{\overline{\{x\}} : x \in A\}, & \text{if } n = 0 \text{ and } \mathcal{U} = A \subseteq \omega, \\ \{\mathcal{D}(\mathcal{V}) : \mathcal{V} \in \mathcal{U}\}, & \text{if } n > 0, \end{cases}$$

is $\mathbf{x}$-c.e.

Theorem (Faizrahmanov, K., 2015) For every $n \in \omega$ there are $(n+1)$-families $\mathcal{U}_n$ and $\mathcal{V}_n$ such that

$$\mathcal{U}_n \text{ is } \mathbf{x}\text{-c.e.} \iff \mathbf{x}^{(2n)} > \mathbf{0}^{(2n)}$$

and

$$\mathcal{V}_n \text{ is } \mathbf{x}\text{-c.e.} \iff \mathbf{x}^{(2n+1)} > \mathbf{0}^{(2n+1)}.$$

Generalized families for infinitely iterated jump inversions

Definition. A $\mathbf{0}$-family is any subset of ω.
An α-family, $0 < \alpha < \omega_1^{CK}$, is a countable set of β-families, $\beta < \alpha$.
An α-family $\mathcal{U}$ is $\mathbf{x}$-c.e. if the β-families $\mathcal{V} \in \mathcal{U}$, $\beta < \alpha$, are uniformly $\mathbf{x}$-c.e.

Generalized families for infinitely iterated jump inversions

Definition. A $\mathbf{0}$-family is any subset of ω.
An α-family, $0 < \alpha < \omega_1^{CK}$, is a countable set of β-families, $\beta < \alpha$.
An α-family $\mathcal{U}$ is $\mathbf{x}$-c.e. if the β-families $\mathcal{V} \in \mathcal{U}$, $\beta < \alpha$, are uniformly $\mathbf{x}$-c.e.

Generalized families for infinitely iterated jump inversions

Definition. A $\mathbf{0}$-family is any subset of ω.
An α-family, $0 < \alpha < \omega_1^{CK}$, is a countable set of β-families, $\beta < \alpha$.
An α-family $\mathcal{U}$ is $\mathbf{x}$-c.e. if the β-families $\mathcal{V} \in \mathcal{U}$, $\beta < \alpha$, are uniformly $\mathbf{x}$-c.e.

Observation. Every α-family $\mathcal{U}$ can be coded into a structure $\mathcal{G}_{\mathcal{U}}$ such that $\mathcal{U}$ is $\mathbf{x}$-c.e. iff $\mathcal{G}_{\mathcal{U}}$ has a $\mathbf{x}$-computable copy.

The $(\omega + 1)$-jump inversion

Let $\mathcal{E}^\omega(A)$ be the $(\omega + 1)$-family containing all ω-families in the form

$$\{\mathcal{E}^n(L(n)) : n \in \omega\},$$

where $L : \omega \to 2^\omega$ is any function such that $L(n)$ is finite for every n and beginning some n we have

$$L(n) = L(n + 1) \subseteq A.$$

Theorem. (Faizrahmanov, K., 2016) A set A is $\mathbf{x}^{(\omega+1)}$-c.e. iff the $(\omega + 1)$-family $\mathcal{E}^{\omega+1}(A)$ is $\mathbf{x}$-c.e.

The non-low$_\omega$ and non-low$_{\omega+1}$ degrees

Corollary (Faizrahmanov, K., 2016). There is an $(\omega+\mathbf{2})$-family $\mathcal{U}$ such that

$$\mathcal{U} \text{ is } \mathbf{x}\text{-c.e.} \iff \mathbf{x}^{(\omega+1)} > \mathbf{0}^{(\omega+1)}.$$

The non-low$_\omega$ and non-low$_{\omega+1}$ degrees

Corollary (Faizrahmanov, K., 2016). There is an $(\omega+2)$-family $\mathcal{U}$ such that

$$\mathcal{U} \text{ is } \mathbf{x}\text{-c.e.} \iff \mathbf{x}^{(\omega+1)} > \mathbf{0}^{(\omega+1)}.$$

Theorem (Faizrahmanov, K., 2016). There is an $(\omega+1)$-family $\mathcal{U}$ such that

$$\mathcal{U} \text{ is } \mathbf{x}\text{-c.e.} \iff \mathbf{x}^{(\omega)} > \mathbf{0}^{(\omega)}.$$

The non-low$_\omega$ and non-low$_{\omega+1}$ degrees

Corollary (Faizrahmanov, K., 2016). There is an $(\omega+2)$-family $\mathcal{U}$ such that

$$\mathcal{U} \text{ is } \mathbf{x}\text{-c.e.} \iff \mathbf{x}^{(\omega+1)} > \mathbf{0}^{(\omega+1)}.$$

Theorem (Faizrahmanov, K., 2016). There is an $(\omega+1)$-family $\mathcal{U}$ such that

$$\mathcal{U} \text{ is } \mathbf{x}\text{-c.e.} \iff \mathbf{x}^{(\omega)} > \mathbf{0}^{(\omega)}.$$

Corollary. There is an algebraic structure $\mathcal{A}$ such that

$$\mathcal{A} \text{ has an } \mathbf{x}\text{-comp. copy} \iff \mathbf{x}^{(\omega)} > \mathbf{0}^{(\omega)}.$$

No ω-jump inversions for structures

No ω-jump inversions for structures

Corollary. There is an algebraic structure $\mathcal{A}$ such that

$$\mathcal{A} \text{ has an } \mathbf{x}\text{-comp. copy} \iff \mathbf{x}^{(\omega)} > \mathbf{0}^{(\omega)}.$$

No ω-jump inversions for structures

Corollary. There is an algebraic structure $\mathcal{A}$ such that

$$\mathcal{A} \text{ has an } \mathbf{x}\text{-comp. copy} \iff \mathbf{x}^{(\omega)} > \mathbf{0}^{(\omega)}.$$

Theorem. (Kach, K., Montalban, Soskov, 2012, unpublished). There is no algebraic structure $\mathcal{A}$ such that

$$\mathcal{A} \text{ has an } \mathbf{x}\text{-comp. copy} \iff \mathbf{x}^{(\omega)} \geq \mathbf{a}$$

if $\mathbf{a} > \mathbf{0}^{(\omega)}$.

No ω-jump inversions for structures

Corollary. There is an algebraic structure $\mathcal{A}$ such that

$$\mathcal{A} \text{ has an } \mathbf{x}\text{-comp. copy} \iff \mathbf{x}^{(\omega)} > \mathbf{0}^{(\omega)}.$$

Theorem. (Kach, K., Montalban, Soskov, 2012, unpublished). There is no algebraic structure $\mathcal{A}$ such that

$$\mathcal{A} \text{ has an } \mathbf{x}\text{-comp. copy} \iff \mathbf{x}^{(\omega)} \geq \mathbf{a}$$

if $\mathbf{a} > \mathbf{0}^{(\omega)}$.

Theorem. (Soskov, 2013). There is a structure $\mathcal{B}$ such that for no algebraic structure $\mathcal{A}$ such that

$$(\exists \mathbf{x})[\mathbf{y} = \mathbf{x}^{(\omega)} \ \&\ \mathcal{A} \text{ has an } \mathbf{x}\text{-comp. copy}]$$
$$\iff \mathbf{y} \geq \mathbf{0}^{(\omega)} \ \&\ \mathcal{B} \text{ has an } \mathbf{y}\text{-comp. copy}.$$

α-jump inversion

Theorem. (Faizrahmanov, K., 2016) For a set A and successive $\alpha < \omega_1^{CK}$ one can define an α-family $E^\alpha(A)$ such that A is $\mathbf{x}^{(\alpha)}$-c.e. iff $\mathcal{E}^\alpha(A)$ is $\mathbf{x}$-c.e.

α-jump inversion

Theorem. (Faizrahmanov, K., 2016) For a set A and successive $\alpha < \omega_1^{CK}$ one can define an α-family $E^\alpha(A)$ such that A is $\mathbf{x}^{(\alpha)}$-c.e. iff $\mathcal{E}^\alpha(A)$ is $\mathbf{x}$-c.e.

Corollary (Faizrahmanov, K., 2016). For a successive $\alpha < \omega_1^{CK}$ there is an $(\alpha + 1)$-family $\mathcal{U}$ such that

$$\mathcal{U} \text{ is } \mathbf{x}\text{-c.e.} \iff \mathbf{x}^{(\alpha)} > \mathbf{0}^{(\alpha)}.$$

α-jump inversion

Theorem. (Faizrahmanov, K., 2016) For a set A and successive $\alpha < \omega_1^{CK}$ one can define an α-family $E^\alpha(A)$ such that A is $\mathbf{x}^{(\alpha)}$-c.e. iff $\mathcal{E}^\alpha(A)$ is $\mathbf{x}$-c.e.

Corollary (Faizrahmanov, K., 2016). For a successive $\alpha < \omega_1^{CK}$ there is an $(\alpha+1)$-family $\mathcal{U}$ such that

$$\mathcal{U} \text{ is } \mathbf{x}\text{-c.e.} \iff \mathbf{x}^{(\alpha)} > \mathbf{0}^{(\alpha)}.$$

Theorem (Faizrahmanov, K., 2016). For a limit $\alpha < \omega_1^{CK}$ there is an $(\alpha+1)$-family $\mathcal{U}$ such that

$$\mathcal{U} \text{ is } \mathbf{x}\text{-c.e.} \iff \mathbf{x}^{(\alpha)} > \mathbf{0}^{(\alpha)}.$$

α-jump inversion

Theorem. (Faizrahmanov, K., 2016) For a set A and successive $\alpha < \omega_1^{CK}$ one can define an α-family $E^\alpha(A)$ such that A is $\mathbf{x}^{(\alpha)}$-c.e. iff $\mathcal{E}^\alpha(A)$ is $\mathbf{x}$-c.e.

Corollary (Faizrahmanov, K., 2016). For a successive $\alpha < \omega_1^{CK}$ there is an $(\alpha+1)$-family $\mathcal{U}$ such that

$$\mathcal{U} \text{ is } \mathbf{x}\text{-c.e.} \iff \mathbf{x}^{(\alpha)} > \mathbf{0}^{(\alpha)}.$$

Theorem (Faizrahmanov, K., 2016). For a limit $\alpha < \omega_1^{CK}$ there is an $(\alpha+1)$-family $\mathcal{U}$ such that

$$\mathcal{U} \text{ is } \mathbf{x}\text{-c.e.} \iff \mathbf{x}^{(\alpha)} > \mathbf{0}^{(\alpha)}.$$

Corollary. For every $\alpha < \omega_1^{CK}$ there is an algebraic structure $\mathcal{A}$ such that

$$\mathcal{A} \text{ has an } \mathbf{x}\text{-comp. copy} \iff \mathbf{x}^{(\alpha)} > \mathbf{0}^{(\alpha)}.$$

Least jump inversions

Least jump inversions

- $\mathcal{A} \leq_{\Sigma} \mathcal{B}$ means that $\mathcal{A}$ is Σ_1^c-interpretable in $\mathcal{B}^{<\omega}$.

Least jump inversions

- $\mathcal{A} \leq_{\Sigma} \mathcal{B}$ means that $\mathcal{A}$ is Σ^c_1-interpretable in $\mathcal{B}^{<\omega}$.
- $\mathcal{A}^{(\alpha)} = (\mathcal{A}, \text{all } \Sigma^c_\alpha\text{-predicates})$.

Least jump inversions

- $\mathcal{A} \leq_\Sigma \mathcal{B}$ means that $\mathcal{A}$ is Σ^c_1-interpretable in $\mathcal{B}^{<\omega}$.
- $\mathcal{A}^{(\alpha)} = (\mathcal{A}, \text{all } \Sigma^c_\alpha\text{-predicates})$.
- A countable structure $\mathcal{B}$ is a least α-jump inversion for a countable structure $\mathcal{A}$ if

$$\mathcal{A} \leq_\Sigma \mathcal{X}^{(\alpha)} \iff \mathcal{B} \leq_\Sigma \mathcal{X}$$

for every countable structure $\mathcal{X}$.

Least jump inversions

- $\mathcal{A} \leq_{\Sigma} \mathcal{B}$ means that $\mathcal{A}$ is Σ^c_1-interpretable in $\mathcal{B}^{<\omega}$.
- $\mathcal{A}^{(\alpha)} = (\mathcal{A}, \text{all } \Sigma^c_\alpha\text{-predicates})$.
- A countable structure $\mathcal{B}$ is a least α-jump inversion for a countable structure $\mathcal{A}$ if

$$\mathcal{A} \leq_{\Sigma} \mathcal{X}^{(\alpha)} \iff \mathcal{B} \leq_{\Sigma} \mathcal{X}$$

  for every countable structure $\mathcal{X}$.
- (Example). The family of all inifinite c.e. sets is a least jump inversion for $\mathbf{0}''$, but the family of all total computable funtions is not.

Existence of least jump inversions

Theorem (Faizrahmanov, K., Montalban, Puzarenko). For every successive $\alpha < \omega_1^{CK}$ and every countable structure $\mathcal{A}$ there is a least jump inversion $\mathcal{A}^{(-\alpha)}$.

Existence of least jump inversions

Theorem (Faizrahmanov, K., Montalban, Puzarenko). For every successive $\alpha < \omega_1^{CK}$ and every countable structure $\mathcal{A}$ there is a least jump inversion $\mathcal{A}^{(-\alpha)}$.

By the definition $\mathcal{A} \leq_\Sigma \mathcal{B}$ implies $\mathcal{A}^{(-\alpha)} \leq_\Sigma \mathcal{B}^{(-\alpha)}$.

Existence of least jump inversions

Theorem (Faizrahmanov, K., Montalban, Puzarenko). For every successive $\alpha < \omega_1^{CK}$ and every countable structure $\mathcal{A}$ there is a least jump inversion $\mathcal{A}^{(-\alpha)}$.

By the definition $\mathcal{A} \leq_\Sigma \mathcal{B}$ implies $\mathcal{A}^{(-\alpha)} \leq_\Sigma \mathcal{B}^{(-\alpha)}$. By the construction $(\mathcal{A} \oplus \mathcal{B})^{(-\alpha)} \leq_\Sigma \mathcal{A}^{(-\alpha)} \oplus \mathcal{B}^{(-\alpha)}$.

Existence of least jump inversions

Theorem (Faizrahmanov, K., Montalban, Puzarenko). For every successive $\alpha < \omega_1^{CK}$ and every countable structure $\mathcal{A}$ there is a least jump inversion $\mathcal{A}^{(-\alpha)}$.

By the definition $\mathcal{A} \leq_\Sigma \mathcal{B}$ implies $\mathcal{A}^{(-\alpha)} \leq_\Sigma \mathcal{B}^{(-\alpha)}$. By the construction $(\mathcal{A} \oplus \mathcal{B})^{(-\alpha)} \leq_\Sigma \mathcal{A}^{(-\alpha)} \oplus \mathcal{B}^{(-\alpha)}$. Thus,

Existence of least jump inversions

Theorem (Faizrahmanov, K., Montalban, Puzarenko). For every successive $\alpha < \omega_1^{CK}$ and every countable structure $\mathcal{A}$ there is a least jump inversion $\mathcal{A}^{(-\alpha)}$.

By the definition $\mathcal{A} \leq_\Sigma \mathcal{B}$ implies $\mathcal{A}^{(-\alpha)} \leq_\Sigma \mathcal{B}^{(-\alpha)}$. By the construction $(\mathcal{A} \oplus \mathcal{B})^{(-\alpha)} \leq_\Sigma \mathcal{A}^{(-\alpha)} \oplus \mathcal{B}^{(-\alpha)}$. Thus,

Corollary. $(\mathcal{A} \oplus \mathcal{B})^{(-\alpha)} \equiv_\Sigma \mathcal{A}^{(-\alpha)} \oplus \mathcal{B}^{(-\alpha)}$.

Least jump inversion for generalized families

Theorem (Faizrahmanov, K., Montalban, Puzarenko). For a set A the family $\mathcal{E}(A) = \{\omega\} \cup \{\{x\} : x \in A\}$ is the least jump inversion for A. Thus, the least jump inversion for an β-family is an $(1 + \beta)$-family.

Least jump inversion for generalized families

Theorem (Faizrahmanov, K., Montalban, Puzarenko). For a set A the family $\mathcal{E}(A) = \{\omega\} \cup \{\{x\} : x \in A\}$ is the least jump inversion for A. Thus, the least jump inversion for an β-family is an $(1 + \beta)$-family.

Remark. For a set A the family $\mathcal{D}(A) = \{\text{all finite sets}\} \cup \{\overline{\{x\}} : x \in A\}$ is not the least double jump inversion for A. Moreover, the 2-family $\mathcal{E}(\mathcal{E}(A))$ is not Σ-equivalent to a 1-family.

Least jump inversion for generalized families

Theorem (Faizrahmanov, K., Montalban, Puzarenko). For a set A the family $\mathcal{E}(A) = \{\omega\} \cup \{\{x\} : x \in A\}$ is the least jump inversion for A. Thus, the least jump inversion for an β-family is an $(1+\beta)$-family.

Remark. For a set A the family $\mathcal{D}(A) = \{\text{all finite sets}\} \cup \{\overline{\{x\}} : x \in A\}$ is not the least double jump inversion for A. Moreover, the 2-family $\mathcal{E}(\mathcal{E}(A))$ is not Σ-equivalent to a 1-family.

Theorem. For a set A and successive $\alpha < \omega_1^{CK}$ the α-family $E^\alpha(A)$ is the least α-jump inversion for A. Thus, the least α-jump inversion for an β-family is an $(\alpha+\beta)$-family.