Degrees of autostability relative to strong constructivizations of structures of finite signature

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Computable structures

- Our signatures (languages) are computable, and our structures have universes contained in ω.
- We identify formulas with their Gödel numbers

A structure $\mathfrak{M}$ is **computable** if its atomic diagram $D(\mathfrak{M})$ is computable.

Autostability

Definition

Let $\mathbf{d}$ be a Turing degree. A computable structure $\mathfrak{M}$ is **d-autostable (d-computably categorical)** if for every computable structure $\mathfrak{N}$ isomorphic to $\mathfrak{M}$, there exists a d-computable isomorphism from $\mathfrak{M}$ onto $\mathfrak{N}$.

Definition(Fokina, Kalimullin, Miller, 2010)

The **autostability spectrum** of a computable structure $\mathfrak{M}$ is the set

$$\mathrm{AutSpec}(\mathfrak{M}) = \{\mathbf{d} : \mathfrak{M}\ \mathbf{d}\text{-autostable}\}.$$

A Turing degree $\mathbf{d_0}$ is the **degree of autostability** of $\mathfrak{M}$ if $\mathbf{d_0}$ is the least degree in $\mathrm{SCAutSpec}(\mathfrak{M})$.

SC-autostability

A structure $\mathfrak{M}$ is **decidable** if its complete digram $D^c(\mathfrak{M})$ is computable, i.e. given a first-order formular $\phi(\bar{x})$ and a tuple $\bar{a}$ from $\mathfrak{M}$, one can effectively determine whether $\phi(\bar{a})$ is true in $\mathfrak{M}$ or not.

A decidable structure $\mathfrak{M}$ is called **d-autostable relative to strong constructivizations** (**d-SC-autostable**) if

every two decidable copies of $\mathfrak{M}$ are d-computably isomorphic.

Prime models and complete formulas

Let $\mathfrak{M}$ be a structure of a signature σ. $\mathrm{Th}(\mathfrak{M})$ denotes the first-order theory of $\mathfrak{M}$.

A structure $\mathfrak{M}$ is a **prime model** (of the theory $\mathrm{Th}(\mathfrak{M})$) if $\mathfrak{M}$ is elementary embeddable into every structure $\mathfrak{N}$ of the theory $\mathrm{Th}(\mathfrak{M})$.

A structure $\mathfrak{M}$ is an **almost prime model** if there exists a finite tuple $\bar{c}$ from $\mathfrak{M}$ such that $(\mathfrak{M}, \bar{c})$ is a prime model.

A first-order formula $\psi(x_0, \dots, x_n)$ is a **complete formula** for the theory $\mathrm{Th}(\mathfrak{M})$ if $\mathfrak{M} \models \exists \bar{x} \psi(\bar{x})$ and, for every σ-formula $\varphi(\bar{x})$, either $\mathfrak{M} \models \forall \bar{x}(\psi(\bar{x}) \to \varphi(\bar{x}))$ or $\mathfrak{M} \models \forall \bar{x}(\psi(\bar{x}) \to \neg\varphi(\bar{x}))$.

Nurtazin's criterion

Theorem (Nurtazin 1974)

Suppose that $\mathfrak{M}$ is a decidable structure of a signature σ. $\mathfrak{M}$ is SC-autostable if and only if there exists a finite tuple $\bar{c}$ from $\mathfrak{M}$ such that the following holds:

(a) The structure $(\mathfrak{M}, \bar{c})$ is a prime model of the theory $\mathrm{Th}(\mathfrak{M}, \bar{c})$.

(b) Given a $(\sigma \cup \{\bar{c}\})$-formula $\psi(\bar{x})$ one can effectively, uniformly in ψ, determine whether ψ is a complete formula for $\mathrm{Th}(\mathfrak{M}, \bar{c})$.

Goncharov's result

Theorem (Goncharov, 2011)

Let $\mathbf{d}$ be a Turing degree. Suppose that $\mathfrak{M}$ is a decidable structure of a signature σ, $\bar{a}$ is a finite tuple from $\mathfrak{M}$ such that the following conditions hold.

(a) The structure $(\mathfrak{M}, \bar{a})$ is a prime model.

(b) Given a $(\sigma \cup \{\bar{a}\})$-formula $\psi(\bar{x})$, one can effectively relative to $\mathbf{d}$, uniformly in ψ, determine whether ψ is a complete formula in the theory $Th(\mathfrak{M}, \bar{a})$.

Then $\mathfrak{M}$ is $\mathbf{d}$-SC-autostable.

In particular, if $\mathfrak{M}$ is a decidable almost prime model (i. e. $\mathfrak{M}$ is decidable and there exists a tuple $\bar{a}$ such that (a) is satisfied), then $\mathfrak{M}$ is $\mathbf{c}$-autostable for some c.e. degree $\mathbf{c}$.

SC-autostability spectrum

Goncharov investigated autostability spectrum restricted to decidable structures.

Definition(Goncharov, 2011)

The **autostability spectrum relative to strong constructivizations** (SC-**autostability spectrum**) of the structure $\mathfrak{M}$ is the set

$$\mathrm{SCAutSpec}(\mathfrak{M}) = \{\mathbf{d} : \mathfrak{M}\ \mathbf{d}\text{-SC-autostable}\}.$$

A Turing degree $\mathbf{d_0}$ is the **degree of SC-autostability** of $\mathfrak{M}$ if $\mathbf{d_0}$ is the least degree in $\mathrm{SCAutSpec}(\mathfrak{M})$.

Directions

- Examples of SC-autostability spectrum.
- Relations between autostability spectrum and SC-autostability spectrum.
- Which autostability spectrums can be witnessed by structures of familiar classes?

Examples of SC-autostability.

Theorem (S.S. Goncharov, 2011)

Every c.e. degree $\mathbf{d}$ is the degree of SC-autostability of some decidable almost prime model of the infinite signature.

Theorem (N.A. Bazhenov, 2016)

- For every computable ordinal α, the Turing degree $\mathbf{0}^{\alpha}$ is a degree of SC-autostability for some decidable Boolean algebra.
- For a computable ordinal α, every Turing degree c.e. in and above $\mathbf{0}^{\alpha+1}$ is the degree of SC-autostability for some decidable structure of the infinite signature.

PA-degrees

A Turing degree $\mathbf{d}$ is called a $\mathbf{PA-degree}$ if it computes some complete extension of Peano arithmetics.

Theorem (N.A. Bazhenov, 2016)

There exists a decidable structure $\mathfrak{M}$ such that $\mathfrak{M}$ is a prime model of the infinite signature of the theory $Th(\mathfrak{M})$, and the SC-autostability spectrum of $\mathfrak{M}$ contains precisely the PA-degrees.

Relations between spectrums

Remark 1

If $\mathfrak{M}$ is a decidable structure, then

- $\mathrm{AutSpec}(\mathfrak{M}) \subseteq \mathrm{SCAutSpec}((\mathfrak{M})$.
- If $\mathbf{0} \in \mathrm{AutSpec}(\mathfrak{M})$, then $\mathbf{0}^{\omega} \in \mathrm{SCAutSpec}(\mathfrak{M})$

Relations between spectrums

Proposition

Let $\mathfrak{M}$ be a decidable structure of a signature σ. Then there exists a computable structure $\mathfrak{M}^*$ of the new signature σ^* such that $\mathrm{AutSpec}(\mathfrak{M}^*) = \mathrm{SCAutSpec}(\mathfrak{M})$. In particular, every degree of SC-autostability is a degree of autostability.

Proof. Consider the structure $\mathfrak{M}^*$ of the signature

$$\sigma^* = \{P^n_\Phi : \Phi(x_1, \ldots, x_n) \text{ is an } \sigma\text{-formula}\}$$

such that $|\mathfrak{M}^*| = |\mathfrak{M}|$ and the predicates of σ^* are interpreted in the natural way.

Relations between spectrums

Theorem (with N.A. Bazhenov 2016)

- Suppose that $0 \leq \alpha \leq \beta \leq \omega$. There exists a decidable structure $\mathfrak{M}$ such that $\mathbf{0}^{\alpha}$ is the degree of SC-autostability of $\mathfrak{M}$ and $\mathbf{0}^{\beta}$ is the degree of autostability of $\mathfrak{M}$.
- Suppose that $0 \leq \beta \leq \omega$. There exists a decidable structure $\mathfrak{M}$ such that $\mathfrak{M}$ has no degree of SC-autostability and $\mathbf{0}^{\beta}$ is the degree of autostability of $\mathfrak{M}$.

Relations between spectrums

Problem 1

Does there exist a decidable structure that has degree of SC-autostaility and has no degree of autostability?

Problem 2

Is every autostability spectrum the SC-autostability spectrum for some decidable structure? In particular, is every degree of autostability a degree of SC-autostability?

Familiar classes

Theorem (Bazhenov, 2016)

For an infinite computable ordinal β, every Turing degree c.e. in and above $\mathbf{0}^{(2\beta+1)}$ is the degree of SC-autostability for some discrete linear order.

Theorem (Goncharov, 2011)

Let $\mathbf{d}$ be a c.e. degree. There exists a decidable structure $\mathfrak{M}$ of the signature $\sigma_1 = \{R_i^1 : i \in \omega\}$ such that:

(1) $\mathfrak{M}$ is a prime model,

(2) $\mathbf{d}$ is the degree of SC-autostabilioty of $\mathfrak{M}$,

(3) every computable copy of $\mathfrak{M}$ is decidable.

We construct the effective transformation Ψ of computable structures of the signature σ_1 into computable structure of finite signature. We ensure that Ψ preserves the key properties. Using this transformation we show the following.

Theorem

Let $\mathbf{d}$ be a c.e. degree. There exists a decidable structure $\mathfrak{M}$ of the finite signature σ_2 such that:

(1) $\mathfrak{M}$ is a prime model,

(2) $\mathbf{d}$ is the degree of SC-autostabilioty of $\mathfrak{M}$,

(3) every computable copy of $\mathfrak{M}$ is decidable.

We use the ideas of Goncharov (1980) to construct a transformation Ψ' of computable structures of the signature σ_2 into computable directed graphs such that Ψ' preserves the key properties.

Corollary 1

Let $\mathbf{d}$ be a c.e. degree. There exists a decidable directed graph $\mathfrak{M}$ such that:

(1) $\mathfrak{M}$ is a prime model,

(2) $\mathbf{d}$ is the degree of SC-autostabilioty of $\mathfrak{M}$,

(3) every computable copy of $\mathfrak{M}$ is decidable.

Recall that a signature σ is **nontrivial** if σ contains a predicate or functional symbol of arity ≥ 2.

Now we can use the standard codings of directed graphs into structures of the nontrivial signature σ.

Corollary 2

Let $\mathbf{d}$ be a c.e. degree. There exists a decidable structure $\mathfrak{M}$ of the nontrivial signature σ such that:

(1) $\mathfrak{M}$ is a prime model,

(2) $\mathbf{d}$ is the degree of SC-autostabilioty of $\mathfrak{M}$,

(3) every computable copy of $\mathfrak{M}$ is decidable.

Familiar classes

Problem 3

Let K be one of the familiar algebraic classes (e.g., directed graphs, partial orders, lattices, groups, fields, etc.). Suppose $\mathfrak{M}$ is a decidable structure (for an arbitrary computable signature). Does there always exist a decidable structure $\mathfrak{N}_{\mathfrak{M}}$ from K such that $\mathrm{SCAutSpec}(\mathfrak{N}_{\mathfrak{M}}) = \mathrm{SCAutSpec}(\mathfrak{M})$

Announcement

Announced result

There exists a decidable undirected graph $\mathcal{G}$ such that $\mathcal{G}$ is a prime model of the theory $Th(\mathcal{G})$, and the SC-autostability spectrum of $\mathcal{G}$ contains precisely the PA-degrees.

Thank you for your attention!