Ordered abelian groups, generalized series and integer parts

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Motivation

In the last several years, Julia Knight and Karen Lange (sometimes with coauthors) have worked towards a computability theoretic analysis of the following theorem and its proof.

Theorem (Mourgues and Ressayre)

Every real closed field has an integer part.

The algebraic method of Mourgues and Ressayre's proof is used in the context of ordered abelian groups to prove Hahn's Theorem, and many of Knight and Lange's results have analogs for ordered abelian groups.

Our goal is to give these analogs and show how to answer one of Knight and Lange's main open questions in the simpler context of ordered abelian groups.

Ordered groups and fields

An *ordered abelian group* is an abelian group G together with a linear order $\leq_G$ on G satisfying

$$x \leq_G y \Rightarrow x + z \leq_G y + z$$

If we reverse the order by

$$x \leq'_G y \Leftrightarrow x \geq_G y$$

then $(G, \leq'_G)$ is also an ordered abelian group.

An *ordered field* is a field F together with a linear order $\leq_F$ on F satisfying

$$x \leq_F y \text{ and } r \geq_F 0_F \Rightarrow x + z \leq_F y + z \text{ and } r \cdot x \leq_F r \cdot y$$

Archimedian classes

G will denote a (computable) ordered abelian group and R will denote a (computable) real closed field. In either case, we define

- x is *archimedean equivalent to* y

$$x \sim y \Leftrightarrow \exists n \in \mathbb{N} \, (|x| \leq n|y| \text{ and } |y| \leq n|x|)$$

- x is *archimedean less than* y

$$x \ll y \Leftrightarrow \forall n \in \mathbb{N} \, (n|x| < |y|)$$

- $[x] = w(x) =$ the equivalence class of x under $\sim$
- $w(X) = \{[x] \mid x \in X\}$

Since we can effectively pass to the divisible closure of G without adding archimedean classes, we can assume G is divisible.

Value group sections in R

$w(R^{\times}) = \{[x] \mid x \in R \text{ and } x \neq 0_R\}$ forms a divisible ordered abelian group

$$[x] < [y] \Leftrightarrow x \ll_R y \qquad [x] + [y] = [x \cdot_R y] \qquad 0_{w(R^{\times})} = [1_R]$$

called the *value group* of R. This group describes the algebraic structure of the archimedean classes in R.

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It is useful to work with an appropriate copy of the value group inside R. For any embedding $t : w(R^\times) \to (R^{>0}, \cdot)$ of ordered groups such that $t([x]) \in [x]$, the image of t is a *value group section* of R.

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Theorem (Knight, Lange)

Every countable R has a value group section which is $\Delta^0_2(R)$. Furthermore, there is a computable R such that every value group section computes $0'$.

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Note: When studying valuations of ordered fields, one typically reverses the order in the value group by setting

$$[x] > [y] \Leftrightarrow x \ll_R y$$

For now, we haven't done this, but we will see why this is done later.

Archimedean representatives in G

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We can find appropriate copies of this linear order inside G. For any embedding $t : w(G^{>0}) \to G^{>0}$ as linear orders such that $t([x]) \in [x]$, the image of t is a *set of archimedean representatives* of G.

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Theorem (Downey, Solomon)

Every countable G has a set of archimedean representatives which is $\Delta^0_2(G)$. Furthermore, there is a computable G such that every set of archimedean representatives computes $0'$.

Theorem (Solomon)

Let L be a r.e. presented linear order. There is a computable G such that $w(G^{>0})$ is isomorphic to L. In particular, for any computable $\widehat{G} \cong G$ and any set $A(\widehat{G})$ of archimedean representatives, $A(\widehat{G}) \cong L$.

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Theorem (Feiner)

There is an r.e. presented linear order with no computable copy.

Corollary

There is a computable G such that no computable copy of G has a computable set of archimedean representatives.

Residue field of R

For a real closed field R,

$$\mathcal{O} = \{r \in R \mid [r] \leq [1_R]\} \text{ is the finite and infinitesimal elements}$$
$$\mathcal{M} = \{r \in R \mid [r] < [1_R]\} \text{ is the infinitesimal elements}$$

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If t is an embedding of the residue field into R such that $t([x]) \in [x]$, then the image of t is a *residue field section* of R.

Theorem (Knight, Lange)

Every R has a residue field section which is $\Pi^0_2(R)$. Furthermore, there is a computable R such that no residue field section is Σ^0_2.

Hahn fields

Let k be an archimedean real closed field and let G be an divisible ordered abelian group. Let t be an indeterminant.

The Hahn field $k((G))$ consists of the formal series

$$\sum_{g \in S} a_g t^g$$

where $S \subseteq G$ is well ordered and each $a_g \in k - \{0_k\}$. The formal series are added and multiplied in $k((G))$ as usual.

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There is a natural order on $k((G))$ defined by

$$\sum_{g \in S_0} a_g t^g < \sum_{g \in S_1} b_g t^g \Leftrightarrow a_g < b_g \text{ for the least } g \text{ such that } a_g \neq b_g$$

In fact, $k((G))$ is a real closed field with residue field k.

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This implies

$$\left[\sum_{g \in S} a_g t^g \right] = [1 t^g] \text{ where } g = \text{least}(S)$$

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$$\sum_{g \in S_0} a_g t^g \ll \sum_{g \in S_1} b_g t^g \Leftrightarrow \text{least}(S_0) >_G \text{least}(S_1)$$

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If we order the elements $[\sum_{g \in S} a_g t^g]$ of the value group by

$$\Big[\sum_{g \in S_0} a_g t^g\Big] > \Big[\sum_{g \in S_1} b_g t^g\Big] \Leftrightarrow \sum_{g \in S_0} a_g t^g \ll \sum_{g \in S_1} b_g t^g$$

then the map sending $[\sum_{g \in S} a_g t^g]$ to least(S) is an isomorphism from the value group onto G.

Recall: The standard definition for the value group on a real closed field orders the archimedean classes by

$$[x] > [y] \Leftrightarrow x \ll y$$

Following this convention, $k((G))$ is a real closed field with residue field k and value group G.

Integer parts

An *integer part* of R is a discrete ordered subring I such that

$$\forall r \in R \, \exists i \in I \, (i \leq r < i + 1)$$

Mourgues and Ressayre proved every R has an integer part.

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Knight and Lange have results on the complexity of this construction. However, it is unknown whether ACA_0 can prove the existence of an integer part, or whether each computable real closed field has a Δ^0_2 integer part.

Hahn's Theorem (Classical Version)

Hahn's Theorem is a similar embedding theorem with a similar proof.

- Switch from real closed fields to (divisible) ordered abelian groups.
- Value group is replaced by a linear order (to represent the structure of the archimedean classes).
- The residue field is replaced by archimedean ordered abelian groups.

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Theorem (Hölder's Theorem)

Every archimedean ordered group is isomorphic to a subgroup of $(\mathbb{R}, +)$.

Let L be a linear order and let $\sum_L \mathbb{R}$ denote the abelian group of all functions $f : L \to \mathbb{R}$ under componentwise addition.

Hahn's Theorem (Classical Version)

A Hahn subgroup of $\sum_L \mathbb{R}$ is a subgroup H such that for all $f \in H$,

$$\text{supp}(f) = \{\ell \in L \mid f(\ell) \neq 0\} \text{ is well ordered}$$

and for all $\ell \in L$ and $f \in H$, the function $C_\ell f$ is also in H where

$$C_\ell f(x) = \begin{cases} f(x) & \text{if } x <_L \ell \\ 0_{\mathbb{R}} & \text{otherwise} \end{cases}$$

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Each $f \in H$ can be represented by $\sum_{\ell \in S} r_\ell t^\ell$ where $S \subseteq L$ is well ordered and each $r_\ell \in \mathbb{R}$ with $r_\ell \neq 0$.

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$$\begin{aligned} f = \;& r_{\ell_0} t^{\ell_0} + r_{\ell_1} t^{\ell_1} + \cdots \quad + r_{\ell_\alpha} t^{\ell_\alpha} + r_{\ell_{\alpha+1}} t^{\ell_{\alpha+1}} + \cdots \\ C_{\ell_\alpha} f = \;& r_{\ell_0} t^{\ell_0} + r_{\ell_1} t^{\ell_1} + \cdots \end{aligned}$$

The requirement that H is closed under the C_ℓ operations is the analog of being truncation closed in the Mourgues and Ressayre construction.

Order H by $f < g$ if and only if $f(\ell) <_{\mathbb{R}} g(\ell)$ where ℓ is the least element of $\{x \in L \mid f(x) \neq g(x)\}$.

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As before, if we flip the order on the equivalence classes to

$$[f] > [g] \Leftrightarrow f \ll g$$

then the map sending $[f]$ to the least element of $\mathrm{supp}(f)$ is an order isomorphism from the archimedean classes of H onto L.

Theorem (Hahn's Theorem)

Let G be a ordered abelian group and let $A \subseteq G$ be a set of archimedean representatives. We order A by $x <_A y$ if and only if $x \gg y$. G is isomorphic to a Hahn subgroup of $\sum_A \mathbb{R}$.

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Theorem (Hölder's Theorem)

Every archimedean ordered group is isomorphic to a subgroup of $\mathbb{R}$.

To study Hahn's Theorem effectively or in reverse math for countable G, we would like to replace $\mathbb{R}$ by countable archimedean groups. Using Hölder's Theorem (in RCA_0), we can always embed these archimedean groups into $\mathbb{R}$ and get the original version.

Let A be a linear order and let K_a for $a \in A$ be a sequence of archimedean ordered groups. A Hahn subgroup of $\sum_A K_a$ (indexed by I) is a sequence of functions $f_i : A \to \bigcup_A K_a$ for $i \in I$ such that

- $f_i(a) \in K_a$ for all $i \in I$ and $a \in A$,
- $\mathrm{supp}(f_i)$ is well ordered,
- $\{f_i \mid i \in I\}$ is group closed under the C_a operations for $a \in A$.

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We say G is isomorphic to a Hahn subgroup of $\sum_A K_a$ if there is a Hahn subgroup $\{f_g \mid g \in G\}$ indexed by G such that

- $f_{0_G}(a) = 0_{K_a}$ for all $a \in A$, so it is the identity element,
- $f_g + f_h = f_{g+h}$ and $f_{-g} = -f_g$, and
- $g <_G h$ if and only if $f_g < f_h$ in the Hahn subgroup.

Theorem (Hahn's Theorem, Version 2)

For every ordered abelian group G, there is a linear order A and a sequence K_a (for $a \in A$) of archimedean ordered subgroups of G such that G is order isomorphic to a Hahn subgroup of $\sum_A K_a$.

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Hahn's Theorem is equivalent to ACA_0 over RCA_0.

In one direction, suppose Hahn's Theorem holds. For $g, h \in G$, if the least element of $\text{supp}(f_g)$ is not equal to the least element of $\text{supp}(f_h)$, then g and h are in different archimedean classes. Using this fact, you can build a group G to recover the range of a given function using the Hahn embedding.

Outline of Hahn's Theorem in ACA_0

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- Since H'_a is divisible, there is a subgroup K_a such that $H_a = H'_a + K_a$. (The groups K_a play the role of the residue field section in the Mourgues and Ressayre construction.)

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- H_a and H'_a are convex and divisible, and H_a/H'_a is archimedean.
- Since H'_a is divisible, there is a subgroup K_a such that $H_a = H'_a + K_a$. (The groups K_a play the role of the residue field section in the Mourgues and Ressayre construction.)
- It remains to construct the Hahn subgroup of $\sum_A K_a$ indexed by G. This is done by starting with the divisible subgroup G_0 generated by $\bigcup_{a \in A} K_a$ (which is easily embeddable in $\sum_A K_a$) and then extending the embedding to larger subgroups formed by adding one element at a time.

Integer parts in G

To define an "integer part" in G, fix $g \in G$ with $g > 0_G$. $I_g \subseteq G$ is an integer part of G relative to g if

- I_g is an ordered subgroup of G with least positive element g, and
- for all $h \in G$, there is a $z \in I_g$ such that $z \leq h < z + g$.

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Using the truncation closed embedding from Hahn's Theorem, it follows that ACA_0 can prove that I_g exists for every positive $g \in G$. However, the proof of Hahn's Theorem uses several jumps.

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Theorem (Lange and Solomon)

Let G be a countable divisible ordered abelian group. For every positive $g \in G$, G has an integer part relative to g which is $\Delta^0_2(G)$.

Thank you!