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Honoring Rod Downey's numerous contributions to the field

# The automorphisms of the lattice of x-computably enumerable vector spaces

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## Computably enumerable (c.e.) sets and their lattice

- $\mathcal{E}$ is the lattice of c.e. sets: $W_0, W_1, ..., W_e, ...$
  $\mathcal{E}^*$ is the lattice of c.e. sets modulo finite sets.

  $\mathcal{E}$ and $\mathcal{E}^*$ are distributive lattices under $\subseteq$, $\cap$, $\cup$.

  Computable sets consist of complemented elements and form a Boolean algebra.

- (Post) $C \subseteq \omega$ is *immune* iff $|C| = \infty$ and

  $(\forall e)[W_e \subseteq C \Rightarrow |W_e| < \infty]$

- $S$ is *simple* iff ($\overline{S}$ is c.e. and co-immune)

- $D_n$ is the finite set with canonical index $n$

  Consider a *strong array* (of finite sets), $(D_{g(i)})_{i\in\omega}$ where $g$ is computable,

  such that $i \neq j \Rightarrow D_{g(i)} \cap D_{g(j)} = \emptyset$

- $C$ is *hyperimmune* ($\mathbf{h}$-*immune*) iff $|C| = \infty$ and there is no such $(D_{g(i)})_{i\in\omega}$ with

  $(\forall i)[D_{g(i)} \cap C \neq \emptyset]$

- $S$ is $\mathbf{h}$-*simple* iff ($S$ is c.e. and co-hyperimmune)

- Similarly, define $\mathbf{hh}$-*immune* and $\mathbf{hh}$-*simple* sets when a strong array $(D_{g(i)})_{i\in\omega}$ is replaced by a weak array of finite sets $(W_{g(i)})_{i\in\omega}$.

- $\mathbf{hh}$-simple $\Rightarrow$ $\mathbf{h}$-simple $\Rightarrow$ simple

  The implications are not reversible.

- (Dekker) Every nonzero c.e. Turing degree contains an $\mathbf{h}$-simple set.

- (Martin) The c.e. Turing degrees that are degrees of $\mathbf{hh}$-simple sets are exactly the high degrees.

  $\mathbf{a}$ is *high* $\Leftrightarrow_{def}$ $\mathbf{a}' = \mathbf{0}''$

- For $A \in \mathcal{E}$, define its principal filter

$$\mathcal{E}(A, \uparrow) = \{E \in \mathcal{E} : A \subseteq E\}$$

- $A$ is $\mathbf{hh}$-simple iff $\mathcal{E}^*(A, \uparrow)$ is a Boolean algebra.

## Cohesive and maximal sets

- A set $C \subseteq \omega$ is *cohesive* iff $|C| = \infty$ and for every c.e. set $W$, either $W \cap C$ or $\overline{W} \cap C$ is finite.

  ($W \cap C$ is infinite $\Rightarrow C \subseteq^* W$
  $\overline{W} \cap C$ is infinite $\Rightarrow C \subseteq^* \overline{W}$)

- Cohesive sets are $\mathbf{hh}$-immune. The converse is not true.

- A set $M \subseteq \omega$ is *maximal* iff $M$ is c.e. and $\overline{M}$ is cohesive.

  Equivalently, $M$ is c.e., $\overline{M}$ is infinite, and for every c.e. set $E$ with $M \subseteq E \subseteq \omega$, either $\omega - E$ or $E - M$ is finite.

- (Friedberg) Maximal sets exist. Hence $\mathcal{E}^*$ has co-atoms.

- $X$ is *atomless* if it has no maximal superset.
  (Lachlan) There is an atomless $\mathbf{hh}$-simple set $H$.
  $\mathcal{E}^*(H, \uparrow)$ is an atomless Boolean algebra.

- (Martin) The c.e. Turing degrees that are degrees of maximal sets are exactly the high degrees.

- (Soare) For any two maximal sets, $M_1$ and $M_2$, there is an automorphism $\Phi$ of $\mathcal{E}$ ($\mathcal{E}^*$) such that $\Phi(M_1) = M_2$ ($\Phi(M_1^*) = M_2^*$).

- Both $\mathcal{E}$ and $\mathcal{E}^*$ have $2^{\aleph_0}$ automorphisms.

- A set $C \subseteq \omega$ is $r$-*cohesive* iff $|C| = \infty$ and if for every computable set $W$ either $W \cap C$ or $\overline{W} \cap C$ is finite.
  A set $M$ is $r$-*maximal* iff $M$ is c.e. and $\overline{M}$ is $r$-*cohesive*.

- Every cohesive set is $r$-cohesive; hence every maximal set is $r$-maximal. The converse is not true.

- $M$ is $r$-maximal and $\mathbf{hh}$-simple $\Rightarrow$ $M$ is maximal

  **Proof**. $\mathcal{E}^*(M, \uparrow)$ contains no nontrivial complemented elements. Every element is complemented, so $\mathcal{E}^*(M, \uparrow)$ is a 2-element Boolean algebra.

- A set $B \subseteq \omega$ is *quasimaximal* iff it is the intersection of finitely many maximal sets: $B = \bigcap\limits_{i=1}^{n} M_i$.
  If $M_i \neq^* M_j$ for $i \neq j$, the number $n$ is called the *rank* of $B$.

- Quasimaximal sets are $\mathbf{hh}$-simple. The converse is not true.

- The principal filter $\mathcal{E}^*(B, \uparrow)$ is isomorphic to the Boolean algebra $\mathbf{B}_n$ of size $2^n$.

- (Soare) For any two quasimaximal sets of the same rank, $B_1$ and $B_2$, there is an automorphism $\Phi$ of $\mathcal{E}$ such that $\Phi(B_1) = B_2$.

## C.e. vector spaces and their lattice

- $V_\infty$: computable $\aleph_0$-dimensional vector space over a computable field (will assume infinite, say $\mathbb{Q}$) with uniformly computable dependence relations $(D_n)_{n\in\omega}$ (dependence algorithm)

- Can think of the elements of $V_\infty$, the vectors, as infinite sequences of elements of $\mathbb{Q}$ with only finitely many nonzero components.

- Pointwise vector addition and scalar multiplication.
  $(a_1, a_2, a_3, 0, 0, \ldots) + (b_1, b_2, 0, 0, 0, \ldots) = (a_1 + b_1, a_2 + b_2, a_3, 0, 0, \ldots)$

  $c(a_1, a_2, a_3, a_4, 0, \ldots) = (ca_1, ca_2, ca_3, ca_4, 0, \ldots)$

- $(1, 0, 0, 0, \ldots), (0, 1, 0, 0, \ldots), \ldots$ a standard (computable) basis for $V_\infty$.

- A (sub)space $V \subseteq V_\infty$ is c.e. if $V$ is a c.e. set.
  $U + W = cl(U \cup W)$
  $(\mathcal{L}(V_\infty), \subseteq, \cap, +)$ is the lattice of c.e. vector subspaces of $V_\infty$:
  nondistributive, modular: $x \leq b \Rightarrow [x \vee (a \wedge b) = (x \vee a) \wedge b]$

- $I_0, I_1, I_2, \ldots$ a computable enumeration of all
  c.e. independent subsets of $V_\infty$
  $V_e = cl(I_e)$
  $V_0, V_1, V_2, \ldots$ a computable enumeration of all spaces in $\mathcal{L}(V_\infty)$

- $V \in \mathcal{L}(V_\infty)$ is complemented iff
  there is a dependence algorithm *mod* $V$
  ($\frac{V_\infty}{V}$ has a dependence algorithm) iff
  $V$ is generated by a computable subset of a computable basis for $V_\infty$.

- Let $V \in \mathcal{L}(V_\infty)$.

  $D_n(V) = \{\langle v_1, \ldots, v_n\rangle : v_1, \ldots, v_n$ are dependent over $V\}$
  $D(V) = \bigcup_{n \geq 1} D_n(V)$
  $D_n(V)$ is c.e.
  $[D_n(V) \leq_T D_{n+1}(V)] \wedge [D_n(V) \leq_T D(V)]$,
  uniformly in $n$.

- (Shore) Assume that $V_\infty$ is over an infinite field.
  Let $C_1, C_2, C_3, \ldots, C$ be a sequence of c.e. sets with
  $C_n \leq_T C_{n+1}$ and $C_n \leq_T C$, uniformly in $n$. Then there is $V \in \mathcal{L}(V_\infty)$
  such that for $n \geq 1$, $D_n(V) \equiv_T C_n$ and $D(V) \equiv_T C$.

- (Dimitrov, Harizanov, and Morozov)
  If $C$ noncomputable and $C_1$ computable, we can also obtain
  that $\frac{V_\infty}{V}$ has trivial computable automorphism group.

## Maximal vector spaces

- Let $V \in \mathcal{L}(V_\infty)$.

  The space $V$ is *maximal* iff $\dim \frac{V_\infty}{V} = \infty$ and for every $W \in \mathcal{L}(V_\infty)$,

  $$V \subseteq W \subseteq V_\infty \Rightarrow [\dim \frac{W}{V} < \infty \vee \dim \frac{V_\infty}{W} < \infty]$$

- (Metakides and Nerode) There are maximal subspaces of $V_\infty$.

- Assume that $\Omega$ is a computable basis of $V_\infty$.
  Identify $\Omega$ with $\omega$.

- (Shore) Every maximal subset $M$ of $\Omega$ spans a maximal subspace of $V_\infty$.

- An independent set $J \subseteq V_\infty$ is *nonextendible* if

  dim $\frac{V_\infty}{cl(J)} = \infty$ and

$$(\forall e)[J \subseteq I_e \Rightarrow |I_e - J| < \infty]$$

- (Metakides and Remmel)

  There exists a maximal subspace $V$ such that no c.e. basis of $V$ is extendible.

## $k$-thin vector spaces

- Let $V \in \mathcal{L}(V_\infty)$ and $k \in \omega$.

  The space $V$ is called is *$k$-thin* iff $\dim \frac{V_\infty}{V} = \infty$,
  $(\forall e)[V \subseteq V_e \Rightarrow (\dim \frac{V_e}{V} < \infty \vee \dim \frac{V_\infty}{V_e} \leq k)]$,
  $(\exists e_0)[V \subseteq V_{e_0} \wedge \dim \frac{V_\infty}{V_{e_0}} = k]$

- (Kalantari and Retzlaff) For $k \geq 0$, there exists a $k$-thin space $\mathcal{T}_k$.

  There exists an infinite sequence of maximal spaces, $(\mathcal{T}_k)_{k\in\omega}$,
  such that for every automorphism $\Phi$ of $\mathcal{L}(V_\infty)$: $i \neq j \Rightarrow \Phi(\mathcal{T}_i) \neq \mathcal{T}_j$.

- Question: Is there an $\mathcal{L}(V_\infty)$-analogue of Soare's theorem?

## Supermaximal vector spaces

- 0-*thin* space $V$ is also called *supermaximal:* for every $W \in \mathcal{L}(V_\infty)$,

$$V \subseteq W \subseteq V_\infty \Rightarrow [\dim \frac{W}{V} < \infty \vee W = V_\infty]$$

- (Kalantari and Retzlaff) Supermaximal subspaces exist.

- (Hird) A space $V$ is called *strongly supermaximal* iff $\dim \frac{V_\infty}{V} = \infty$ and for every c.e. subset $X \subseteq V_\infty - V$:

$$(\exists a_0, \ldots, a_{n-1} \in V_\infty)[X \subseteq cl(V \cup \{a_0, \ldots, a_{n-1}\})]$$

- (Downey and Hird) Strongly supermaximal subspaces exist.

- Every strongly supermaximal space $V$ is supermaximal.
  The converse is not true.

- (Downey and Hird)
  Every nonzero c.e. Turing degree contains two strongly supermaximal subspaces, $U$ and $V$, such that for every automorphism $\Phi$ of $\mathcal{L}(V_\infty)$:

$$\Phi(U) \neq V$$

## Principal filters of quasimaximal spaces

- Let $\Omega$ be a computable basis of $V_\infty$.

  Let $B$ be a quasimaximal subset of $\Omega$ of rank $n > 1$.
  Let $V = cl(B)$.

- (Dimitrov) $\mathcal{L}^*(B, \uparrow)$ is isomorphic to one of the following:
  (1) Boolean algebra $\mathbf{B}_n$,

  (2) the lattice of all subspaces of an $n$-dimensional space over a corresponding field (to be described later),

  (3) a finite product of structures from the previous two cases.

- “Suitable fields” are of independent interest and related to effective products previously studied by:

  S. Feferman, D. Scott, and S. Tennenbaum

  M. Lerman

  Y. Hirshfeld and W. Wheeler

  T. McLaughlin

## Cohesive powers of computable structures

- Let $\mathcal{A}$ be a computable structure for $L$ with domain $A$, and let $C \subseteq \omega$ be a cohesive set.

  The *cohesive power of* $\mathcal{A}$ *over* $C$, denoted by $\prod_{C} \mathcal{A}$, is a structure $\mathcal{B}$ for $L$ with domain $B = (D \,/ =_C)$, where

  $D = \{\varphi \mid \varphi : \omega \to A$ is partial computable and $C \subseteq^* dom(\varphi)\}$.

  For $\varphi_1, \varphi_2 \in D$:

  $$\varphi_1 =_C \varphi_2 \quad \text{iff} \quad C \subseteq^* \{x : \varphi_1(x) \downarrow = \varphi_2(x) \downarrow\}$$

  The equivalence class of $\varphi$ is denoted by $[\varphi]_C$, or simply by $[\varphi]$.

- If $f \in L$ is an $n$-ary function symbol, then

$$f^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_n]) = [\varphi],$$

where for every $x \in \omega$,

$$\varphi(x) \simeq f^{\mathcal{A}}(\varphi_1(x), \ldots, \varphi_n(x))$$

- If $P \in L$ is an $m$-ary predicate symbol, then

$$P^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_m]) \quad \text{iff} \quad C \subseteq^* \{x \in \omega \mid P^{\mathcal{A}}(\varphi_1(x), \ldots, \varphi_m(x))\}$$

- If $c \in L$ is a constant symbol, then $c^{\mathcal{B}}$ is the equivalence class of the computable function with constant value $c^{\mathcal{A}}$.

- If $C$ is co-c.e., then for every $[\varphi] \in \prod_C \mathcal{A}$ there is a computable function $f$ such that $f =_C \varphi$.

- For a finite structure $\mathcal{A}$, we have $\prod_C \mathcal{A} \cong \mathcal{A}$.

  **Proof**. Let $[\varphi] \in \prod_C \mathcal{A}$.
  For $a \in A$, let $X_a = \{x \in dom(\varphi) : \varphi(x) = a\}$.
  Since $A$ is finite and $C \subseteq^* dom(\varphi)$, for some $a_0 \in A$, $X_{a_0} \cap C$ is infinite.
  Since $C$ is cohesive and $X_{a_0}$ is c.e., we have $C \subseteq^* X_{a_0}$.
  Thus, $[\varphi] = [\varphi_{a_0}]$, where $(\forall x)[\varphi_a(x) =_{\text{def}} a]$.

  The canonical embedding: $a \rightarrow [\varphi_a]$ is an isomorphism.

- **Theorem** (Dimitrov)

  (i) If $\alpha(y_1, \ldots, y_n)$ is a formula in $L$, which is a Boolean combination of $\mathbf{\Sigma}^0_1$ (or $\mathbf{\Pi}^0_1$) formulas, then

  $$\prod_C \mathcal{A} \models \alpha([\varphi_1], \ldots, [\varphi_n]) \text{ iff } C \subseteq^* \{x : \mathcal{A} \models \alpha(\varphi_1(x), \ldots, \varphi_n(x))\}$$

  (ii) If $\sigma$ is a $\mathbf{\Pi}^0_2$ (or $\mathbf{\Sigma}^0_2$) sentence in $L$, then

  $$\prod_C \mathcal{A} \models \sigma \quad \text{iff} \quad \mathcal{A} \models \sigma$$

  (iii) If $\sigma$ is a $\mathbf{\Pi}^0_3$ sentence in $L$, then

  $$\text{if } \prod_C \mathcal{A} \models \sigma \quad \text{then} \quad \mathcal{A} \models \sigma$$

- The structures $\prod_C \mathbb{Q}$ and $\mathbb{Q}$ are not elementary equivalent.

  **Proof idea**. Consider the sentence

$$\forall x \exists s \forall e \leq x\ [\varphi_e(x) \downarrow \Rightarrow \varphi_{e,s}(x) \downarrow]$$

- The transcendence degree of $\prod_C \mathbb{Q}$ over $\mathbb{Q}$ is infinite.

  **Proof idea**. Let $2 = p_1 < p_2 < \cdots$ be the sequence of all primes. Define $\psi_i : \omega \to \mathbb{Q}$ for $i \geq 1$ by:

$$\psi_i(n) = p_i^n$$

  Then the set $\{[\psi_i] : i \geq 1\}$ of elements of $\prod_C \mathbb{Q}$ is algebraically independent over $\mathbb{Q}$.

- $X \leq_m Y$ if there is a computable function $f : \omega \to \omega$ such that

$$x \in X \Leftrightarrow f(x) \in Y$$

  $f(X) \subseteq Y \wedge f(\overline{X}) \subseteq \overline{Y}$

- $X \leq_1 Y$ if there is such $1-1$ function $f$.

- The sets $X$ and $Y$ have the same $m$-*degree*, denoted by

$$X \equiv_m Y$$

  iff $X \leq_m Y$ and $Y \leq_m X$.

  Similarly, 1-*degree*: $X \equiv_1 Y$

- $X \equiv_1^* Y$ iff there are $P =^* X$ and $R =^* Y$ such that $P \equiv_1 R$.

- (Myhill's Isomorphism Theorem)

  $X \equiv_1^* Y$ iff there is a computable permutation $\sigma$ of $\omega$ such that $\sigma(X) =^* Y$.

- **Fact**. $M_1 \equiv_m M_2$ iff $M_1 \equiv_1^* M_2$ where $M_1, M_2$ are maximal sets.

## Isomorphisms of cohesive powers

(Dimitrov, Harizanov, R. Miller, and Mourad)

Let $M_1, M_2 \subseteq \omega$ be maximal sets. Consider field $\mathbb{Q}$.

- **Theorem.** $\prod_{\overline{M_1}} \mathbb{Q} \cong \prod_{\overline{M_2}} \mathbb{Q}$ iff $deg_m(M_1) = deg_m(M_2)$

- **Theorem.** The cohesive power $\prod_{\overline{M_1}} \mathbb{Q}$ has only the trivial automorphism (i.e., it is rigid).

  The proof uses a recent result by Koenigsmann that $\mathbb{Z}$ is $\forall$-definable in $\mathbb{Q}$ (in the language of rings $\{+, \cdot, 0, 1\}$).

- Koenigsmann proved that there is a polynomial $k \in \mathbb{Z}[t, x_1, \ldots, x_{418}]$ such that

$$t \in \mathbb{Z} \Leftrightarrow \mathbb{Q} \models \forall x_1 \cdots \forall x_{418} \; [k(t, x_1, \ldots, x_{418}) \neq 0]$$

- Previously, Poonen had a $\forall\exists$-definition with 2 universal and 7 existential quantifiers.

- It is still open whether $\mathbb{Z}$ is existentially definable in $\mathbb{Q}$.

## Application of cohesive power results to $\mathcal{L}^*(V_\infty)$

(assuming $V_\infty$ is over $\mathbb{Q}$)

- Let $V$ be spanned by a rank $n$ quasimaximal subset of a computable basis of $V_\infty$. Assume $n \geq 3$.
  Consider an isomorphism type of $\mathcal{L}^*(V, \uparrow)$, which is the lattice $L(n, \prod_C \mathbb{Q})$ of all subspaces of an $n$-dimensional space over a cohesive power of $\mathbb{Q}$.

- These principal filters fall into infinitely many non-isomorphic classes, even when the filters are isomorphic to the lattices of subspaces of finite dimensional vector spaces of the same dimension.

- Every automorphism of $\mathcal{L}^*(V, \uparrow) \cong L(n, \prod_C \mathbb{Q})$ can be extended to an automorphism of $\mathcal{L}^*(V_\infty)$, which is of Ash type (see below).

- (Guichard) The automorphisms of $\mathcal{L}(V_\infty)$ are induced by $1-1$ and onto *computable* semilinear transformations.
Hence there are countably many automorphisms of $\mathcal{L}(V_\infty)$.

- $(\mu, \sigma)$ is a *semilinear* transformation if $\mu : V_\infty \rightarrow V_\infty$,
$\sigma$ is an automorphism of $F$, and for every $u, v \in V_\infty$ and $a, b \in F$:

$$\mu(au + bv) = \sigma(a)\mu(u) + \sigma(b)\mu(v)$$

- **Conjecture** (Ash) The automorphisms of $\mathcal{L}^*(V_\infty)$ are induced by semilinear transformations with finite dimensional kernels and co-finite dimensional images in $V_\infty$.

## Automorphism results for $\mathcal{L}^*(V_\infty)$

(Dimitrov and Harizanov)

- **Theorem**. Let $M_1$ and $M_2$ be maximal subsets of computable bases $\Omega_1$ and $\Omega_2$ of $V_\infty$, respectively. Then there is an automorphism $\Phi$ of $\mathcal{L}^*(V_\infty)$ such that: $\Phi(cl(M_1)^*) = cl(M_2)^*$ iff $deg_m(M_1) = deg_m(M_2)$.

- We introduce the notion of a *type* of a quasimaximal set $B = \bigcap_{i=1}^{n} M_i$, which captures the number and the $m$-degrees of the maximal sets $M_i$'s.

- **Theorem**. Let $B_1$ and $B_2$ be quasimaximal subsets of computable bases $\Omega_1$ and $\Omega_2$ of $V_\infty$, respectively. There is an automorphism $\Phi$ of $\mathcal{L}^*(V_\infty)$ such that: $\Phi(cl(B_1)^*) = cl(B_2)^*$ iff $type_{\Omega_1}(B_1) = type_{\Omega_2}(B_2)$.

- **Theorem**. If a modular lattice $\mathbf{1}-\mathbf{3}-\mathbf{1}$ is a principal filter in $\mathcal{L}^*(V_\infty)$, then either all co-atoms in the filter have c.e. extendable bases, or no co-atom has a c.e. extendable basis.

  The same dichotomy holds if the modular lattice $\mathbf{1}-\infty-\mathbf{1}$ is a principal filter.

- **Corollary**. If $V_1$ and $V_2$ are two maximal spaces such that $V_1$ has an extendable c.e. basis, while no c.e. basis of $V_2$ is extendable, then

$$\mathcal{L}^*(V_1 \cap V_2, \uparrow) \cong \mathbf{B_2}$$

## Lattice $\mathcal{L}_{\mathbf{d}}(V_\infty)$ and its automorphisms

- Let $\mathcal{L}$ denote the lattice of all subspaces of $V_\infty$.

- Let $\mathcal{L}_{\mathbf{d}}(V_\infty) = \{V \in \mathcal{L} : V$ is $\mathbf{d}$-computably enumerable$\}$.

- By $GSL_{\mathbf{d}}$ we denote the group of $\mathbf{1}{-}\mathbf{1}$ and onto semilinear transformations $(\mu, \sigma)$ such that $deg(\mu) \leq \mathbf{d}$ and $deg(\sigma) \leq \mathbf{d}$.

- Every $\Phi \in Aut(\mathcal{L}_{\mathbf{d}}(V_\infty))$ is induced by some $(\mu, \sigma) \in GSL_{\mathbf{d}}$.

- If $\Phi \in Aut(\mathcal{L}_{\mathbf{d}}(V_\infty))$ is induced by $(\mu, \sigma) \in GSL_{\mathbf{d}}$ and by some other $(\mu_{\mathbf{1}}, \sigma_{\mathbf{1}}) \in GSL_{\mathbf{d}}$, then there is $\gamma \in F$ such that

$$(\forall v \in V_\infty)[\mu(v) = \gamma\mu_{\mathbf{1}}(v)]$$

- (Dimitrov, Harizanov and Morozov)

  For any pair $\mathbf{a}, \mathbf{b}$ of Turing degrees we have

$$(Aut(\mathcal{L}_{\mathbf{a}}(V_\infty)) \hookrightarrow Aut(\mathcal{L}_{\mathbf{b}}(V_\infty))) \Leftrightarrow \mathbf{a} \leq \mathbf{b}$$

## Turing degree spectrum of $GSL_{\mathbf{d}}$

- *Turing degree spectrum* of a structure $\mathcal{A}$:

$$DgSp(\mathcal{A}) = \{deg(\mathcal{B}) : \mathcal{B} \cong \mathcal{A}\}$$

- (Knight) Turing degree spectrum of a structure is either a singleton or is closed upward in the set $\mathcal{D}$ of all Turing degrees.

- (Dimitrov, Harizanov and Morozov)

$$DgSp(GSL_{\mathbf{d}}) = \{\mathbf{c} \in \mathcal{D} : \mathbf{c} \geq \mathbf{d}''\}$$

# THANK YOU!