Array noncomputability for left-c.e. reals and not totally ω-c.e. degrees

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ABSTRACT

We extend the notion of an array noncomputable c.e. set to the setting of left-c.e. reals. Given a very strong array $\mathcal{F} = \{F_n\}_{n\geq 0}$, a set (real) B is $\mathcal{F}$-compatibly left-c.e. if there is a computable approximation $\{B_s\}_{s\geq 0}$ of B such that, for any numbers x and n such that $x \in F_n$ and $x \in B_s \setminus B_{s+1}$, there is an $x' < x$ such that $x' \in F_n$ and $x' \in B_{s+1} \setminus B_s$; and a real A is $\mathcal{F}$-left-c.e.-a.n.c. if A is left-c.e. and, for any $\mathcal{F}$-compatibly left-c.e. real B, there are infinitely many numbers n such that $A \cap F_n = B \cap F_n$.

We show that, for any v.s.a. $\mathcal{F}$, there is an $\mathcal{F}$-left-c.e.-a.n.c. real A and that the (Turing and wtt) degrees of the $\mathcal{F}$-left-c.e.-a.n.c. sets coincide with the a.n.c. c.e. degrees. Moreover, we show that there are left c.e. reals which are universally left-c.e.-a.n.c., i.e., which are $\mathcal{F}$-left-c.e.-a.n.c. for all very strong arrays $\mathcal{F}$, and that the Turing degrees of the sets with this property are just the c.e. degrees which are not totally ω-c.e.

By the above results, our new notions provide a new simplified approach to dealing with left-c.e. reals in the a.n.c. and not-totally ω-c.e. degrees and give new multiple permitting properties pertaining to these degrees. We illustrate this by giving some properties of the left-c.e.-a.n.c. and universally left-c.e.-a.n.c. reals. For instance we show that there is a v.s.a. $\mathcal{F}$ and a c.e. set B such that, for any $\mathcal{F}$-left-c.e.-a.n.c. set A, there is no left-c.e. real C which is cl-above A and B whence A is not cl-reducible to any left-c.e. random real. Moreover, universally left-c.e.-a.n.c. reals are bounded random but not complex (hence not random and not wtt-complete). In fact, universally left-c.e.-a.n.c. reals are not cl-reducible to any complex left-c.e. reals. By the latter we can argue that the c.e. Turing degrees which contain left-c.e. reals which are not cl-reducible to any complex left-c.e. reals are just the not totally ω-c.e. degrees which answers a question of Noam Greenberg.

Rearrangements

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ABSTRACT

Let $\sum_n a_n$ be a conditionally convergent series of real numbers. The Riemann rearrangement theorem says that by choosing a permutation p of the natural numbers $\mathbb{N}$ appropriately, the rearranged series $\sum_n a_{p(n)}$ can be made to diverge or to converge to any prescribed real number. The rearrangement number $\mathfrak{rr}$ is the least size of a family $\mathcal{P}$ of permutations such that for every conditionally convergent $\sum_n a_n$ there is $p \in \mathcal{P}$ such that $\sum_n a_{p(n)}$ no longer converges to the same limit. We compare $\mathfrak{rr}$ to other cardinal invariants of the continuum and also discuss some of its relatives. This is joint work with A. Blass, W. Brian, J. Hamkins, M. Hardy, and P. Larson [1].

References

[1] Blass A., Brendle J., Brian W., Hamkins J., Hardy M., Larson P., *The rearrangement number*, preprint.

Lowness Notions in the C.E. Sets

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ABSTRACT

We will explore the lowness notions of low, low_2, nonhigh, semilow, $\text{semilow}_{1.5}$, the outer splitting property, and semilow_2 in the c.e. sets. For example, it is known that that every low_2 c.e. set has a maximal superset but there are lots of nonlow_2 c.e. set without maximal supersets. We show that there are semilow_2 c.e. sets without maximal supersets. We will discuss how this impacts a conjecture by Soare and others on low_2 c.e. sets.

Wadge-like classifications of real-valued functions

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ABSTRACT

Throughout this talk, we assume AD^+, which is an extension of the axiom of determinacy introduced by Woodin. Of course, if we restrict our attention to Borel sets and Baire functions, every result presented in this talk is provable within ZFC.

Recently, Day-Downey-Westrick [1] introduced the notions of m-, tt-, and T-reducibility for real-valued functions. They connected these reducibility notions with the Bourgain hierarhcy, the Kechris-Louveau ranks, etc.

In this talk, we give a full description of DDW's m-degrees, and clarify the relationship between DDW's T-degrees and the uniform Martin conjecture.

Theorem 1 *The DDW-m-degrees of real-valued functions form a semi-well-order of length* Θ*, where* Θ *is the least nonzero ordinal* α *such that there is no surjection from the reals onto* α*.*

For a limit ordinal $\alpha < \Theta$ *and finite* $n < \omega$*, the DDW-m-rank* $\alpha + 3n + c_\alpha$ *consists of two incomparable degrees, and each of the other ranks consists of a single degree, where* $c_\alpha = 2$ *if* $\alpha = 0$*;* $c_\alpha = 1$ *if the cofinality of* α *is* ω*; and* $c_\alpha = 0$ *if the cofinality of* α *is* $\geq \omega_1$*.*

Theorem 2 *The DDW-T-degrees (a.k.a. the parallel continuous Weihrauch degrees) of real-valued functions is isomorphic to the Martin ordering of the uniformly Turing degree invariant operators.*

Indeed, the identity map induces an isomorphism between the Martin ordering of the uniformly $\leq_T$*-preserving operators and the DDW-T-degrees.*

References

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Algorithmically random structures

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ABSTRACT

We present an axiomatic approach that introduces measure, and hence algorithmic randomness into various classes of structures. The central concept is the notion of a branching class. This technical yet simple notion allows one to introduce natural measure, metric, and topology into many classes of structures. These classes include graphs, trees, relational structures, and algebras. As a consequence we define and study algorithmically random structures. We prove the existence of algorithmically random structures with various computability-theoretic properties. We show that any nontrivial variety of algebras has an effective measure 0. This implies that (1) the classes of all groups and semigroups have effective measure 0 in the class of all algebras, and (2) no finitely presented algebra, such as finitely presented group, is algorithmically random. We also prove a counter-intuitive result that there are algorithmically random yet computable structures. This establishes a connection between algorithmic randomness and computable model theory. Our results exhibit important differences between algorithmically random infinite strings and algorithmically random infinite structures.

The talk is based on two papers published by the speaker in the proceedings of LICS-CSL-2014 and LFCS-2015 conferences, and the current work with D. Turetsky.

Reducibility of metrics on the real line

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ABSTRACT

Existence of effectively inequivalent computable copies of a structure (and the number of them) is a central subject of study in computable model theory. Particularly, in computable analysis, different *representations* of a given topological space and their reducibilities are of interest. In this talk we discuss the following problem: for the space of real numbers with standard topology and rationals as a dense subset how many computably inequivalent computable metrics can be constructed?

Let ρ and ρ' be complete metrics on a separable space X inducing the same topology on it, let W be a countable dense subset of X, enumerated by integers. We say that ρ is *computably reducible* to ρ' (and write $\rho \leq_c \rho'$) if Cauchy representation δ_ρ of effective metric space (X, ρ, W) is computably reducible to representation $\delta_{\rho'}$ of the space (X, ρ', W); precise definitions can be found in [1].

Obviously, $\rho \leq_c \rho'$ iff the identity homeomorphism id_X is $(\delta_\rho, \delta_{\rho'})$-computable. This motivates the following definition: ρ is weakly reducible to ρ' ($\rho \leq_{ch} \rho'$) if there is a $(\delta_\rho, \delta_{\rho'})$-computable autohomeomorphism of X.

Theorem. *All convex computable metrics on the reals are c-equivalent.*

Theorem. *Any countable tree T can be isomorphically embedded into the ordering of computable metrics on the reals under c-reducibility.*

Theorem. *There exists a countable sequence of computable metrics on $\mathbb{R}$ which are not ch-reducible to each other. Informally, copies of the real line, equipped with these metrics, are pairwise homeomorphic, but not computably homeomorphic.*

References

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Degrees of autostability relative to strong constructivizations of structures of finite signature

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ABSTRACT

Goncharov[1] investigated autostability (computable categoricity) restricted to decidable structures. Let $\mathbf{d}$ be a Turing degree. A decidable structure $\mathfrak{M}$ is $\mathbf{d}$-*autostable relative to strong constructivizations* ($\mathbf{d}$-*SC-autostable*) if for every decidable copy $\mathfrak{N}$ of $\mathfrak{M}$, there exists a $\mathbf{d}$-computable isomorphism from $\mathfrak{M}$ onto $\mathfrak{N}$. $\mathbf{0}$-SC-autostable structures are also called SC-autostable. The *autostability spectrum relative to strong constructivizations* of the structure $\mathfrak{M}$ is the set

$$\mathrm{SCAutSpec}(\mathfrak{M}) = \{\mathbf{d} : \mathfrak{M}\ \mathbf{d}\text{-SC-autostable}\}.$$

A Turing degree $\mathbf{d_0}$ is the *degree of SC-autostability* of $\mathfrak{M}$ if $\mathbf{d_0}$ is the least degree in SCAutSpec($\mathfrak{M}$).

Goncharov[1] proved that every c.e. degree $\mathbf{d}$ is the degree of SC-autostability of some decidable almost prime model of infinite signature σ_0. We will make overview of the results associated with the degrees of SC-autostability [1, 2, 3, 4] and also show that every c.e. degree $\mathbf{d}$ is the degree of SC-autostability of some decidable almost prime model of finite signature.

References

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Can we fish with Mathias forcing?

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ABSTRACT

Ramsey's theorem and its consequences received a great attention in reverse mathematics and computable analysis, due to their chaotic computability-theoretic features. Many simply stated questions became long-standing open questions and the topic quickly became a major subject of research in reverse mathematics.

Many tools were invented to answer these questions. Over the years, the technics were simplified, uniformized, and many questions were answered about Ramsey's theorem and its variants. Besides the nature of these answers, this systematic study led to the important observation that the minimalistic framework of *Mathias forcing* combined with a *CJS argument* was sufficient to give very precise and even often optimal answers to most questions.

This observation can be considered as a sign of maturity of the domain. From a personal point of view, the ultimate goal of a field is not to have all the questions answered, but to find a general technic which would enable one to answer easily any further question that one might have, following the famous proverb "give a man a fish and you feed him for a day; teach a man to fish and you feed him for a lifetime". However, some open questions seem to resist this framework, and in particular the question of the relationship between stability and cohesiveness. Can these questions be answered by a careful use of the existing framework, or are they requesting new tools? Have we found with Mathias forcing the right framework enabling us to fish in the sea of Ramsey's theory?[a]

[a] The metaphor is dubious, and may explain why the author ended-up doing mathematics.

Cototal enumeration degrees and the skip operator

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ABSTRACT

This is a joint work with U. Andrews, H. Ganchev, R. Kuyper, S. Lempp, J. Miller and M. Soskova. The talk will be an overview on the properties of the cototal enumeration degrees which form a proper substructure of the degree structure of the enumeration degrees closed under least upper bound and the enumeration jump operator. The cototal enumeration degrees properly extend the substructure of the total enumeration degrees. A set A is cototal if it is enumeration reducible to its complement. An enumeration degree is cototal if it contains a cototal set. The skip is a monotone operator such that the skip of the set A is the uniform upper bound of the complements of all sets enumeration reducible to A. These are closely connected: A has cototal degree if and only if it is enumeration reducible to its skip. We study cototality, using the skip operator and give some examples of classes of enumeration degrees that either guarantee or prohibit cototality. The skip has many of the nice properties of the Turing jump, even though the skip of A is not always above A. In fact, there is a set that is its own double skip.

Andrews et al. [1] provide the first paper substantially focused on the cototal enumeration degrees. The skip inversion theorem proved there shows that the range of the skip operator is the upper cone with base $\mathbf{0}'$. So, not every e-degree is reducible to its skip. The e-degrees reducible to its skip are exactly the cototal degrees. The cototal enumeration degrees have many further characterizations: they are the enumeration degrees of complements of maximal independent sets for infinite computable graphs on ω [1]; McCarthy [4] gave three more characterizations — the enumeration degrees of complements of maximal antichains in $\omega^{<\omega}$, the enumeration degrees of uniformly enumeration pointed trees, the enumeration degrees of languages of minimal subshifts. Another characterization comes from computable analysis. Miller [5] introduced a reducibility between points in computable separable metric spaces. This reducibility gave rise to the structure of the continuous degrees and Miller showed that this structure also properly embeds in the structure of the enumeration degrees, and forms a proper extension of the Turing degrees. Andrews et al. [1] showed that the image of the continuous degrees is contained in the cototal enumeration degrees. Kihara and Pauly [2] extended Miller's reducibility to capture points in any represented metric space.

Recently Miller and M. Soskova [6] proved that the cototal enumeration degrees form a dense substructure of the structure of the enumeration degrees. Moreover they proved that these are the only enumeration degrees which have a good approximations in the sense of Lachlan and Shore [3].

References

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Active Learning of Classes of Recursive Functions by Ultrametric Algorithms

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ABSTRACT

We study active learning of classes of recursive functions by asking value queries about the target function f, where f is from the target class. That is, the query is a natural number x, and the answer to the query is $f(x)$. The complexity measure in this paper is the worst-case number of queries asked. We prove that for some classes of recursive functions *ultrametric active learning* algorithms can achieve the learning goal by asking *significantly fewer* queries than deterministic, probabilistic, and even nondeterministic active learning algorithms. This is the first ever example of a problem, where ultrametric algorithms have advantages over nondeterministic algorithms.