Wavelet frames via duality and constant matrix completion

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[1] Fan, Heinecke, Shen: *Duality for frames*, JFAA, 2016
[2] Fan, Ji, Shen: *Dual Gramian Analysis: Duality principle and unitary extension principle*, Math of Comp, 2016

"It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out."

- Emil Artin, *Geometric Algebra*

Duality for frames from truism

- Knowing the columns of a matrix is as good as knowing its rows

General

Dual questions

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Given system $X \subseteq H$

Analysis

$T_X^* \colon H \to \ell_2$
$\quad f \mapsto (\langle f, x \rangle)_{x \in X}$

X **frame**
$A\|f\| \leq \|T_X^* f\| \leq B\|f\| \quad \forall f$

tight $A = B = 1$

Synthesis

$T_X \colon \ell_2 \to H$
$\quad c \mapsto \sum_{x \in X} c(x)x$

X **Riesz sequence**
$A\|c\| \leq \|T_X c\| \leq B\|c\| \quad \forall c$

orthonormal $A = B = 1$

Call X and X^* **adjoint** if $T_X^* = T_{X^*}$

E.g.: X (tight) frame $\iff$ X^* (orth) Riesz sequence

Dual questions

Duality principle

X, X^* adjoint if for **matrix representation** of T_X, columns associated with X while rows associated with X^*

Analysis properties of X characterized by synthesis properties of X^*
E.g.:

- N vectors, M dimensions

$$\begin{pmatrix} x_1(1) & \cdots & x_N(1) \\ \vdots & \ddots & \vdots \\ x_1(M) & \cdots & x_N(M) \end{pmatrix}$$

- Ron-Shen fiber matrices

Dual frames

- Recall, X is
  - Bessel $\iff T_X T_X^*$ bounded
  - frame $\iff T_X T_X^*$ bounded with bounded inverse
  - **tight frame** $\iff T_X T_X^* = I$
  - Riesz sequence $\iff T_X^* T_X$ bounded with bounded inverse
  - orthonormal $\iff T_X^* T_X = I$
- Bessel X, Y are **dual frames** $:\iff T_Y T_X^* = I$

Duality Principle (cont'd)

$$T_Y T_X^* = T_{Y^*}^* T_{X^*}$$

(Mixed) **dual Gramian** of systems is (mixed) **Gramian** of adjoint systems

$$X, Y \text{ dual frames} \iff X^*, Y^* \text{ biorthonormal}$$

Abstract pre-Gramian

- Infinite matrix for $T : H' \to H$

$$Te' = \sum_{e \in \mathcal{O}} \langle Te', e \rangle e \text{ for } e' \in \mathcal{O}'$$

$$J_X := \begin{pmatrix} & \vdots & \\ \cdots & \langle x, e \rangle & \cdots \\ & \vdots & \end{pmatrix}_{e \in \mathcal{O}, x \in X} \qquad \text{if } \sum_{x \in X} |\langle x, e \rangle|^2 < \infty \quad \forall e$$

$$UJ_X c = T_X c \quad \forall c \in \ell_0(X) \qquad\qquad T_X^* U d = J_X^* d \quad \forall d \in \ell_0(\mathcal{O})$$

- Adjoint systems via **columns vs rows**: $J_{X^*} = UJ_X^* V$
- Gramian $G_{X,Y} := J_X^* J_Y$ and dual Gramian $\widetilde{G}_{X,Y} := J_X J_Y^*$

Abstract pre-Gramian

Example Casazza's R-*dual sequence*: for ONB $\{v_x\}_{x\in X}$

$$X^* := \left\{ \sum_{x\in X} \langle x, e\rangle v_x \right\}_{e\in\mathcal{O}}$$

Example Let X, Y, Z Bessel

$$J_X J_Y^* J_Z = V(J_{Z^*} J_{Y^*}^* J_{X^*})^* U$$

$\Longrightarrow T_X T_Y^* Z$ adjoint system of $T_{Z^*} T_{Y^*}^* X^*$ (think Wexler-Raz)

Wavelet systems

Wavelet systems

$$(s,t) \mapsto \int f(x) 2^{sd/2} \overline{\psi(2^s x - t)}\, dx =: \langle f, D^s E^t \psi \rangle$$

- **Scaling**: $\widehat{\varphi}(2\cdot) = \widehat{a}_0 \widehat{\varphi}$

Construct **MRA wavelet system**

$$\mathcal{X} = \{D^k E^j \psi_\ell\}_{k \in \mathbb{Z}, j \in \mathbb{Z}^d, \ell = 1, \ldots, r}$$

via finding **masks** for $\widehat{\psi}_\ell(2\cdot) = \widehat{a}_\ell \widehat{\varphi}$

MRA-wavelets with Ron-Shen

Unitary extension principle

- MRA given via mask a_0
- Find wavelet system $\mathcal{X} \longleftrightarrow$ find masks $\{a_\ell\}$
- For $\{\nu_i\}_{i=1}^{2^d} = \{0,\pi\}^d$ let

$$\mathcal{M}_a(\omega) = \begin{pmatrix} \widehat{a}_0(\omega+\nu_1) & \widehat{a}_1(\omega+\nu_1) & \dots & \widehat{a}_r(\omega+\nu_1) \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{a}_0(\omega+\nu_{2^d}) & \widehat{a}_1(\omega+\nu_{2^d}) & \dots & \widehat{a}_r(\omega+\nu_{2^d}) \end{pmatrix}$$

$$\mathcal{M}_a(\omega)\mathcal{M}_a^*(\omega) = I \implies \mathcal{X} \text{ tight frame}$$

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$$\mathcal{M}_a(\omega)\mathcal{M}_a^*(\omega) = I \Longrightarrow \mathcal{X} \text{ tight frame}$$

Mixed extension principle

$$\mathcal{X},\mathcal{Y} \text{ Bessel and } \mathcal{M}_b(\omega)\mathcal{M}_a^*(\omega) = I \Longrightarrow \mathcal{X},\mathcal{Y} \text{ dual frames}$$

- Bessel guaranteed if entries of each mask sum to zero

Example (Ron-Shen)

B-spline mask $\widehat{a}_0(\omega) = e^{-i\omega/2}\cos^m(\omega/2) \qquad m$ odd

Unitary extension principle is trig poly matrix completion

$$\mathcal{M}(\omega) = \begin{pmatrix} \widehat{a}_0(\omega) & \widehat{a}_1(\omega) & \cdots & \widehat{a}_r(\omega) \\ \widehat{a}_0(\omega+\pi) & \widehat{a}_1(\omega+\pi) & \cdots & \widehat{a}_r(\omega+\pi) \end{pmatrix}$$

$\mathcal{M}(\omega)\mathcal{M}^*(\omega) = I \quad \Longrightarrow \quad$ tight wavelet frame

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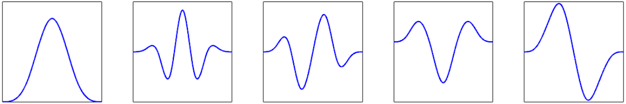
$$\mathcal{M}(\omega)\mathcal{M}^*(\omega) = I \quad \Longrightarrow \quad \text{tight wavelet frame}$$

Idea for B-splines $(\cos^2 \omega + \sin^2 \omega)^m = 1$



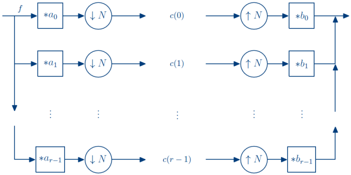
Higher dimensions.. non-tensor product.. other MRA's.. ?

UEP and duality principle

MEP in spatial domain

$$2^d \sum_{l=0}^{r} \sum_{k \in \mathbb{Z}^d} a_l(n+2k+\ell)\overline{b_l(2k+\ell)} = \delta_{n,0} \quad \forall n, \ell \in \mathbb{Z}^d$$

is **filter bank** dual frame condition.



$$X = \{(a_l(n-kN))_{n\in\mathbb{Z}^d}\}_{l\in\mathbb{Z}_r, k\in\mathbb{Z}^d}$$

$$X^* = \{(a_l(n))_{(l,n)\in\mathbb{Z}_r\times N\mathbb{Z}^d+j}\}_{j\in\mathbb{Z}^d}$$

$$G_{Y^*,X^*} = I \Longleftrightarrow \sum_{l=0}^{r-1} \sum_{n\in j+N\mathbb{Z}^d} \overline{a_l(n)}b_l(n+k) = \delta_{k,0} \quad \forall j,k \in \mathbb{Z}^d/N\mathbb{Z}^d$$

Example: Filter bank design

For **finite filters** $\{a_l\}_{l=0}^{r-1}$ and $\{b_l\}_{l=0}^{r-1} \subset \ell_2(\mathbb{Z}^d)$ let $\{n_1, \ldots, n_m\} = \mathcal{B} \cap \mathbb{Z}^d$ where $\mathcal{B}$ box containing their support

$$\begin{pmatrix} a_0(n_1) & \cdots & a_0(n_m) \\ \vdots & \ddots & \vdots \\ a_{r-1}(n_1) & \cdots & a_{r-1}(n_m) \end{pmatrix} \qquad \begin{pmatrix} b_0(n_1) & \cdots & b_0(n_m) \\ \vdots & \ddots & \vdots \\ b_{r-1}(n_1) & \cdots & b_{r-1}(n_m) \end{pmatrix}$$

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Corollary FIR filters provide perfect reconstruction filterbank with sampling rate N whenever

$$\sum_{l=0}^{r-1} \overline{a_l(n)} b_l(n') = 0 \text{ and } \sum_{l=0}^{r-1} \sum_{n \in j + N\mathbb{Z}^d} \overline{a_l(n)} b_l(n) = 1 \ \forall n \neq n',\ j \in \mathbb{Z}^d / N\mathbb{Z}^d$$

Example: Filter bank design

Construction Let $A = (a_l(n_j))_{l,j=1,\dots,r}$ invertible and $M = \text{diag}(d(n_1),\dots,d(n_r))$ such that

$$\sum_{n\in j+N\mathbb{Z}^d} d(n) = 1 \qquad \forall j \in \mathbb{Z}^d/N\mathbb{Z}^d$$

and $B = (b_l(n_j))_{l,j=1,\dots,r} = (A^*)^{-1}M$.

Then A and B define filters $\{a_l\}_{l=1,\dots,r}$ and $\{b_l\}_{l=1,\dots,r} \subset \ell_2(\mathbb{Z}^d)$ for perfect reconstruction filterbank with subsampling rate N.

Constant matrix completion

Filter bank construction + Extension principle $\Longrightarrow$

Construction Let a_0 finite ref mask with $\widehat{\varphi}_{a_0}(0) = 1$ and $\sum_{n\in j+2\mathbb{Z}^d} a_0(n) = 2^{-d} \quad \forall j \in \mathbb{Z}^d/2\mathbb{Z}^d$.

- Let a_0 define first row of A and diagonal of diagonal matrix M.
- **Primary masks** Complete A invertible, each row summing to zero
- **Dual masks** $B := (A^*)^{-1}M$

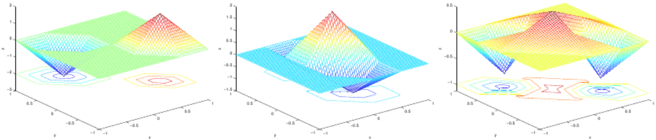
$\Longrightarrow$ **dual MRA-wavelet frames**

Example

Box spline of directions $\{(1,0)^\top, (0,1)^\top, (1,1)^\top\}$

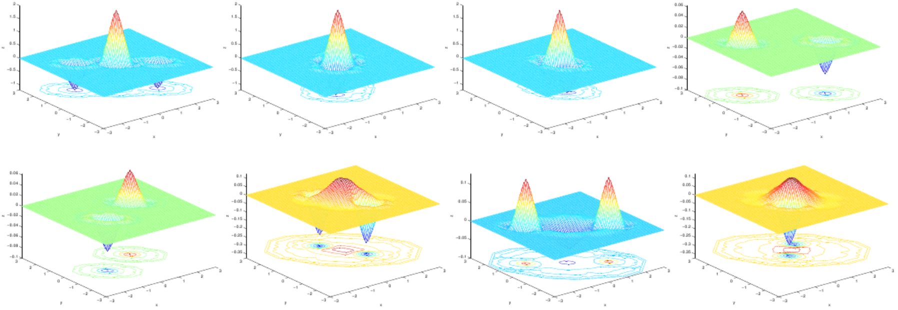
$$a_0 = \frac{1}{8}\begin{pmatrix} & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & \end{pmatrix} \qquad A = \frac{1}{8}\begin{pmatrix} 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 4 & & & & & & -4 \\ & 4 & & & & -4 & \\ & & 4 & & -4 & & \\ -2 & & & 4 & & & -2 \\ & -2 & & 4 & & -2 & \\ & & -2 & 4 & -2 & & \end{pmatrix}$$

$$B = (A^*)^{-1}\mathsf{diag}(a_0) = \frac{1}{16}\begin{pmatrix} 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 2 & & & & & & -2 \\ & 2 & & & & -2 & \\ & & 2 & & -2 & & \\ -3 & 1 & 1 & 2 & 1 & 1 & -3 \\ 1 & -3 & 1 & 2 & 1 & -3 & 1 \\ 1 & 1 & -3 & 2 & -3 & 1 & 1 \end{pmatrix}$$



Example: Butterfly subdivision

$$
\frac{1}{64}\begin{pmatrix}
 & & & & -1 & -1 & \\
 & & -1 & & 2 & & -1 \\
 & -1 & 2 & 8 & 8 & 2 & -1 \\
 & & 8 & 16 & 8 & & \\
-1 & 2 & 8 & 8 & 2 & -1 & \\
-1 & & 2 & & -1 & & \\
 & -1 & -1 & & & &
\end{pmatrix}
$$



Constant matrix completion

Theorem For any MRA of $L_2(\mathbb{R}^d)$ from finite real mask a_0 s.t. $\widehat{\varphi}_{a_0}(0) = 1$ and $\sum_{n \in j + 2\mathbb{Z}^d} a_0(n) = 2^{-d} \quad \forall j \in \mathbb{Z}^d / 2\mathbb{Z}^d$, there exist dual wavelet frames with:

- Number of primary/dual wavelets one less than support size of ref mask
- Support of wavelet masks contained in any box containing support of ref mask

Simpler adaption for **tight** case if refinement mask has nonnegative entries

Assumption is mild:

Sufficient that $\widehat{a}_0(0) = 1$ and $\widehat{a}_0(j\pi) = 0 \quad \forall j \in \mathbb{Z}^d / 2\mathbb{Z}^d \setminus \{0\}$

Gabor systems

General Gabor systems

Short-time F-trafo: $(t,\omega) \mapsto \langle f, M^\omega E^t \varphi \rangle := \int f(x) e^{-i\omega\cdot x} \overline{\varphi(x-t)}\, dx$

$X = \{E^\gamma M^\eta \varphi\}_{(\gamma,\eta)\in\Lambda}$ for **unstructured** discrete $\Lambda \subset \mathbb{R}^{2d}$

- Available structure: shift-modulation

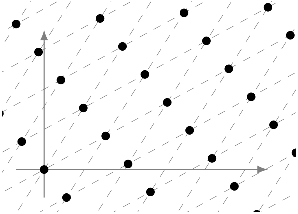
$$J_X = |\widetilde{K}|^{-1/2} \left(e^{i\eta\cdot(\tilde{k}-\gamma)} \langle M^{\eta-k} E^{\gamma-\tilde{k}} \varphi, \chi_{\Omega_{\widetilde{K}}} \rangle \right)_{(k,\tilde{k})\in K\times\widetilde{K},(\gamma,\eta)\in\Lambda}$$

Rows $\Longrightarrow$ adjoint system

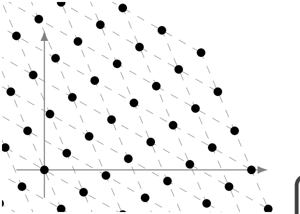
- Choose bijection $R\colon \Lambda \to K \times \widetilde{K}$ and ONB to map back to function space.
Duality principle with **adjoint system**: $(k,\tilde{k}) \in K \times \widetilde{K}$

$$f_{k,\tilde{k}} = |\widetilde{K}|^{-1} \sum_{(\gamma,\eta)\in\Lambda} e^{i\eta\cdot(\tilde{k}-\gamma)} \langle M^{\eta-k} E^{\gamma-\tilde{k}} \varphi, \chi_{\Omega_{\widetilde{K}}} \rangle M^{R\gamma} E^{R\eta} \chi_{\Omega_{\widetilde{K}}}$$

Regular Gabor systems (Ron-Shen)



K



$\widetilde{K}$

If more structure:

$$X = \{E^k M^l \varphi\}_{(k,l)\in K\times L} \text{ for } \textbf{lattices } K, L \subset \mathbb{R}^d$$

- Fiber pre-Gramians of Ron-Shen

$$((T_X c)^\wedge(\omega - \tilde{k}))_{\tilde{k}\in\widetilde{K}} = \mathcal{J}_X(\omega)\hat{c}(\omega)$$

$$\mathcal{J}_X(\omega) := \Big(\hat{\varphi}(\omega - \tilde{k} - l)\Big)_{\tilde{k}\in\widetilde{K}, l\in L}$$

- Adjoint system has Gabor structure

$$X^* = (\text{den}\,(K,L))^{1/2}\{E^{\tilde{l}} M^{\tilde{k}} \varphi\}_{(\tilde{l},\tilde{k})\in\widetilde{L}\times\widetilde{K}}$$

$$\mathcal{J}_X(\omega)\hat{c}(\omega) = |K|^{1/2}(J_X c)^\wedge(\omega)$$

DGA for regular Gabor systems

Let $X = (K, L)_\phi$ and $Y = (K, L)_\psi$ Bessel

$$\widetilde{\mathcal{G}}_{Y,X}(\omega) := \mathcal{J}_Y(\omega)\mathcal{J}_X^*(\omega) = |K|^{-1}\left(\sum_{l \in L} E^{\tilde{k}+l}\hat{\psi}(\omega)\overline{E^{\tilde{k}'+l}\hat{\phi}(\omega)}\right)_{\tilde{k},\tilde{k}' \in \widetilde{K}}$$

- **Fourier domain (Ron-Shen)**

$$((T_Y T_X^* f)^\wedge(\omega - \tilde{k}))_{\tilde{k} \in \widetilde{K}} = \widetilde{\mathcal{G}}_{Y,X}(\omega)(\hat{f}(\omega - \tilde{k}))_{\tilde{k} \in \widetilde{K}}$$

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- **Time domain** Letting $\widehat{X} = (L, K)_{\hat{\phi}}$ and $\widehat{Y} = (L, K)_{\hat{\psi}}$

$$((T_Y T_X^* f)(-\omega + \tilde{l}))_{\tilde{l}\in\widetilde{L}} = (2\pi)^{-d}\widetilde{\mathcal{G}}_{\widehat{Y},\widehat{X}}(\omega)(f(-\omega + \tilde{l}))_{\tilde{l}\in\widetilde{L}}$$

View on classical duality identities

Let $X = (K, L)_\phi$, $Y = (K, L)_\psi$, $Z = (K, L)_g$ Bessel

- **Walnut representation**

$$T_Y T_X^* f = |\widetilde{L}| \sum_{\tilde{l} \in \widetilde{L}} \sum_{k \in K} E^k \psi \overline{E^{\tilde{l}+k} \phi} E^{\tilde{l}} f$$

is 0-entry of DGA time domain representation

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- **Wexler-Raz biorthogonality relations**

$$(\text{den}\,(K, L)) \langle \psi, E^{\tilde{l}} M^{\tilde{k}} \phi \rangle = \delta_{\tilde{l},0} \delta_{\tilde{k},0} \quad \text{for all } (\tilde{l}, \tilde{k}) \in \widetilde{L} \times \widetilde{K}$$

are one aspect of duality principle $\widetilde{\mathcal{G}}_{X,Y}(\omega) = \overline{\mathcal{G}_{X^*,Y^*}(\omega)}$

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- **Janssen/Wexler-Raz identity**

$$T_Y T_Z^* \phi = T_{X^*} T_{Z^*}^* \psi$$

from $(0, 0)$-entry of $\mathcal{J}_Y(\omega) \mathcal{J}_Z^*(\omega) \mathcal{J}_X(\omega) = (\overline{\mathcal{J}_{X^*}(\omega) \mathcal{J}_{Z^*}^*(\omega) \mathcal{J}_{Y^*}(\omega)})^*$

View on painless constructions

Daubechies-Grossman-Meyer, Ron-Shen, ...:

- Guarantee orthogonality by disjointness of support
- Let $X = (K, L)_\phi$, $Y = (K, L)_\psi$ Bessel

$$
\begin{aligned}
X, Y \text{ dual frames } &\iff \text{pre-Gramians have } \textbf{biorthogonal rows} \\
&\iff |K|^{-1} \sum_{l \in L} E^l \hat{\phi} E^{\tilde{k}+l} \overline{\hat{\psi}} = \delta_{\tilde{k},0} \quad \forall \tilde{k} \in \widetilde{K} \\
&\iff |\widetilde{L}| \sum_{k \in K} E^k \phi E^{k+\tilde{l}} \overline{\psi} = \delta_{\tilde{l},0} \quad \forall \tilde{l} \in \widetilde{L}
\end{aligned}
$$