Sparse Approximation: from Image Restoration to High Dimensional Classification

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Workshop on Frame Theory and Sparse Representation for Complex Data

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Outlines

I. Brief review of image restoration models

II. Wavelet frame transforms and differential operators under variational and PDE framework

III. Sparse approximation for high-dimensional data classification

IV. Conclusions and Future work

Image Restoration Model

- Image Restoration Problems

$$f = Au + \eta$$

  - Denoising, when A is identity operator
  - Deblurring, when A is some blurring operator
  - Inpainting, when A is some restriction operator
  - CT/MR Imaging, when A is partial Radon/Fourier transform

- Chal ... ill-posed

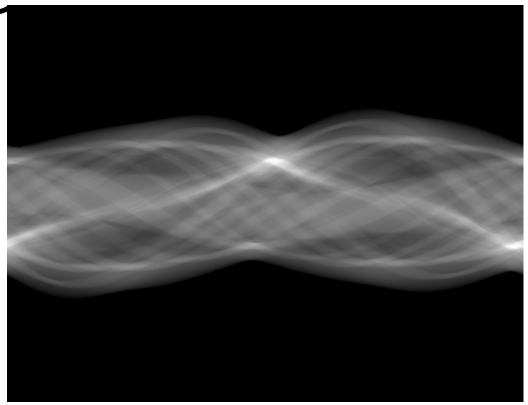


Image Restoration Models: A Quick Review

- Image restoration: $f = Au + \eta$
- Variational and Optimization Models

$$\min_u \lambda R(u) + \|Au - f\|^2$$

  - Total variation (TV) and generalizations: $R(u) = \|\nabla u\|_1 \quad \text{or} \quad \|\boldsymbol{D}u\|_1$
  - Wavelet frame based: $R(u) = \|Wu\|_1 \quad \text{or} \quad \|Wu\|_0$
  - Others: total generalized variation, low rank, NLM, BM3D, dictionary learning, etc.
- PDEs and Iterative Algorithms
  - Perona-Malik equation, shock-filtering (Rudin & Osher), etc

$$u_t = \sum_{\ell=1}^{L} \frac{\partial^{\boldsymbol{\alpha}_\ell}}{\partial x^{\boldsymbol{\alpha}_\ell}} \Phi_\ell(\boldsymbol{D}u, u) - A^*(Au - f), \qquad \text{with } \boldsymbol{D} = \left(\frac{\partial^{\boldsymbol{\beta}_1}}{\partial x^{\boldsymbol{\beta}_1}}, \ldots, \frac{\partial^{\boldsymbol{\beta}_L}}{\partial x^{\boldsymbol{\beta}_L}}\right)$$

  - Iterative shrinkage algorithm

$$\boldsymbol{u}^k = \widetilde{\boldsymbol{W}}^\top \boldsymbol{S}_{\boldsymbol{\alpha}^{k-1}}(\boldsymbol{W}\boldsymbol{u}^{k-1}) - \boldsymbol{A}^\top(\boldsymbol{A}\boldsymbol{u}^{k-1} - \boldsymbol{f}), \quad k = 1, 2, \cdots$$

- What do they have in common?

“Dong and Shen, *Image restoration: a data-driven perspective*, Proceedings of the International Congress of Industrial and Applied Mathematics (ICIAM), 2015”

<u>Shrinkage</u> in <u>sparse</u> domain under <u>transformation</u>!

Bridging discrete and continuum

WAVELET FRAME TRANSFORMS AND DIFFERENTIAL OPERATORS

❖ J. Cai, B. Dong, S. Osher and Z. Shen, *Image restoration: total variation; wavelet frames; and beyond*, **Journal of American Mathematical Society**, 25(4), 1033-1089, 2012.

❖ B. Dong, Q. Jiang and Z. Shen, *Image restoration: wavelet frame shrinkage, nonlinear evolution PDEs, and beyond*, MMS, 15(1)， 606-660, 2017.

❖ Jian-Feng Cai, B. Dong and Zuowei Shen, *Image restorations: a wavelet frame based model for piecewise smooth functions and beyond*, **Applied and Computational Harmonic Analysis**, 41(1), 94-138, 2016.

❖ Bin Dong, Zuowei Shen and Peichu Xie, Image restoration: a general wavelet frame based model and its asymptotic analysis, **SIAM Journal on Mathematical Analysis**, 49(1), 421-445, 2017.

MRA-Based Tight Wavelet Frames

- Refinable and wavelet functions

$$\phi = 2^d \sum \boldsymbol{a}_0[\boldsymbol{k}]\phi(2\cdot - \boldsymbol{k}) \quad \psi_\ell = 2^d \sum \boldsymbol{a}_\ell[\boldsymbol{k}]\phi(2\cdot - \boldsymbol{k}), \quad \ell = 1, 2, \ldots, q.$$

- Unitary extension principle (UEP)

$$\sum_{\ell=0}^{q} |\widehat{\boldsymbol{a}}_\ell(\xi)|^2 = 1 \quad \text{and} \quad \sum_{\ell=0}^{q} \widehat{\boldsymbol{a}}_\ell(\xi)\overline{\widehat{\boldsymbol{a}}_\ell(\xi+\nu)} = 0,$$

$$\nu \in \{0,\pi\}^d \setminus \{\boldsymbol{0}\} \text{ and } \xi \in [-\pi,\pi]^d$$

- Discrete 2D transformation: $\boldsymbol{W}\boldsymbol{u} = \{\boldsymbol{W}_{l,\boldsymbol{i}}\boldsymbol{u} : 0 \le l \le L-1, 0 \le i_1, i_2 \le r\}$

$$\boxed{\boldsymbol{W}_{l,\boldsymbol{i}}\boldsymbol{u} := \boldsymbol{a}_{l,\boldsymbol{i}}[-\cdot] \circledast \boldsymbol{u},}$$

$$\boldsymbol{a}_{\boldsymbol{i}}[\boldsymbol{k}] := \boldsymbol{a}_{i_1}[k_1]\boldsymbol{a}_{i_2}[k_2], \quad 0 \le i_1, i_2 \le r; \ (k_1,k_2) \in \mathbb{Z}^2.$$

$$\boldsymbol{a}_{l,\boldsymbol{i}} = \tilde{\boldsymbol{a}}_{l,\boldsymbol{i}} \circledast \tilde{\boldsymbol{a}}_{l-1,0} \circledast \ldots \circledast \tilde{\boldsymbol{a}}_{0,0} \quad \text{with} \quad \tilde{\boldsymbol{a}}_{l,\boldsymbol{i}}[\boldsymbol{k}] = \begin{cases} \boldsymbol{a}_{\boldsymbol{i}}[2^{-l}\boldsymbol{k}], & \boldsymbol{k} \in 2^l\mathbb{Z}^2; \\ 0, & \boldsymbol{k} \notin 2^l\mathbb{Z}^2. \end{cases}$$

- Perfect reconstruction: $\boldsymbol{W}^\top \boldsymbol{W} = \boldsymbol{I}$

- Further reading: **[Dong and Shen, *MRA-Based Wavelet Frames and Applications*, IAS Lecture Notes Series,2011]**

Connections: Motivation

- Difference operators in wavelet frame transform:

Haar Filters $h_{0,1} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, h_{1,0} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, h_{1,1} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

Transform $Wu = \{h_{0,1}[-\cdot] \circledast u; h_{1,0}[-\cdot] \circledast u; h_{1,1}[-\cdot] \circledast u\}$

Approximation

$$h_{0,1}[-\cdot] \circledast u \approx \frac{1}{2}\delta u_x, \quad h_{1,0}[-\cdot] \circledast u \approx \frac{1}{2}\delta u_y, \quad h_{1,1}[-\cdot] \circledast u \approx \frac{1}{4}\delta^2 u_{xy}$$

- Thus, $\boxed{\frac{2}{\delta} Wu \approx \nabla u}$

$$|\nabla u| \approx \left(\frac{1}{4}\left[(D_x^+ u_{i,j})^2 + (D_x^+ u_{i,j+1})^2 + (D_y^+ u_{i,j})^2 + (D_y^+ u_{i+1,j})^2\right] + \left[\frac{(D_x^+ u_{i,j} + D_y^- u_{i,j+1})^2}{4} + \frac{(D_x^+ u_{i,j} + D_y^+ u_{i+1,j})^2}{4}\right]\right)^{\frac{1}{2}}$$

- More rigorously **[Choi, Dong and Zhang, preprint, 2017]**

Proposition 2.2. *Let a tensor product framelet function $\psi_{\boldsymbol{\alpha}} \in L_2(\mathbb{R}^2)$ have vanishing moments of order $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| \le s$, and let $\mathrm{supp}(\psi_{\boldsymbol{\alpha}}) = [a_1, a_2] \times [b_1, b_2]$. For $n \in \mathbb{N}$ and $\boldsymbol{k} \in \mathbb{Z}^2$ with $\mathrm{supp}(\psi_{\boldsymbol{\alpha},n-1,\boldsymbol{k}}) \subseteq \overline{\Omega}$, we have*

$$\langle u, \psi_{\boldsymbol{\alpha},n-1,\boldsymbol{k}} \rangle = (-1)^{|\boldsymbol{\alpha}|} 2^{|\boldsymbol{\alpha}|(1-n)} \langle \partial^{\boldsymbol{\alpha}} u, \varphi_{\boldsymbol{\alpha},n-1,\boldsymbol{k}} \rangle$$

for every $u \in W_1^s(\Omega)$.

$\int_{\mathbb{R}^2} \varphi_{\boldsymbol{\alpha}} \mathrm{d}\boldsymbol{x} \neq 0, \quad \mathrm{supp}(\varphi_{\boldsymbol{\alpha}}) = \mathrm{supp}(\psi_{\boldsymbol{\alpha}})$

Connections: Analysis Based Model and Variational Model

- **[Cai, Dong, Osher and Shen, *JAMS*, 2012]**:

$$\|\lambda W u\|_1 + \frac{1}{2}\|Au - f\|_2^2 \xrightarrow{\text{Converges}} \lambda\|D(u)\|_1 + \frac{1}{2}\|Au - f\|_{L_2(\Omega)}^2$$

For any differential operator when proper parameter is chosen.

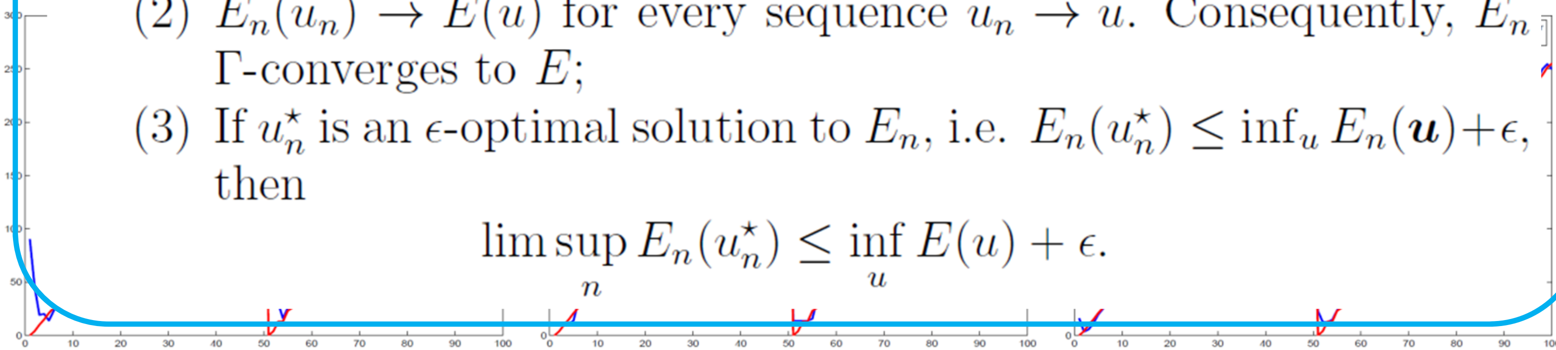
Theorem. Let the objective functionals of the analysis based model and the variational model be $E_n(u)$ and $E(u)$ respectively, then:

(1) $E_n(u) \to E(u)$ for each $u \in W_1^s(\Omega)$;

(2) $E_n(u_n) \to E(u)$ for every sequence $u_n \to u$. Consequently, E_n Γ-converges to E;

(3) If $u_n^\star$ is an ϵ-optimal solution to E_n, i.e. $E_n(u_n^\star) \le \inf_u E_n(\boldsymbol{u}) + \epsilon$, then

$$\limsup_n E_n(u_n^\star) \le \inf_u E(u) + \epsilon.$$



- Image segmentation: **[Dong, Chien and Shen, 2010]**
- Surface reconstruction from point clouds: **[Dong and Shen, 2011]**

Relations: Wavelet Shrinkage and Nonlinear PDEs

- **[Dong, Jiang and Shen, MMS, 2017]**

$$\boldsymbol{u}^k = \widetilde{\boldsymbol{W}}^\top \boldsymbol{S}_{\boldsymbol{\alpha}^{k-1}}(\boldsymbol{W}\boldsymbol{u}^{k-1}), \quad k = 1, 2, \cdots$$

$$u_t = \sum_{\ell=1}^{L} \frac{\partial^{\boldsymbol{\alpha}_\ell}}{\partial x^{\boldsymbol{\alpha}_\ell}} \Phi_\ell(\boldsymbol{D}u, u), \quad \text{with } \boldsymbol{D}u = \left(\frac{\partial^{\boldsymbol{\beta}_1}}{\partial x^{\boldsymbol{\beta}_1}}, \ldots, \frac{\partial^{\boldsymbol{\beta}_L}}{\partial x^{\boldsymbol{\beta}_L}}\right)$$

- Theoretical justification available for quasilinear parabolic equations.
- Lead to new PDE models such as:

$$u_{tt} + Cu_t = \sum_{\ell=1}^{L} (-1)^{1+|\boldsymbol{\beta}_\ell|} \frac{\partial^{\boldsymbol{\beta}_\ell}}{\partial \boldsymbol{x}^{\boldsymbol{\beta}_\ell}} \left[g_\ell\left(u, \frac{\partial^{\boldsymbol{\beta}_1} u}{\partial \boldsymbol{x}^{\boldsymbol{\beta}_1}}, \cdots, \frac{\partial^{\boldsymbol{\beta}_L} u}{\partial \boldsymbol{x}^{\boldsymbol{\beta}_L}}\right) \frac{\partial^{\boldsymbol{\beta}_\ell}}{\partial \boldsymbol{x}^{\boldsymbol{\beta}_\ell}} u \right] - \kappa A^\top (Au - f)$$

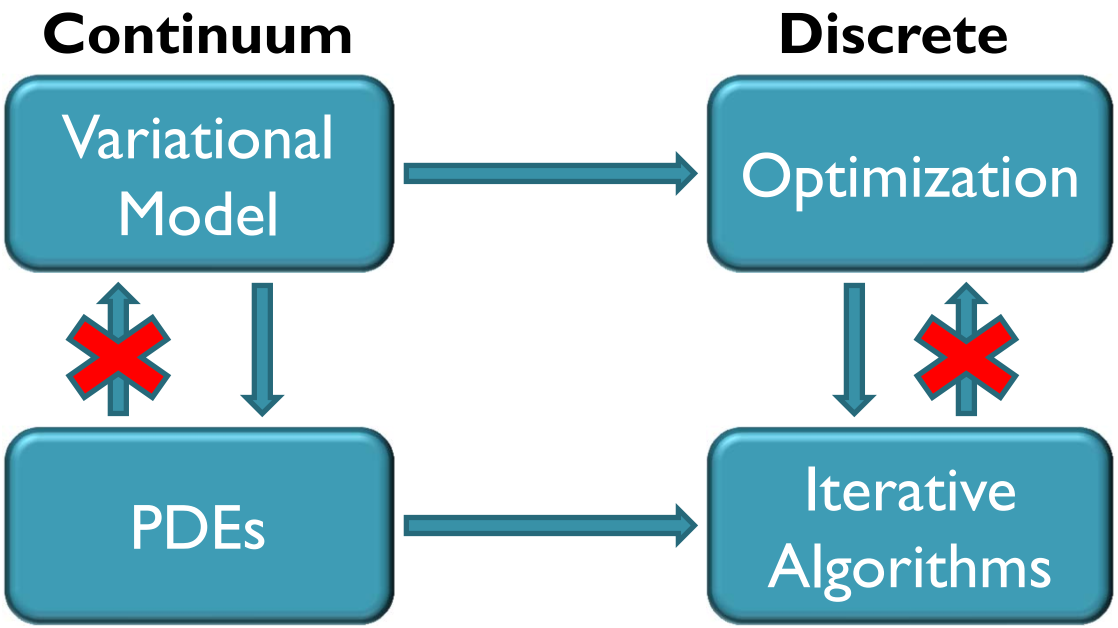
$$\boldsymbol{u}^k = (I - \mu \boldsymbol{A}^\top \boldsymbol{A}) \boldsymbol{W}^\top \boldsymbol{S}_{\boldsymbol{\alpha}^{k-1}}(\boldsymbol{W}\boldsymbol{u}^{k-1}) + \mu \boldsymbol{A}^\top \boldsymbol{f}$$

where

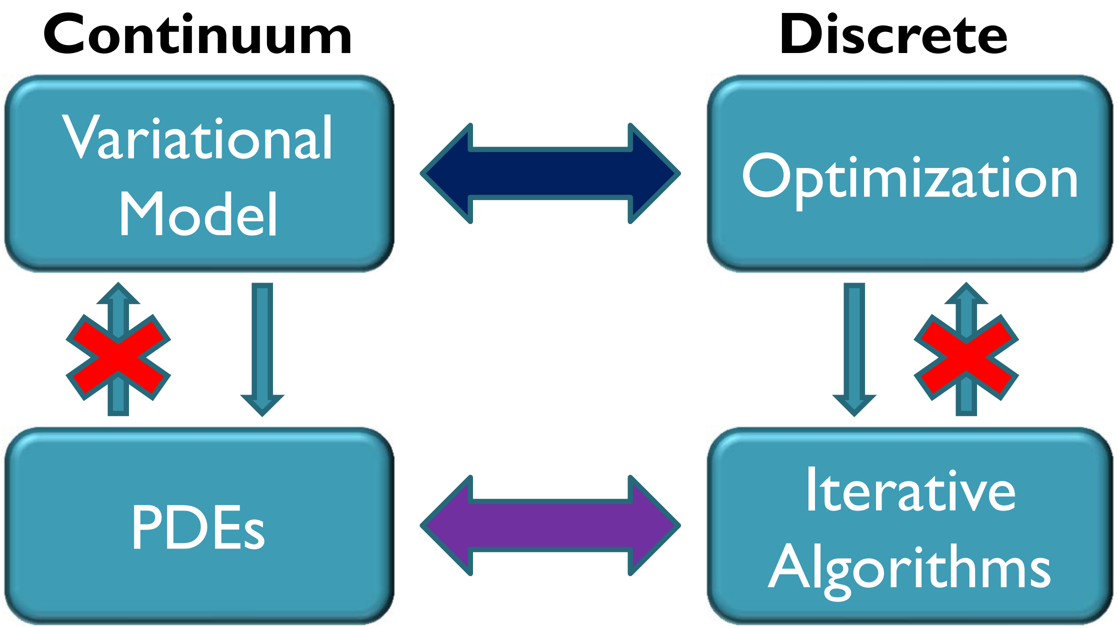
$$\boldsymbol{S}_{\boldsymbol{\alpha}^{k-1}}(\boldsymbol{W}\boldsymbol{u}^{k-1}) = \{S_{\alpha_{l,\ell,\boldsymbol{n}}(\boldsymbol{W}_l \boldsymbol{u}^{k-1})}(\boldsymbol{W}_l \boldsymbol{u}^{k-1}) : \ 0 \le l \le \text{Lev} - 1, 1 \le \ell \le L\}$$

$$S_{\boldsymbol{\alpha}_{\ell,\boldsymbol{n}}(\boldsymbol{d})}(d_{1,\boldsymbol{n}}, d_{2,\boldsymbol{n}}) = d_{\ell,\boldsymbol{n}} \left(1 - \frac{4\tau}{h^2} g\left(\frac{4(d_{1,\boldsymbol{n}})^2 + 4(d_{2,\boldsymbol{n}})^2}{h^2}\right)\right)$$

Summary


Continuum
Discrete
Variational Model
Optimization
PDEs
Iterative Algorithms

Summary



❖ J. Cai, B. Dong, S. Osher and Z. Shen, *Image restoration: total variation; wavelet frames; and beyond*, **Journal of American Mathematical Society**, 25(4), 1033-1089, 2012.
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❖B. Dong, Q. Jiang and Z. Shen, *Image restoration: wavelet frame shrinkage, nonlinear evolution PDEs, and beyond*, **Multiscale Modeling & Simulation**, 15(1)， 606-660, 2017.

SPARSE APPROXIMATION IN HIGH-DIMENSIONAL DATA CLASSIFICATION

❖ Ning Hao, Bin Dong and Jianqing Fan, *Sparsifying the Fisher Linear Discriminant by Rotation*, **Journal of the Royal Statistical Society Series B**, 2015.

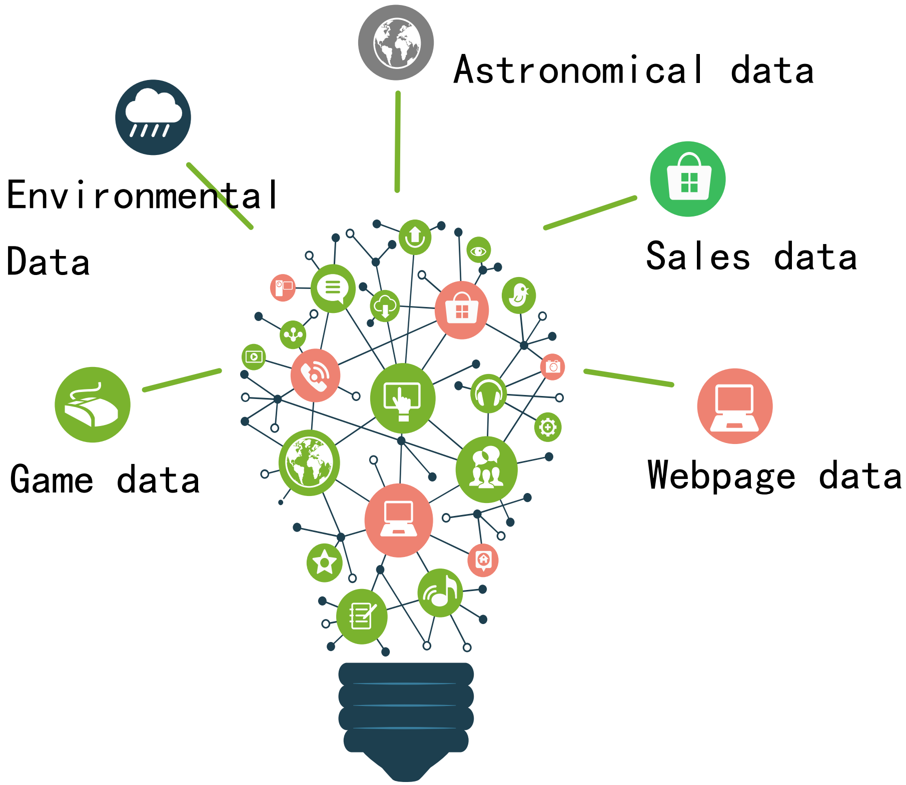
❖ Bin Dong, *Sparse Representation on Graphs by Tight Wavelet Frames and Applications*, **Applied and Computational Harmonic Analysis**, 2015.

❖ Bin Dong and Ning Hao, *Semi-supervised high dimensional clustering by tight wavelet frames*, **Proceedings of SPIE**, 2015.

Introduction

What is **data science**? Extracting knowledge from data to make intelligent observations and decisions.

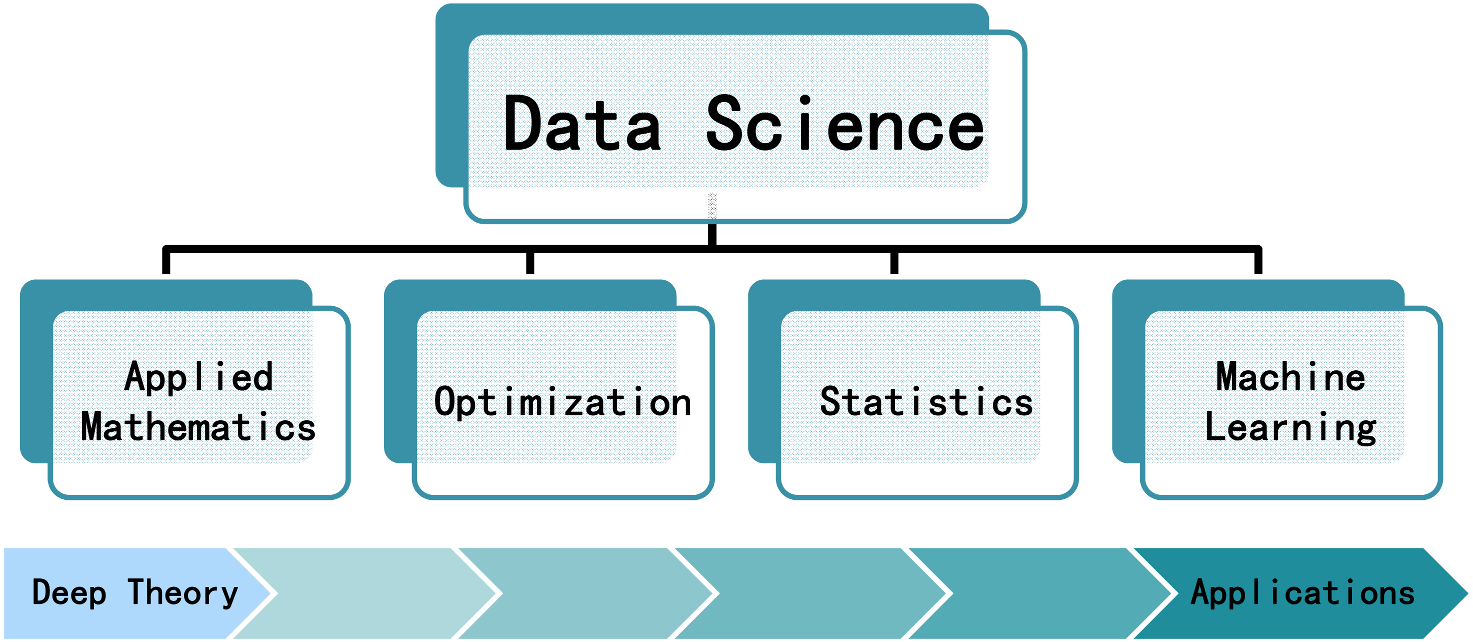
1. Broad applications



2. Variety

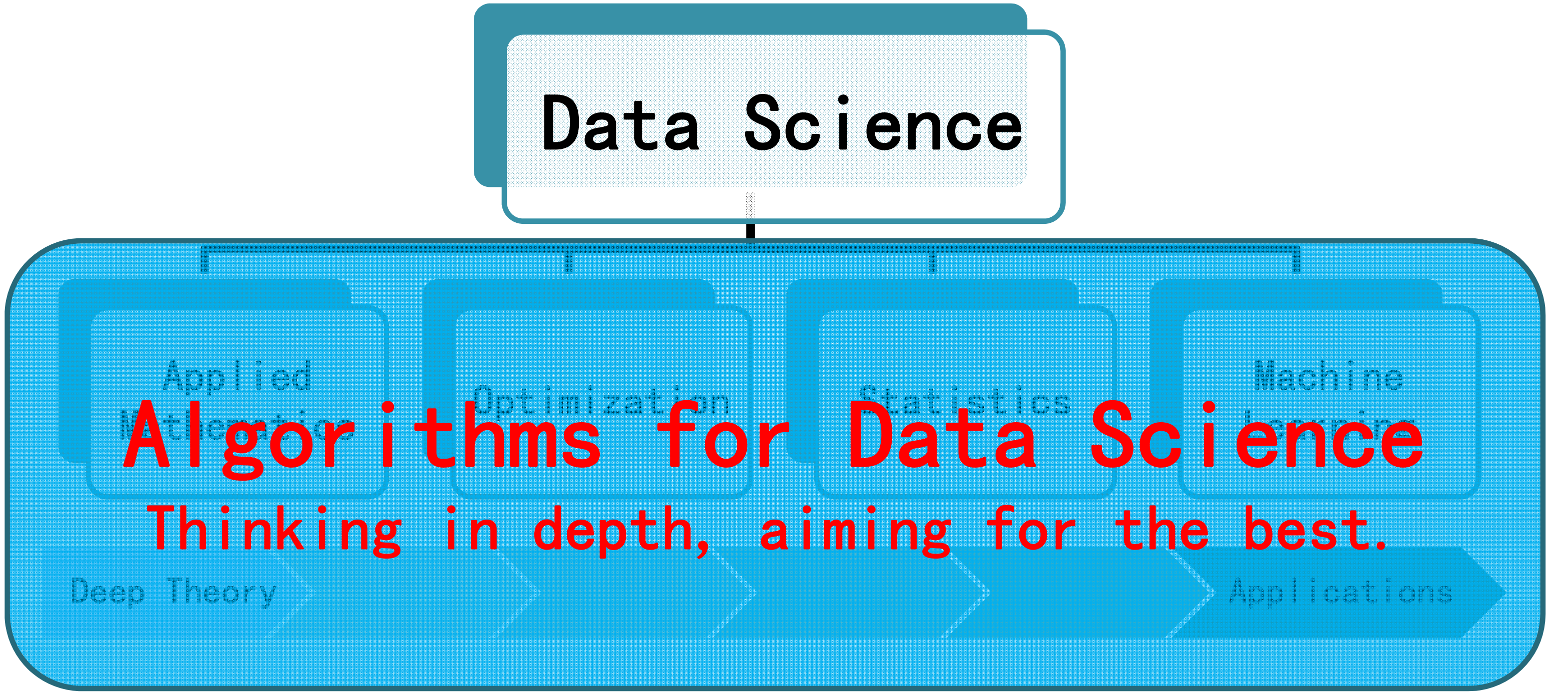
- Network data
- Webpage data
- Text
- Image
- Video
- Audio

Introduction



- Different area has different focus. Some has tight link with another.
- Broader links?

Introduction



Importance of the merge:

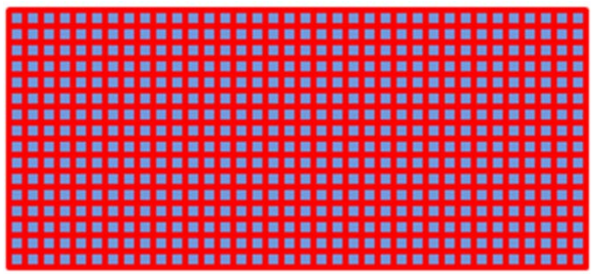
- Combining merits
- New insights on classical problems

Introduction

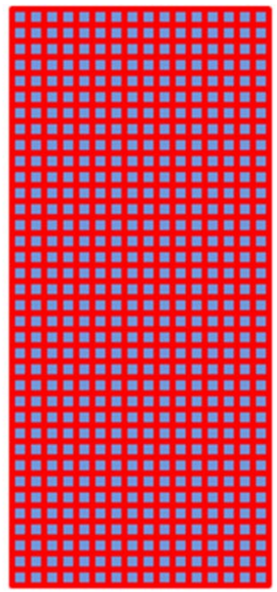
- Typical big data set: with $n \times p$ huge

$$X \in \mathbb{K}^{n \times p}, \quad \mathbb{K} = \mathbb{Z}, \mathbb{R}, \mathbb{C}, \text{ etc.}$$

- Classical v.s. modern



Classical: n<p



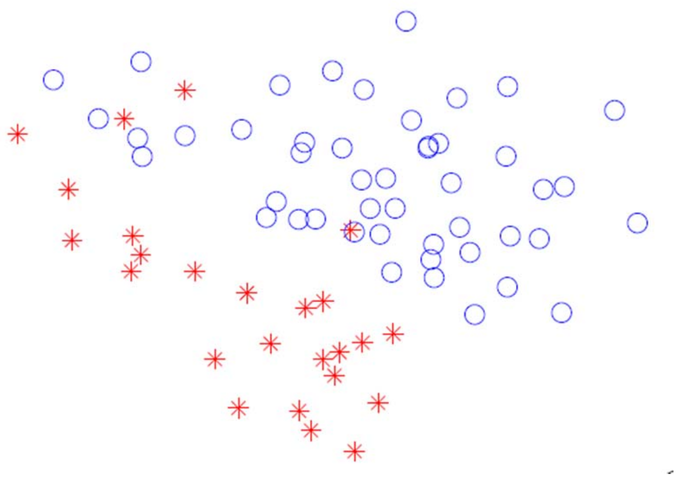
Modern: n>p

Introduction

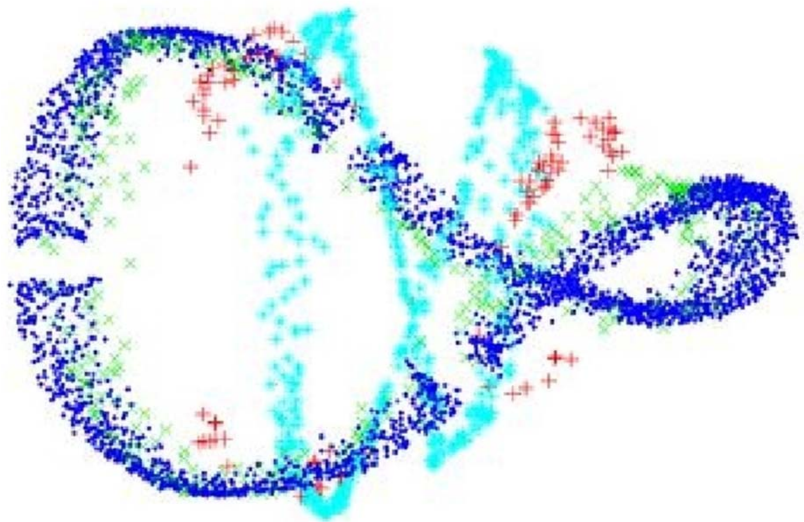
- Typical big data set: with $n \times p$ huge

$$X \in \mathbb{K}^{n \times p}, \quad \mathbb{K} = \mathbb{Z}, \mathbb{R}, \mathbb{C}, \text{ etc.}$$

- Classical v.s. modern



Classical: n<p



Modern: n>p

Introduction

- Sparsity is key
  - What is sparsity for general data sets?

    Essential information is of much lower dimension than the dimension of the data itself.

  - How do we harvest sparsity?

    Sparse under certain (nonlinear) transformation.

  - Examples:
    - PCA and its siblings
    - Low rank approximation
    - Wavelet frame transform
    - Dictionary learning
    - Isomaps, LLE, diffusion maps
    - Autoencoder
    - … etc.

Nonlinear Classification

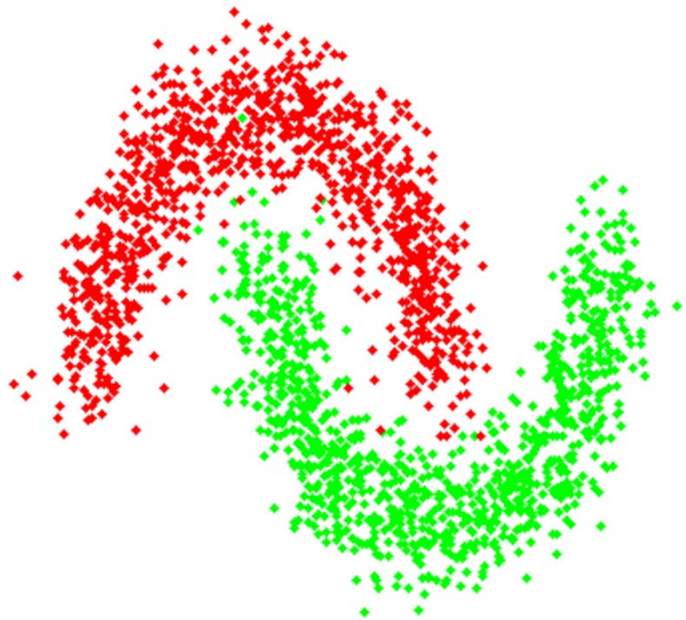
MODERN SCENARIO

❖ Bin Dong, *Sparse Representation on Graphs by Tight Wavelet Frames and Applications*, **Applied and Computational Harmonic Analysis**, 2015.

❖ Bin Dong and Ning Hao, *Semi-supervised high dimensional clustering by tight wavelet frames*, **Proceedings of SPIE**,2015.

Nonlinear Classifier

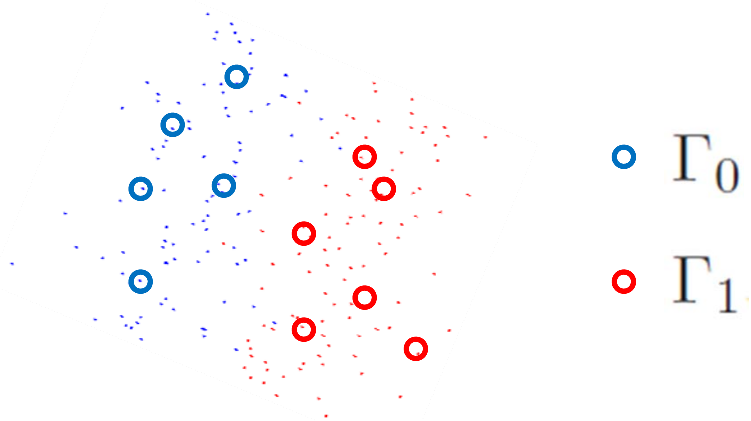
- When we have enough observations, nonlinear classifier leads to more accurate classification.



Explicit Classifier

PDE Method

- Ginzburg–Landau (GL) functional [Andrea and Flenner, 2011]



$$f[i] = \begin{cases} 0 & \text{for } i \in \Gamma_0, \\ 1 & \text{for } i \in \Gamma_1. \end{cases}$$

- GL model

$$E(u) = \frac{\epsilon}{2}\langle u, \mathcal{L}_s u\rangle + \frac{1}{4\epsilon}\|u^2 - 1\|_{2,G}^2 + \frac{\mu}{2}\|u_{|\Gamma} - f\|_{2,G}^2$$

where

$$\mathcal{L}_s = I - D^{-1/2} A D^{-1/2} \qquad \|f\|_{p,G} := \left(\sum_{k=1}^{K} |f[k]|^p d[k]\right)^{\frac{1}{p}}$$

PDE Method

- Splitting *E* to a sum of convex and concave parts

$$\boxed{E(u) = E_1(u) - E_2(u)}$$

$$E_1(u) = \frac{\epsilon}{2}\int |\nabla u(x)|^2 dx + \frac{c}{2}\int |u(x)|^2 dx,$$
$$E_2(u) = -\frac{1}{4\epsilon}\int (u(x)^2 - 1)^2 dx + \frac{c}{2}\int |u(x)|^2 dx - \int \frac{\lambda(x)}{2}(u(x) - u_0(x))^2 dx.$$

- Convex splitting scheme

$$\frac{u^{n+1} - u^n}{dt} = -\frac{\partial E_1}{\partial u}(u^{n+1}) + \frac{\partial E_2}{\partial u}(u^n)$$

❖ At each iteration, we need to solve a Laplace equation on graph.
❖ Fast graph Laplacian solver is needed, such as Nystrom's method.

Wavelet Frame Method

- Key idea: Eigenfunctions of Laplace-Beltrami operator (graph Laplacian in discrete setting) are understood as Fourier basis on manifolds (graphs in discrete setting) and the associated eigenvalues as frequency components.
- Spectrum of Laplace-Beltrami operator on $\{\mathcal{M}, g\}$

$$\Delta u + \lambda u = 0, \qquad u_{|S} = 0.$$

Eigenvalues and eigenfunctions: $0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$

$$\langle u_p, u_{p'} \rangle_{L_2(\mathcal{M})} = \int_{\mathcal{M}} u_p(x) u_{p'}^*(x) dx = \delta_{p,p'}$$

- Fourier transform $\widehat{f}[p] = \langle f, u_p \rangle_{L_2(\mathcal{M})}$
- Plancherel and Parseval's identities

$$\langle f, g \rangle_{L_2(\mathcal{M})} = \langle \widehat{f}, \widehat{g} \rangle_{\ell_2(\mathbb{Z}^+)} \quad \text{for } f, g \in L_2(\mathcal{M})$$
$$\|f\|^2_{L_2(\mathcal{M})} = \|\widehat{f}\|^2_{\ell_2(\mathbb{Z}^+)}.$$

Wavelet Frame Method

- Asymptotic properties of eigenfunctions and eigenvalues:
  - Weyl's asymptotic formula（1912）： $\lambda_p \asymp p^{\frac{2}{m}}$
  - Uniform bound (Grieser, 2002): $\|u_p\|_{L_\infty(\mathcal{M})} \leq C\lambda_p^{\frac{m-1}{4}}$

- Wavelet system (semi-continuous) on manifold $\mathcal{M}$:

$$X(\Psi) = \{\psi^{\mathcal{M}}_{j,n,y} \in L_2(\mathcal{M}) : \ 1 \leq j \leq r, n \in \mathbb{Z}, y \in \mathcal{M}\},$$

where $\psi^{\mathcal{M}}_{j,n,y} \in L_2(\mathcal{M})$ is generated by $\Psi = \{\psi_j : \ 1 \leq j \leq r\} \subset L_2(\mathbb{R})$ as

Eigenvalues Eigenfunctions

$$\psi^{\mathcal{M}}_{j,n,y}(x) = \sum_{p=0}^{\infty} \widehat{\psi}_j(2^{-n}\lambda_p)u_p^*(y)u_p(x), \quad \text{with } n \in \mathbb{Z}, \ x \in \mathcal{M}, \ y \in \mathcal{M},$$

Dilation Translation

where $\widehat{\psi}_j$ denotes that Fourier transform of $\psi_j \in L_2(\mathbb{R})$.

- Question: how to construct ψ_j so that $X(\Psi)$ is a tight frame on $\mathcal{M}$?

Wavelet Frame Method

- Further restriction on ψ_j :

Given $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ and $a_j \in \ell_0(\mathbb{Z})$

let $\widehat{\psi}_j(2\xi) := \widehat{a}_j(\xi)\widehat{\phi}(\xi), \quad 1 \leq j \leq r.$

- Question: how to construct a_j so that $X(\Psi)$ is a tight frame on $\mathcal{M}$?
- Benefits of such restriction
  - Grants a natural transition from continuum to discrete setting
  - Makes construction of tight frames on manifolds/graphs painless
  - Grants fast decomposition and reconstruction algorithms (Chebyshev polynomial approximation)

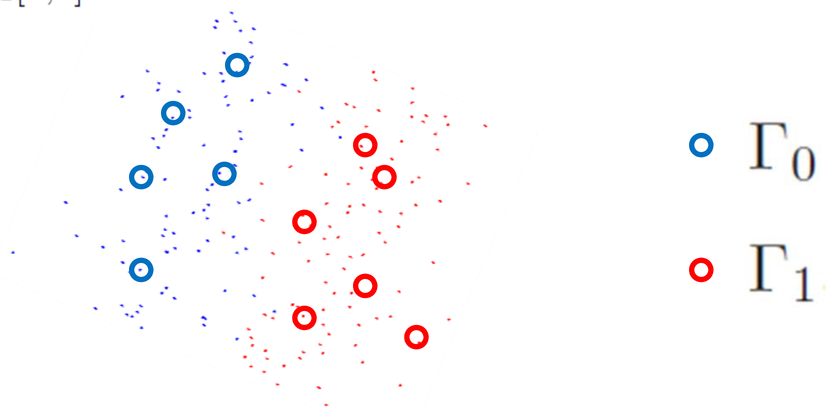
Wavelet Frame Method

- Sparsity based semi-supervised learning models

Model L2: $$\min_{u\in[0,1]} \|\boldsymbol{\nu}\cdot\boldsymbol{W}u\|_{1,G}+\frac{1}{2}\|u_{|\Gamma}-f\|_{2,G}^2,$$

Exact Model: $$\begin{array}{ll}\min_{u\in[0,1]} & \|\boldsymbol{\nu}\cdot\boldsymbol{W}u\|_{1,G} \\ \text{s.t.} & u_{|\Gamma}=f\end{array} \qquad f[i]=\begin{cases}0 & \text{for } i\in\Gamma_0,\\ 1 & \text{for } i\in\Gamma_1.\end{cases}$$

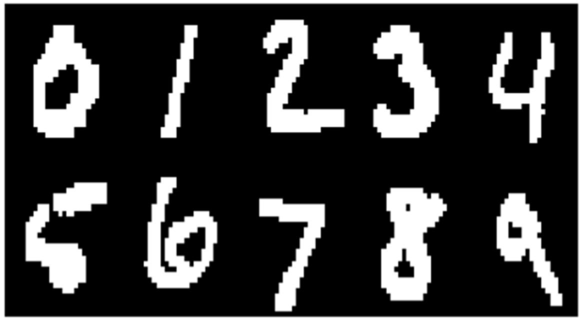
Robust Model: $$\min_{u\in[0,1]} \|\boldsymbol{\nu}\cdot\boldsymbol{W}u\|_{1,G}+\|u_{|\Gamma}-f\|_{1,G}$$



- Transform ***W*** is the fast tight wavelet frame transform on graphs **[B. Dong, ACHA, 2015]**.

Wavelet Frame Method

- Classification – Real Datasets
  - MNIST data set (http://yann.lecun.com/exdb/mnist/)
  - Banknote authentication dataset (UCI machine learning repository)



Machine Learning Repository

- Results: Model L2 **[Dong, ACHA, 2015]**

~3-5% label

Errors (%)	Our Method	Max-Flow	PAL	Binary MBO	GL
MNIST	2.76 (8.5 sec.)	1.52	1.56	1.64	1.75
Banknote	1.64 (2.9 sec.)	1.17	1.71	6.52	3.90

- Max-Flow & PAL : [Merkurjev, Bae, Bertozzi, and Tai, preprint, 2014]
- Binary MBO: [Merkurjev, Kostic, and Bertozzi, 2013]
- GL: [Bertozzi and Flenner, 2012]

Further Studies of Wavelet Frame Transform on Graphs

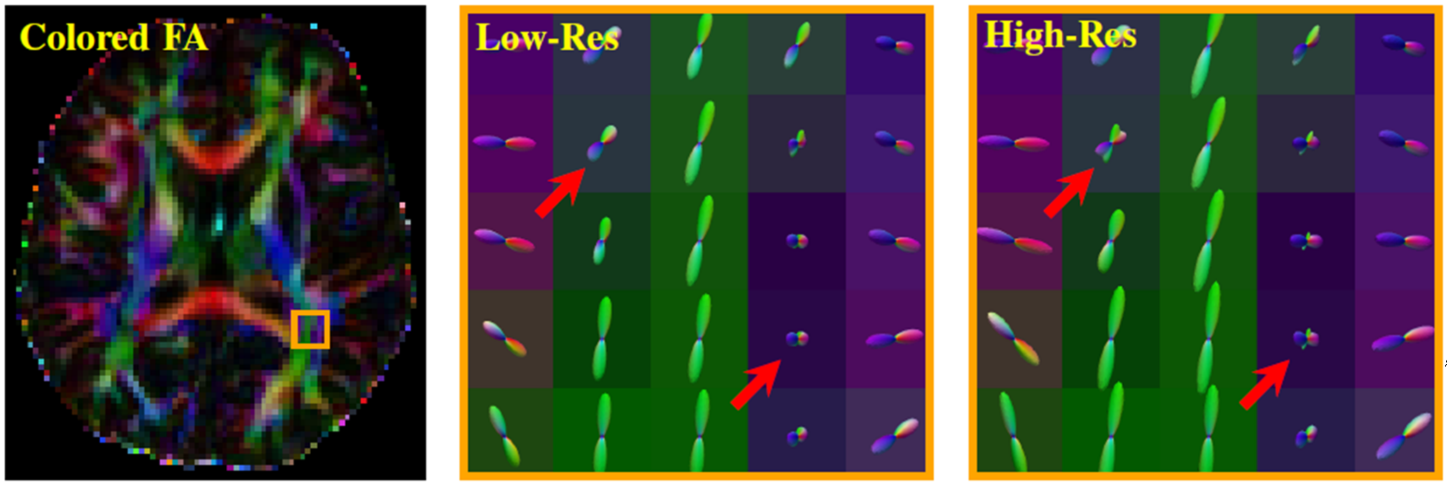
- High dimensional classification **[Dong and Hao, SPIE 2015]**
  - V.S. LDA methods: Leukemia (n=72, p=7129) and Lung (n=181, p=12533)

Error % (Std %)	NSC	IR	ROAD	RS-ROAD	Exact Model
Leukemia	8.51 (3.0)	4.27 (8.4)	6.35 (6.0)	4.46 (3.1)	5.57 (4.2)
Lung Cancer	10.44 (1.4)	3.47 (7.3)	1.37 (1.1)	0.93 (0.9)	0.59 (0.6)

- Application in super-resolution diffusion MRI **[Yap, Dong, Zhang,**

Leukemia

Lung



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CONCLUDING REMARKS

Conclusions and Future Work

- **Conclusions**
  - Bridging wavelet frame transforms and differential operators
  - New insights, models/algorithms and applications
  - Sparse approximation for general data analysis
- **Future work**
  - Idea of "end-to-end" in classical problems such as imaging
  - Learning PDEs from data

Thanks for Your Attention
and
Questions?

http://bicmr.pku.edu.cn/~dongbin