An approach to statistical shape analysis

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- ~~Deep learning~~
- ~~Deep mathematics~~
- *You might learn about new problem*
- *You might do the above afterwards*
- *You make an impact in this particular industry*

How I got involved

A hearing-aid company approached the IMA.

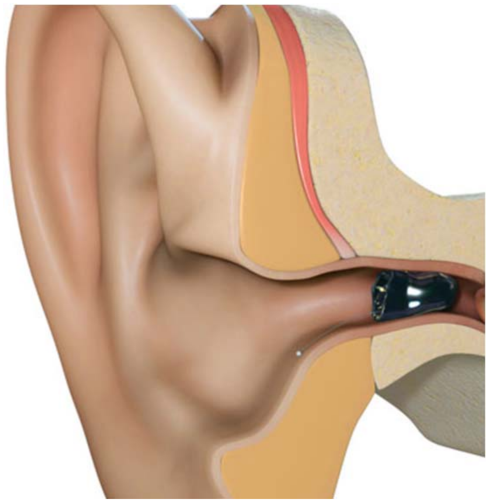
Statistical Shape Analysis of the Human Ear Canal with Application to In-the-Ear Hearing Aid Design

Rasmus R. Paulsen

Kongens Lyngby 2004
IMM-PHD-2004-134

In-ear-canal hearing aid

- A technician makes an impression of the patient's ear canal.
- Impression is scanned.
- File is sent to company.
- A custom shell (3D printed) in the shape of the ear canal is made; electronics is packed inside shell.
- Patient gets the hearing aid.



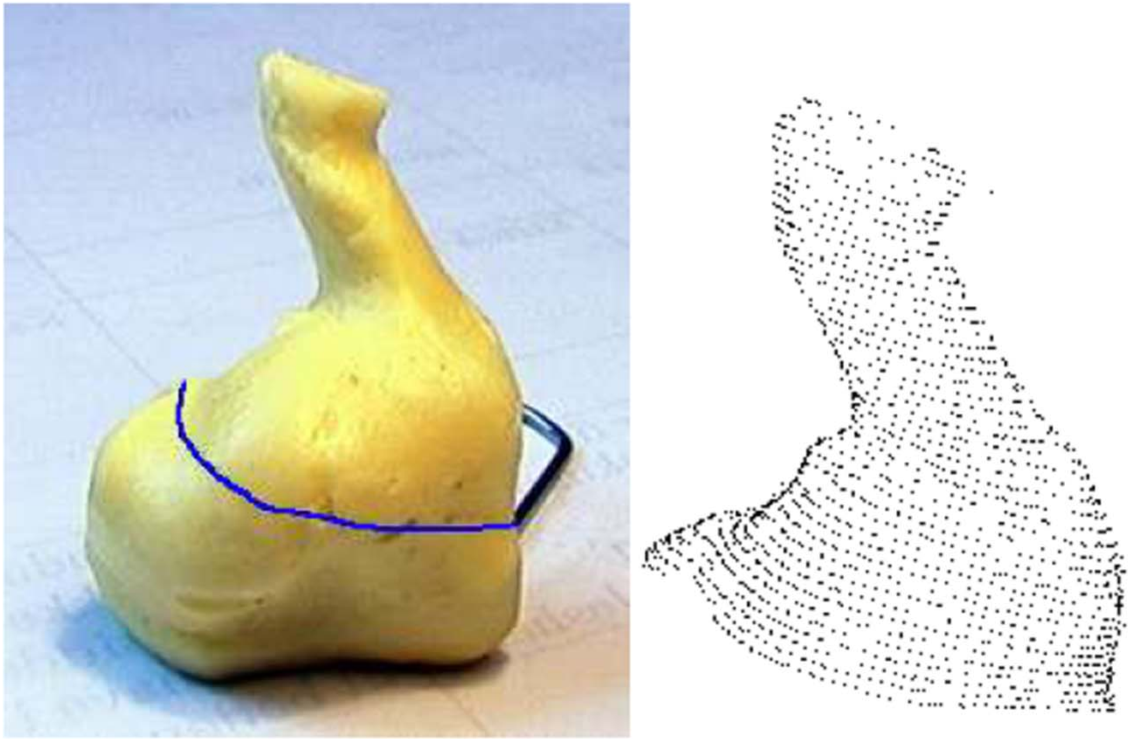


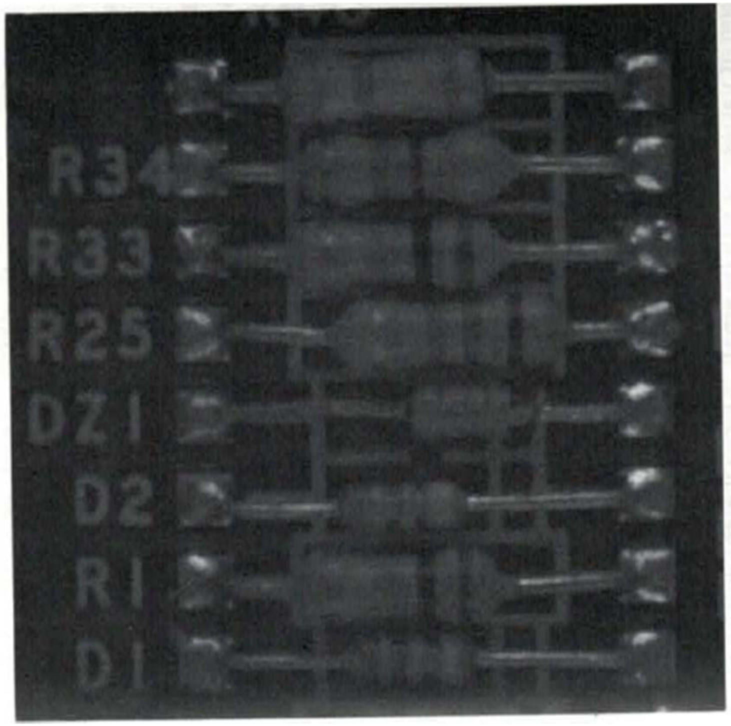
Figure 3.1: An ear impression and the corresponding point cloud. For clarity, only the points on the visible part of the surface are shown. The line on the ear impression corresponds to the lowest samples of the point cloud.

- In the thesis, Paulsen did a statistical analysis of 29 ear canal impressions.
- The company wanted to do something similar … for the millions of samples available in their data base.

Workflow for shape analysis*

- Given a set of objects, tag n landmark points on each.
- What is a landmark?
  - Points with application-dependent significance
  - Points without, but with geometric importance such as high curvature, highest/lowest point, etc.
  - Other points which are interpolated from above types.

*After Cootes, Taylor, Cooper, and Graham (1995)



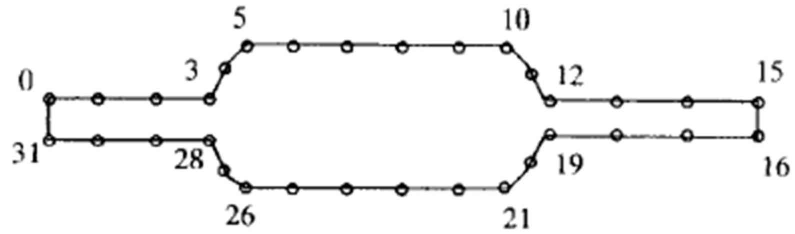
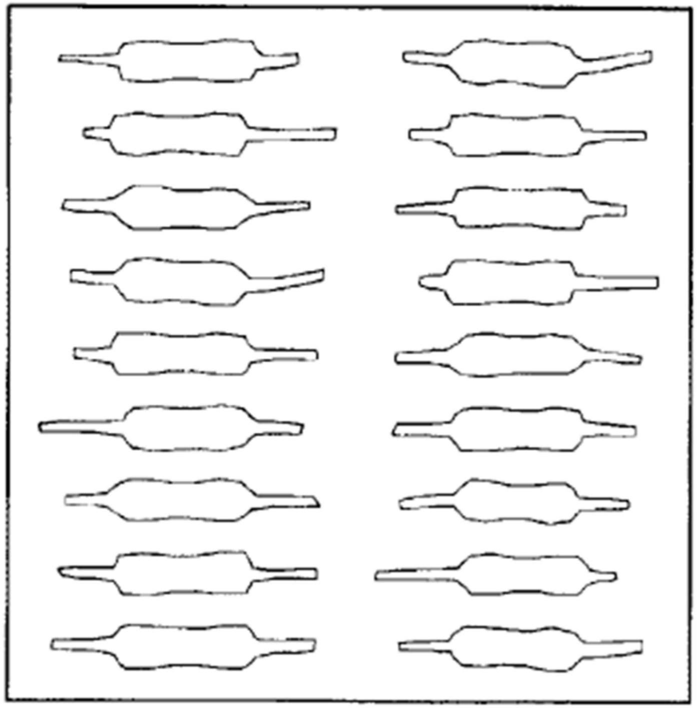


FIG. 4. Thirty-two point model of the boundary of a resistor.

A shape is represented by points. The labels are important: 0 & 31, 15 & 16 always represent the ends of the wire, 3-4-5, 10-11-12, 19-20-21, 26-27-18, represent the ends of the body of the resistor; the other points are interpolants. We must use the same number of points and same labeling scheme for each object.



From Cootes et al, examples of resistor shapes segmented from an image.

Let $\mathbf{x}_j = \left[x_{j0}, y_{j0}, x_{j1}, y_{j1}, \cdots, x_{j(n-1)}, y_{j(n-1)}\right]^T$ for $j = 1, \cdots, m$.

Alignment means picking a reference, say $\mathbf{x}_1$ and translating and rotating $\mathbf{x}_j$ so that it is "as close as possible" to $\mathbf{x}_1$.

Let us assume that we have done this for each vector and $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_m\}$ are aligned.

The data matrix for our objects is

$$\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_m]$$

Objects can be clustered, e.g., using *k*-means. Within each cluster, we can compute the “mean”. Assume there is only one cluster,

$$\overline{\mathbf{x}} = \frac{1}{m}\sum_{j=1}^{m}\mathbf{x}_j.$$

Remove the mean from the data matrix

$$\Delta\mathbf{X} = [\mathbf{x}_1 - \overline{\mathbf{x}}, \mathbf{x}_2 - \overline{\mathbf{x}}, \cdots, \mathbf{x}_m - \overline{\mathbf{x}}]$$

Now compute the SVD

$$\Delta\mathbf{X} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$$

If the shapes are similar, the singular values should drop off rapidly.

Now one can do simple statistics.

1. Distribution in the most important variations. Compute

$$b_j^{(k)} = \mathbf{u}_k^T(\mathbf{x}_j - \overline{\mathbf{x}}),$$

   and look at the histogram.

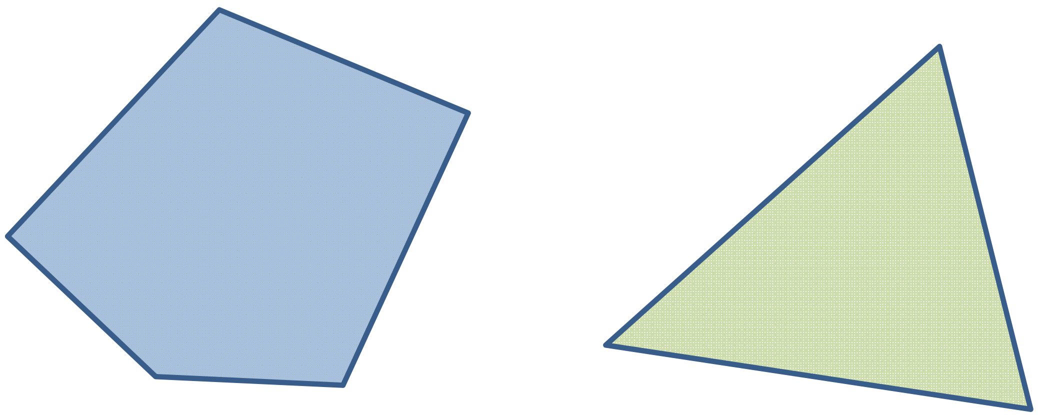
2. Compute the standard deviation σ_k and visualize what one standard deviation shape looks like

$$\mathbf{x}^{(k)} = \overline{\mathbf{x}} \pm \sigma^{(k)} \mathbf{u}_k.$$

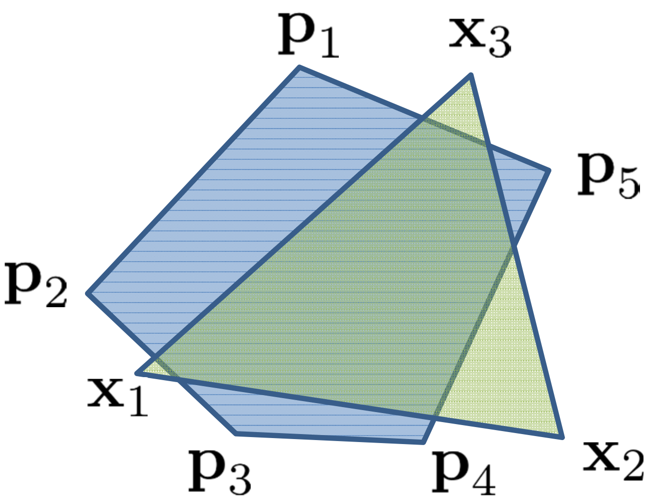
Splitting alignment from landmarking

One can use ICP* (Iterative Closest Points) algorithm to align and then landmark the align shapes.

Align polygon $P = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5\}$ with polygon $X = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ which is fixed in space.



*Besl and McKay (1992)



Distance from point $\mathbf{p}$ to X

$$d(\mathbf{p}, X) = \min_{\mathbf{x} \in X} \|\mathbf{x} - \mathbf{p}\|$$

Closest point operator

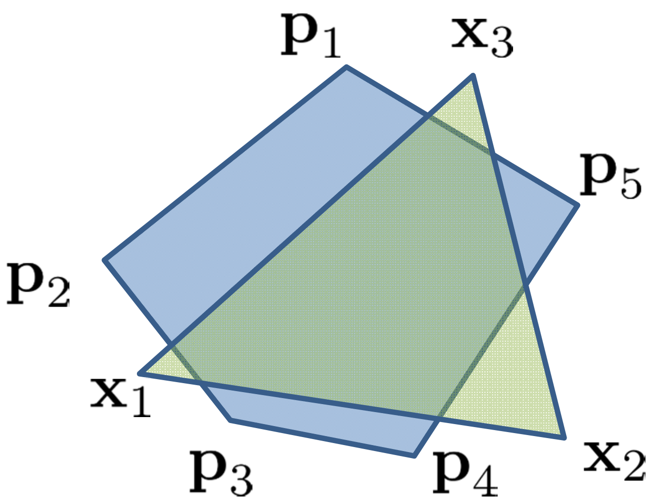
$$Y = \mathcal{C}(P, X)$$

In this example $Y = [\mathbf{x}_3, \mathbf{x}_1, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]$

Next step: find rigid rotation and translation that makes P close to Y

$$\min_{\mathbf{q}_R,\mathbf{q}_T} \sum_{i=1}^{5} \|\mathbf{y}_i - R(\mathbf{q}_R)\mathbf{p}_i - \mathbf{q}_T\|^2$$

Can be solved by SVD.



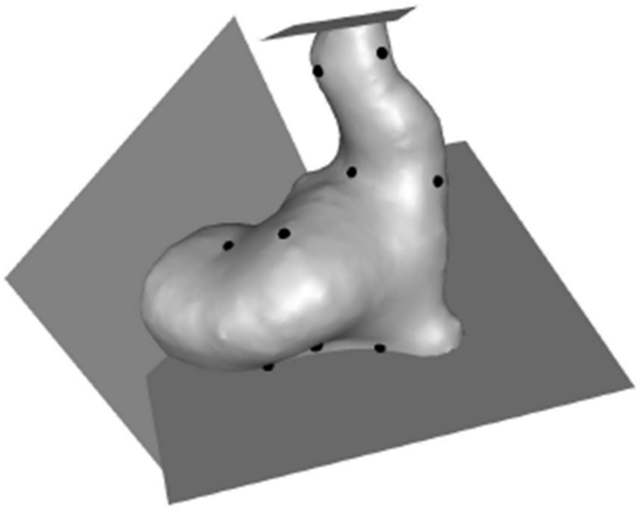
$$Y = [\mathbf{x}_3, \mathbf{x}_1, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]$$

$$(\mathbf{q}, d) = \mathcal{Q}(Y, P)$$

Repeat until change in d is below some threshold.

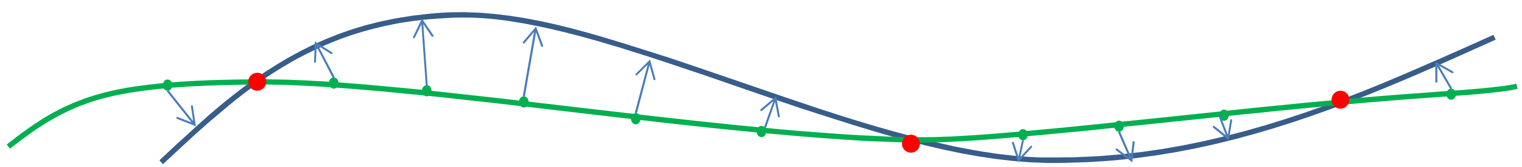
What Paulsen did

- Each of the 29 (20 male and 9 female subjects) ear canal shapes was annotated with landmarks by an <u>expert</u>.
- There were 18 <u>anatomically significant</u> landmarks.

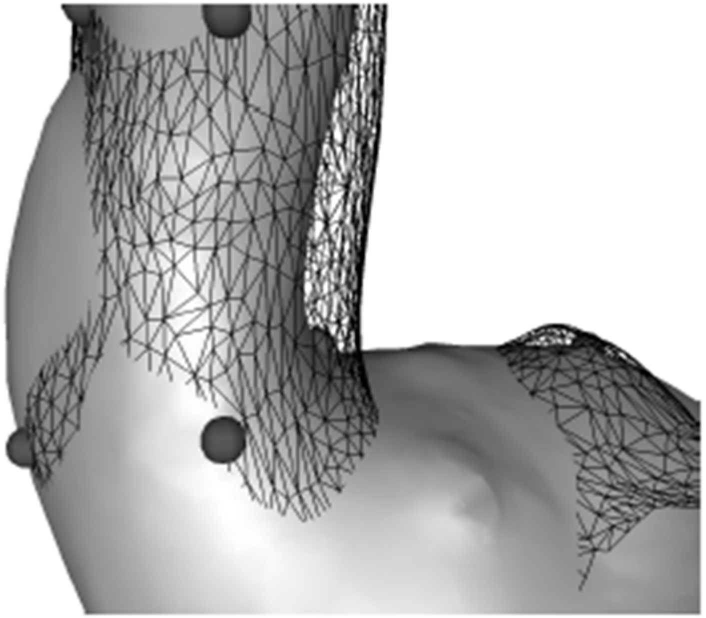


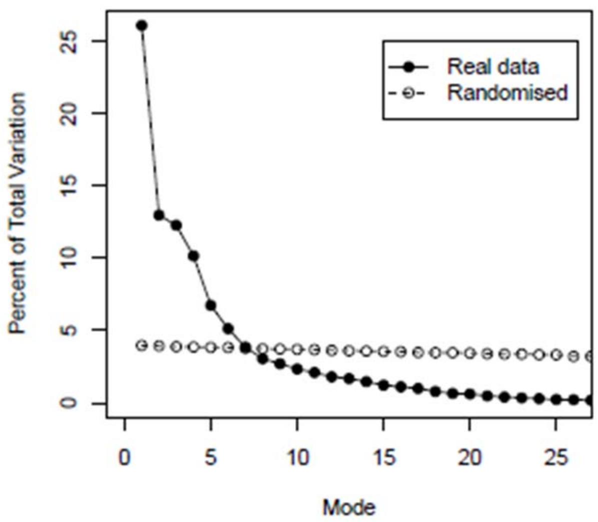
Create dense surface correspondence using Thin Plate Spline warping:

- Form a spline surface that goes through each of the 18 landmark points
- "Project" nodes on the spline onto the nearest point on the surface

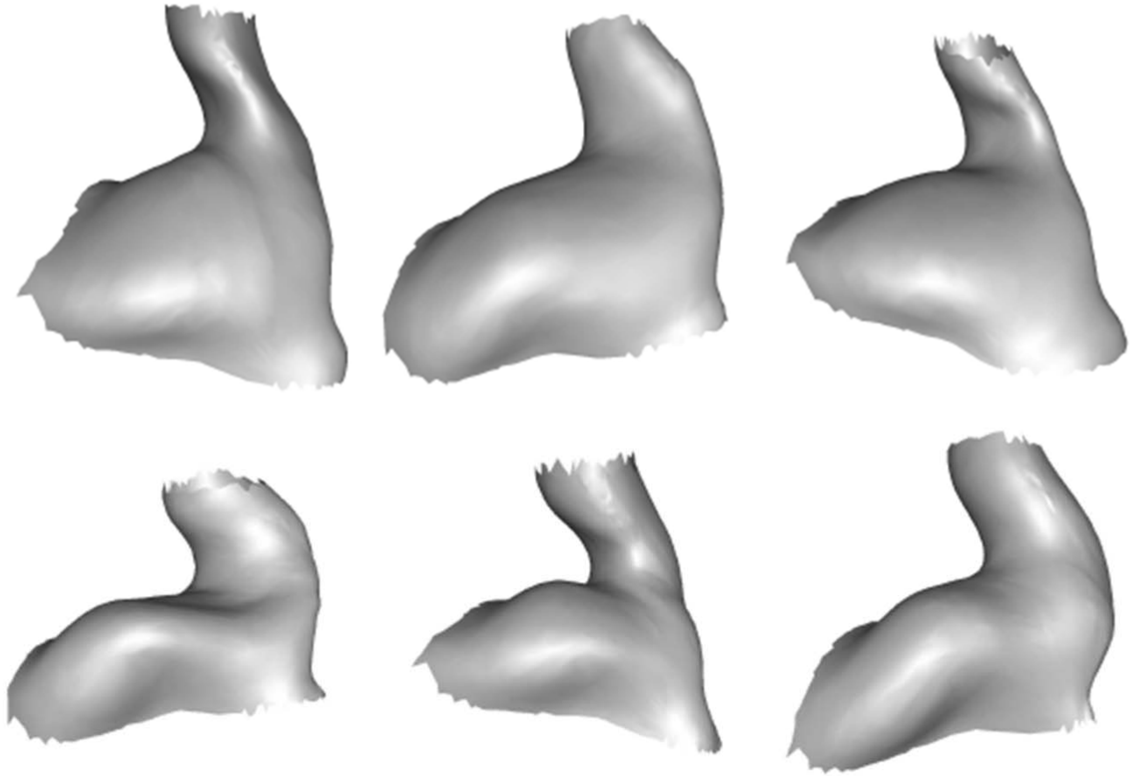


- Projected points on the surface of one shape can now be compared to projected points on another shape.





$\pm 3\sigma$ shapes in the first 3 modes



(a) Mode 1 (b) Mode 2 (c) Mode 3

Challenges in scaling up

- Landmarking samples by hand not practical.
- Too many samples to manually edit.
- Building dense surface correspondence via Thin Plate Spline too slow.

Need to find a way to get correspondence without landmarking.

Extracting shape properties from global features

Several ideas

- Fourier representation of surfaces
- Solid object properties
  - Zernike polynomial (and 3-D generalization)
  - Fourier series

Surface representation using Fourier basis*

In 2D, $\mathbf{x} = (x, y)$

$$\mathbf{x}(t) = \sum_{m=-(K-1)}^{K-1} \mathbf{a}_m e^{imt}$$

We are given a shape represented by points

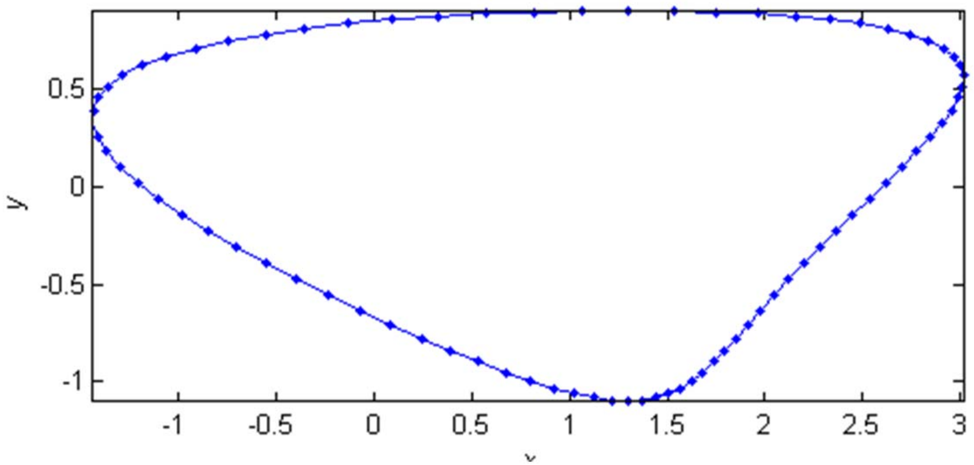
$$\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N\}$$

We want to find

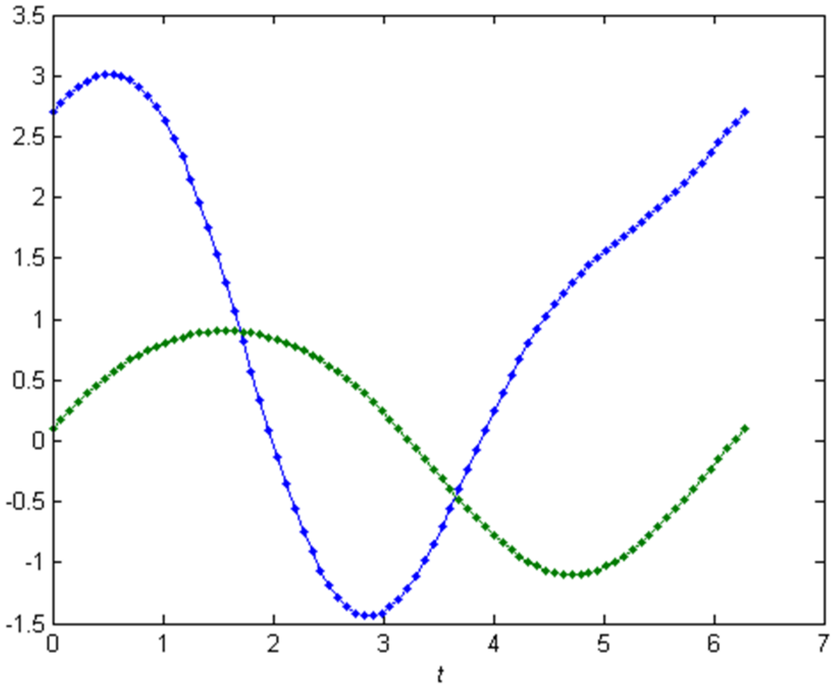
$$\{\mathbf{a}_{-(K-1)}, \mathbf{a}_{-(K-2)}, \cdots\cdots, \mathbf{a}_{(K-2)}, \mathbf{a}_{(K-1)}\}$$

This can be done by resampling the data on a regular grid and inverting the Fourier series.

*Staib and Duncan (1996)



2D shape given by points



$$(t_i, x_i), \quad i = 1, \cdots, n$$
$$(t_i, y_i), \quad i = 1, \cdots, n$$

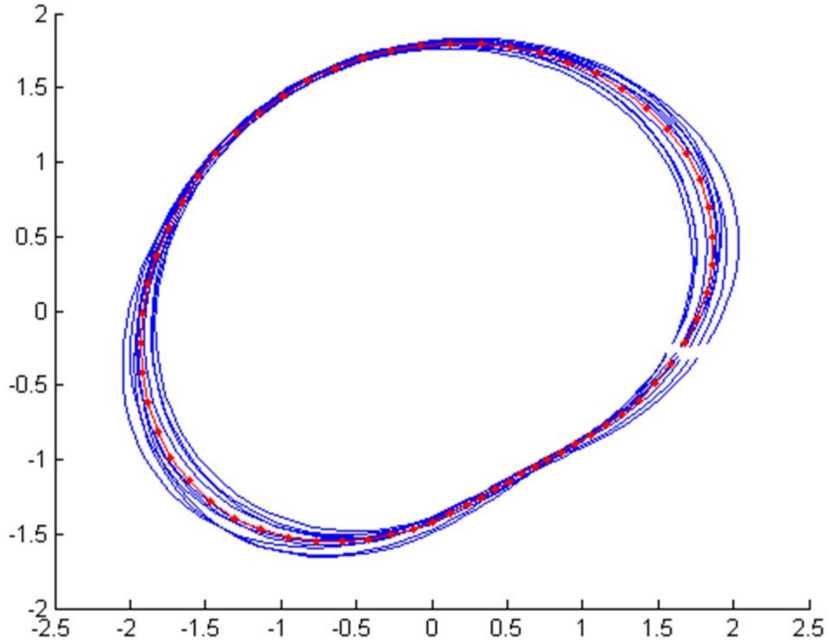
Assume t_i regularly spaced.

Nice properties:

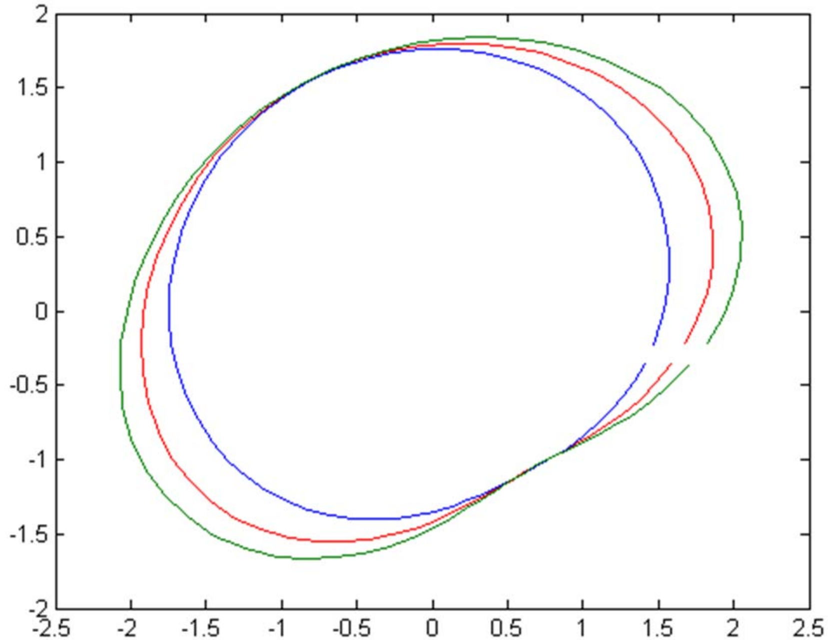
- Linear dependence on parameters means that if we have n objects, the average object has parameters which are the average of the parameters of the set

$$\overline{\mathbf{x}}(t) = \sum_{m=-(K-1)}^{K-1} \overline{\mathbf{a}}_m e^{imt}$$

- Average of circles is a circle of average radius; average of ellipses of same orientation is an ellipse whose major and minor axes are the averages of the major and minor axes
- The distribution of the coefficients give a distribution of shapes, which can be further simplified using PCA.



Red is average



2 σ deviations from average in the first principal direction

Not-so-nice properties

- Not clear how one should do this in 3D

$$\mathbf{x}(u,v) = \sum_{m=-(K_u-1)}^{(K_u-1)} \sum_{l=-(K_v-1)}^{(K_v-1)} g_{ml} e^{i(mu+lv)}$$

- We are given point cloud

$$\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N\}$$

- We need to find both g_{ml} and (u_i, v_i)
- One possible way is to turn point cloud into a binary voxel image, then use the cost function proposed by Staib and Duncan – a nonconvex optimization.

Zernike polynomials*

- View object as characteristic functions, in 2D

$$u(x,y) = \begin{cases} 1 & (x,y) \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

- Consider orthonormal functions on unit disc

$$V_{mn}(r,\theta) = R_{mn}(r)e^{in\theta}$$

$$R_{mn}(r) = \sum_{s=0}^{n-|m|/2} (-1)^s \frac{(n-s)!}{s!\left(\frac{n+|m|}{2} - s\right)\left(\frac{n-|m|}{2} - s\right)} r^{(n-2s)}$$

*Kohstanzad and Hong (1990)

- Compute

$$A_{mn} = \frac{n+1}{\pi} \int_0^{2\pi} \int_0^1 u(r,\theta)\, R_{mn}(r) e^{in\theta}\, r dr\, d\theta$$

- They propose to use $|A_{mn}|$ as a feature vector for shape retrieval.
- We propose to use A_{mn} as feature vector after shapes are aligned.
- Can be extended to 3D using functions that are orthonormal on the unit sphere.
- Computational complexity forces us to abandon the approach.

Fourier series

Periodic function over $[-L, L]^2$, coefficients are

$$A_{mn} = \lambda_{mn} \int_{-L}^{L} \int_{-L}^{L} u(x,y) e^{i\frac{m\pi x}{L} + i\frac{n\pi y}{L}}\, dx\, dy$$

$$\lambda_{mn} = \begin{cases} 1/(4L^2) & m = 0 \text{ and } n = 0 \\ 1/(2L^2) & m = 0 \text{ or } n = 0 \\ 1/(L^2) & m > 0 \text{ and } n > 0 \end{cases}$$

Feature vector of a shape

$$\begin{aligned} \mathbf{A} = [&A_{-M,-N}, A_{-M+1,-N}, \cdots, A_{M-1,-N}, A_{M,-N}, \\ &A_{-M,-N+1}, A_{-M+1,-N+1}, \cdots, A_{M-1,-N+1}, A_{M,-N+1}, \\ &\cdots, \cdots, A_{-M,N}, A_{-M+1,N}, \cdots, A_{M-1,N}, A_{M,N}]^T \end{aligned}$$

Nice property – fast to compute, can be used for clustering

Not-so-nice properties

- Averaged and other extracted Fourier coefficients do not necessarily lead to binary functions. This can be "fixed" this by binarizing using Modica-Mortola functional

$$\min_u \int\int \epsilon|\nabla u|^2 + \frac{1}{\epsilon}u^2(1-u)^2\,dxdy + \lambda\|u-u_0\|^2$$

- Averaged coefficients are not those of the averaged shape.

Average of discs is not disc of average radius

$$A_{mn} = \lambda_{mn} \int_0^a \int_0^{2\pi} e^{im\pi \frac{r}{L}\cos\theta + in\pi\frac{r}{L}\sin\theta}\, d\theta r dr$$

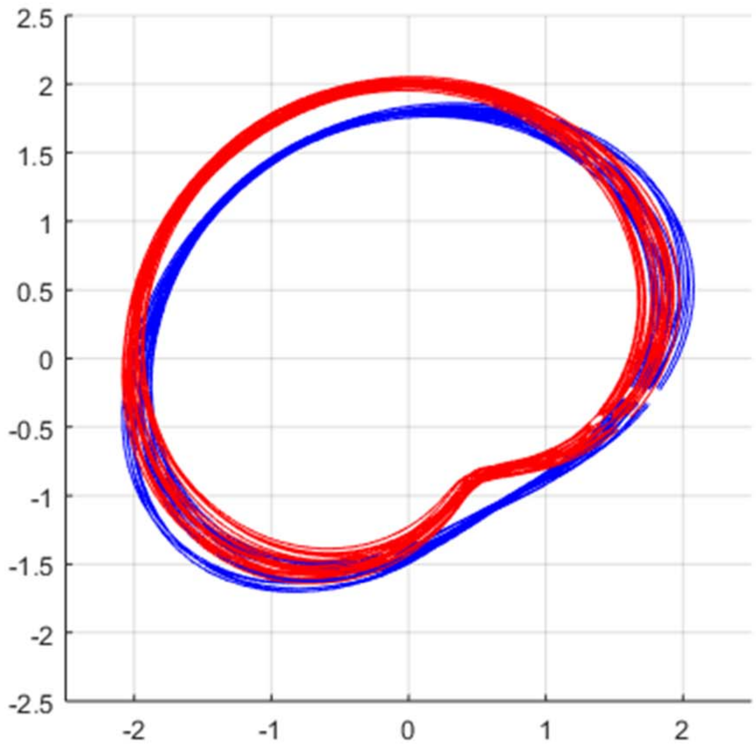
$$A_{00} = \frac{\pi a^2}{4L^2} \quad \text{and} \quad \mathrm{Re}\left(A_{mn}\right) = \frac{2\pi\kappa a}{L^2}\frac{J_1(\alpha a)}{\alpha}$$

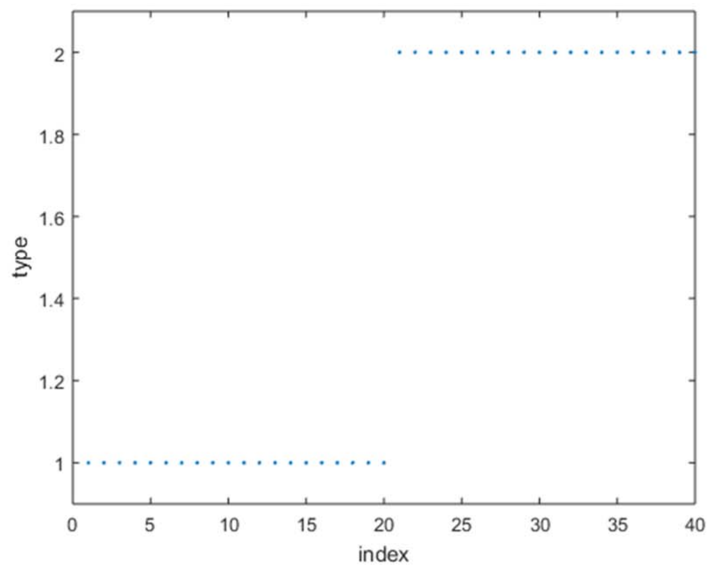
where $\alpha = \sqrt{m^2+n^2},\ \ \kappa = 2 \text{ or } 4$

If $\overline{a}$ is the average radius,

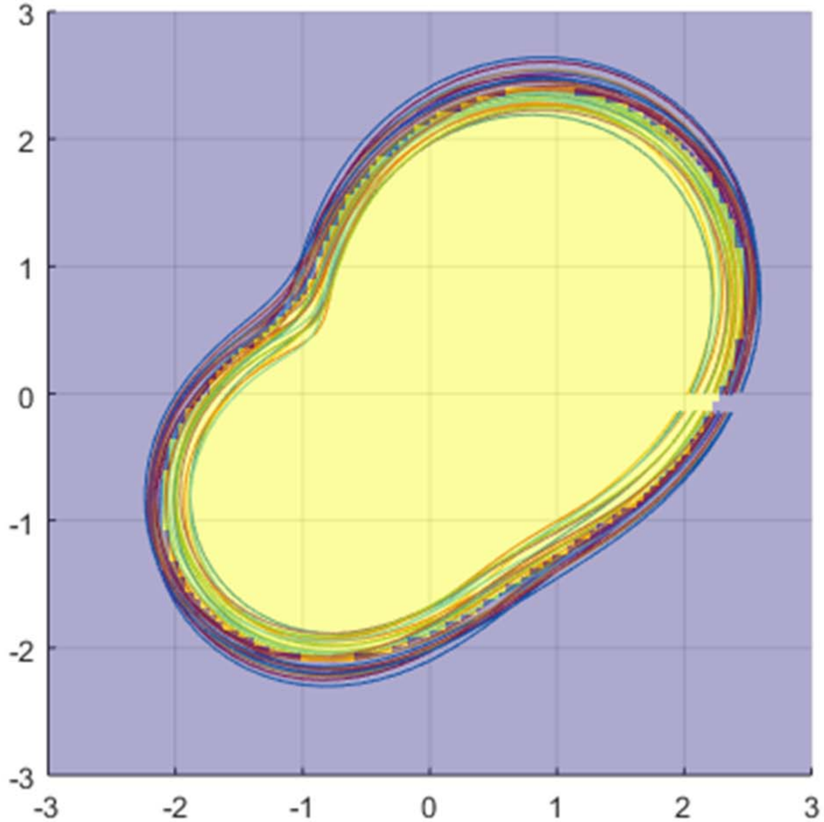
$$\overline{A_{00}} \neq \frac{\pi\overline{a}^2}{4L^2} \quad \text{and} \quad \overline{\mathrm{Re}\left(A_{mn}\right)} \neq \frac{2\pi\kappa\overline{a}}{L^2}\frac{J_1(\alpha\overline{a})}{\alpha}$$

Example of clustering



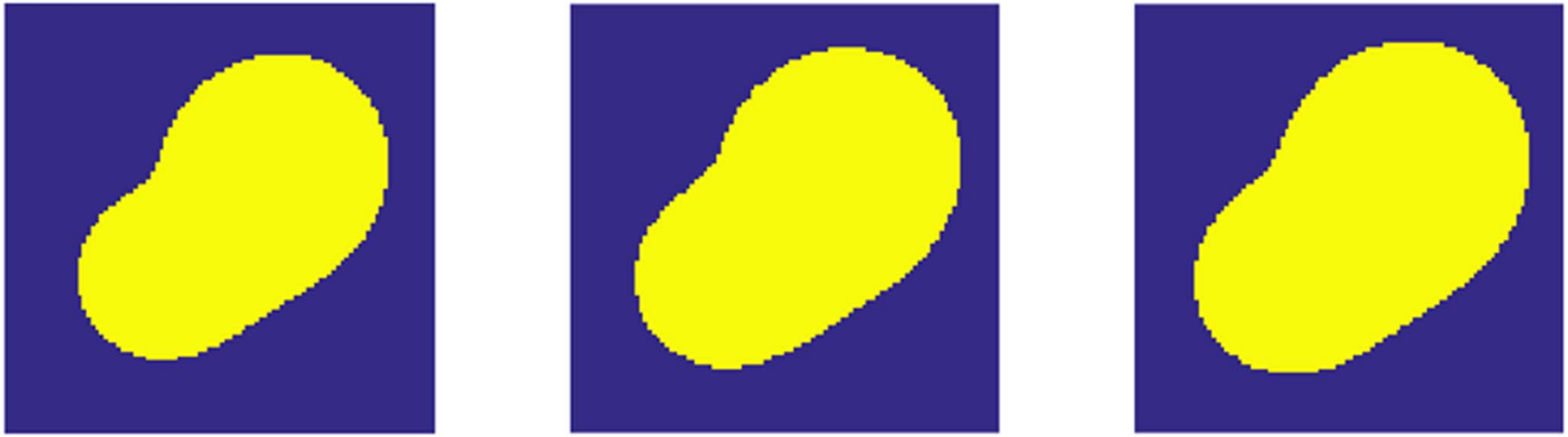


Shape average



Output is binary function.

Example of statistical analysis



Average shape and shapes one standard deviation from average.

Discussion

- Presented simple ideas for shape analysis that is computationally reasonable.
- Prospects are good that we can scale up and perform analysis on 10,000s of shapes.
- We are developing the infrastructure to do this.
- There is a lot of data clean up to do.
- We think we can learn a lot about the shapes of human ear canals by clustering and performing analysis on the clusters. However, the clustered shapes may not have anatomical significance.