A discrete uniformization for polyhedral surfaces and its applications

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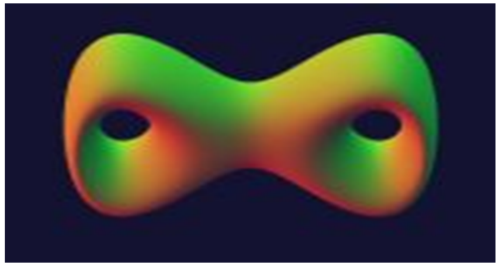
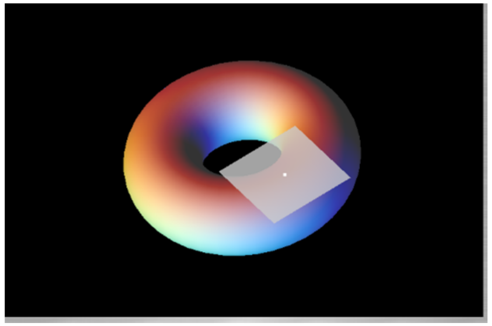
June 12-16, 2017

classical theory of
smooth surfaces

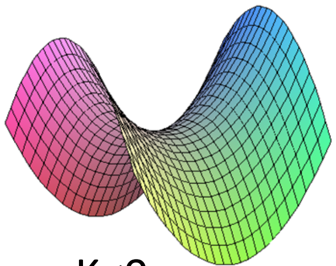


Metric = Riemannian metric

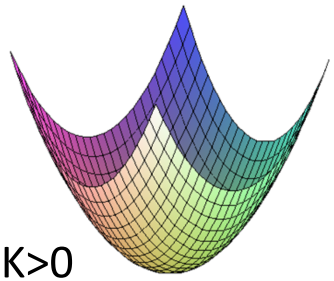


curvature=Gaussian curvature

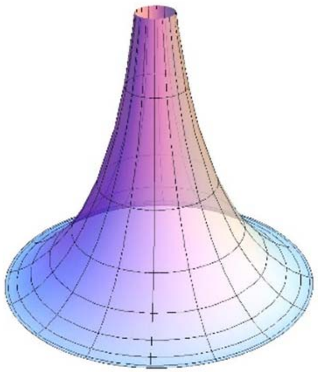
$$K = \lim_{r \to 0^+} 12 \frac{\pi r^2 - A(r)}{\pi r^4}$$



K<0



K>0

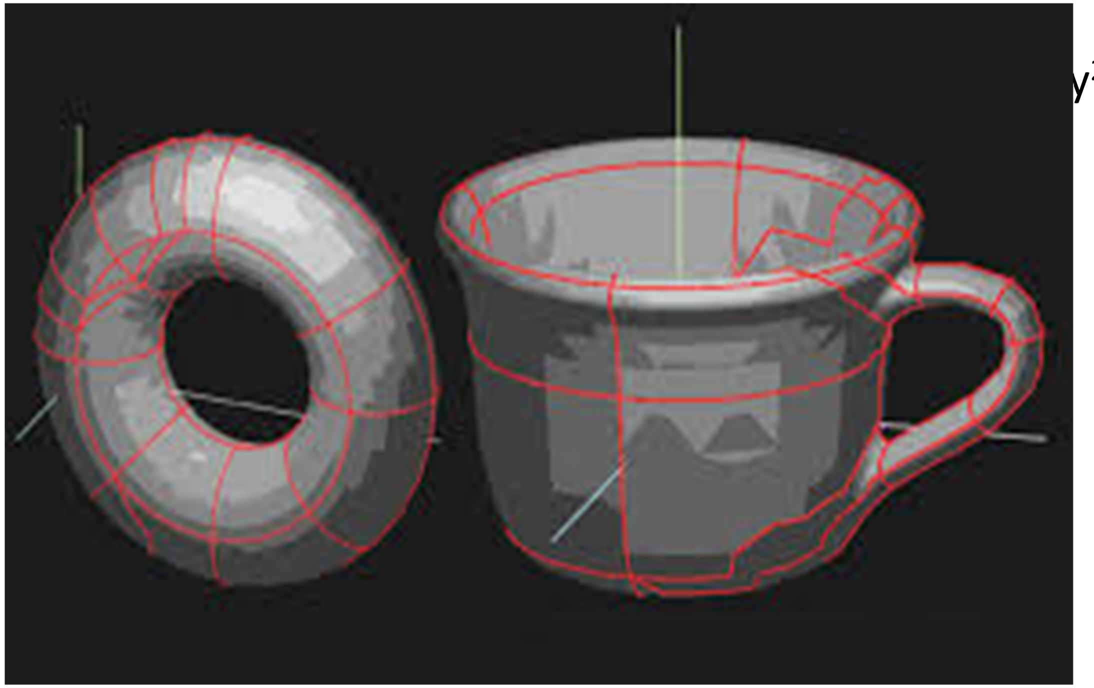


pseudosphere, K≡-1

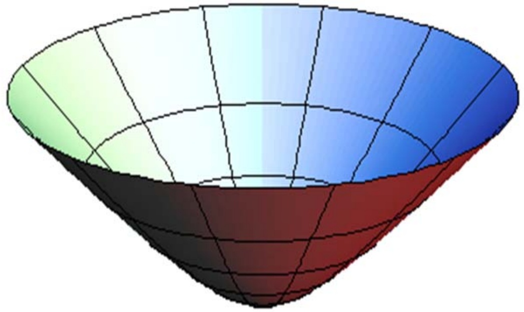
k≡1

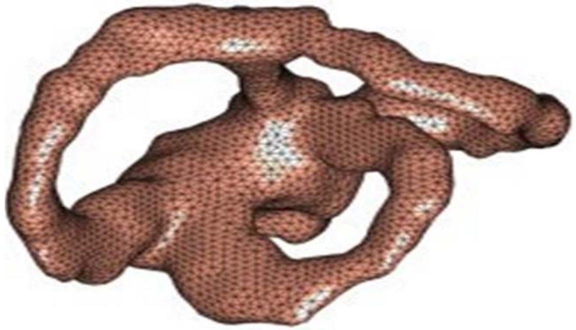
Basic question:

relationship between curvature and metric



y^2-z^2

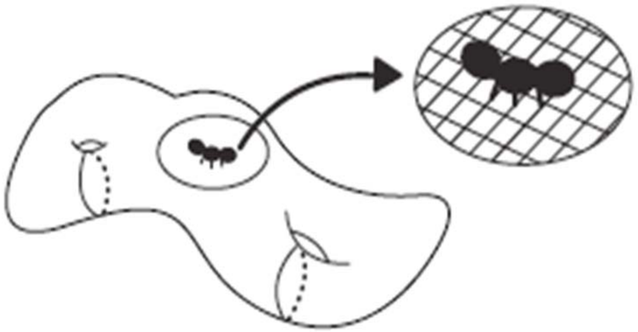


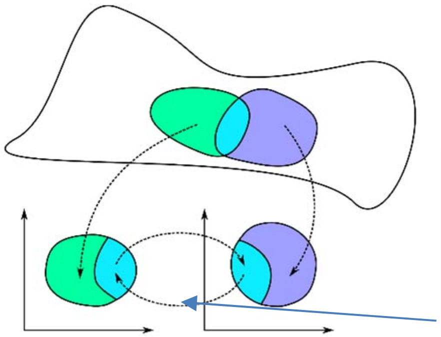


Uniformization thm (Poincare-Koebe, 1907) ∀ Riemannian metric g on S,

∃ λ: S → $\mathbf{R}_{>0}$ s.t., (S, λg) is a complete metric of curvature 1, 0, -1.

λg and g have the same notion of angles, i.e., conformal.



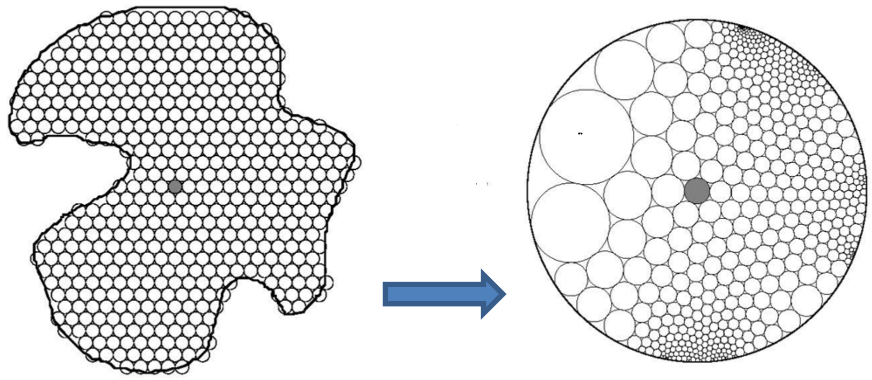


$$z = e^{it} \frac{z - z_0}{1 - \overline{z}_0 z}$$

6g-6 invariants

Corollary. (Riemann mapping)

Any simply connected bounded domain in the plane is conformal to the unit disk.

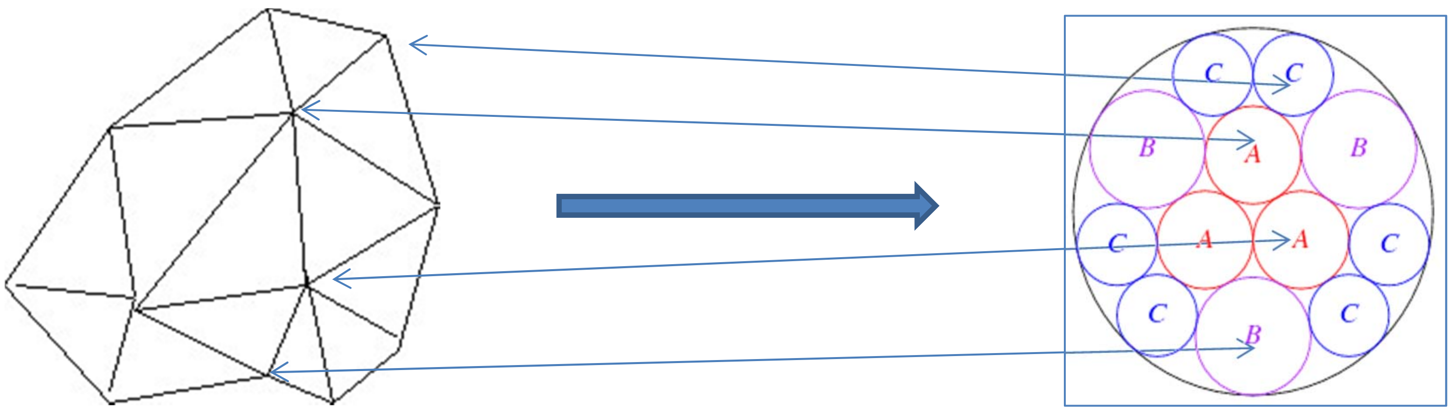


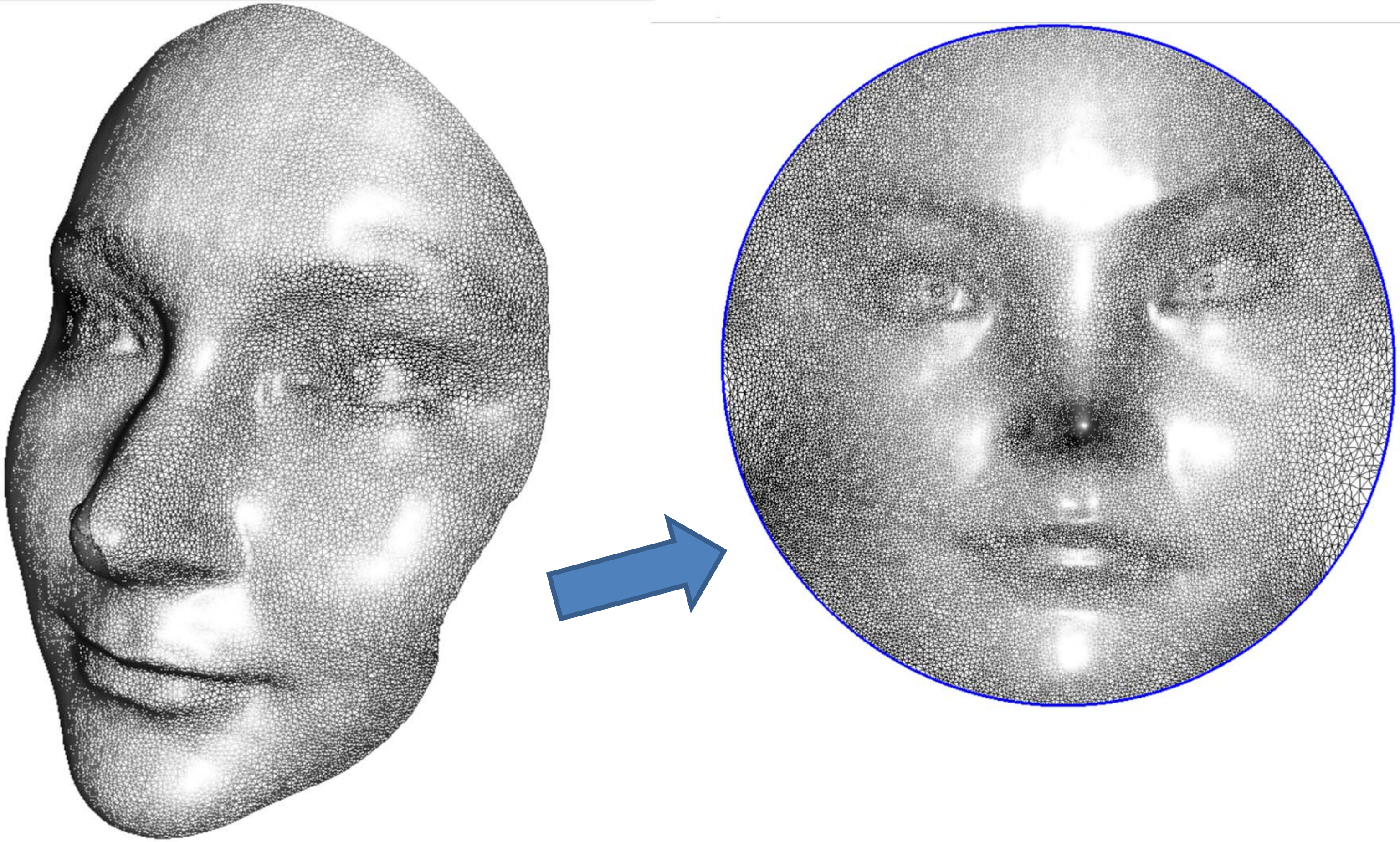
circle packing metric

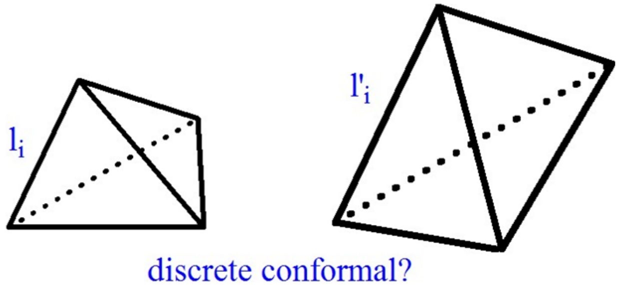
K. Stephenson

Andreev-Koebe-Thurston Theorem

A simplicial triangulation of a disk can be *realized* by a circle packing of the unit disk.







Key issue:

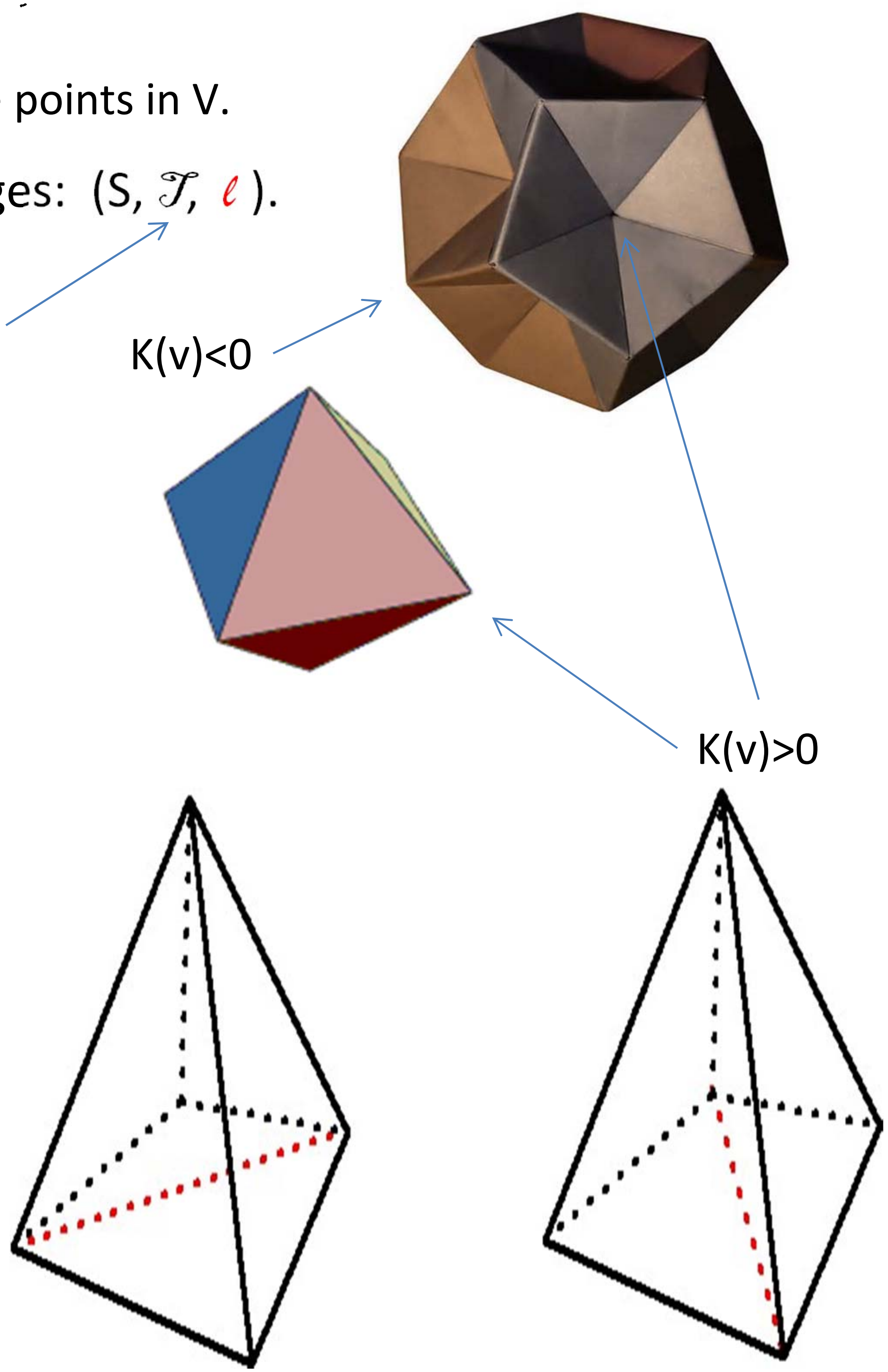
what is a discrete conformal equivalence for PL metrics?

Polyhedral surface

PL metric d on (S,V) is a flat cone metric, cone points in V.

Isometric gluing of $\mathbf{E}^2$ triangles along edges: $(S, \mathcal{T}, \ell)$.

triangulation

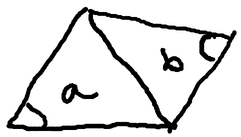


Curvature $K=K_d: V \to \mathbf{R}$,

$K(v) = 2\pi$-sum of angles at v

Gauss-Bonnet $\sum K_d(v) = 2\pi\, \chi(S)$

(Closed surfaces)

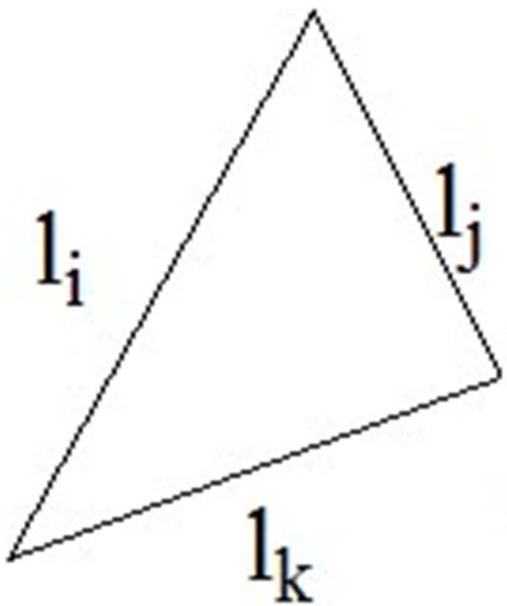


A triangulated PL metric $(S, \mathcal{T}, \ell)$

is *Delaunay*: $a+b \leq \pi$ at each edge e.

discretization

Smooth world	Discrete world
Smooth surface S	Triangulated surfaces (S, T)
Functions on S	Functions on V=V(T)
Riemannian metrics	Polyhedral metrics
Gauss Bonnet	Curvature K_d: V -> $(-\infty, 2\pi)$

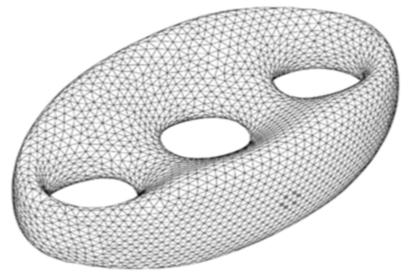


$$l_i + l_j > l_k$$

- S closed surface,

  $V=\{v_1, \ldots, v_n\}$ in S, n>0.

- Triangulation T of (S,V): V(T)=V, E=E(T)



- PL metric d on (S, T, V):

isometric gluing of triangles, singularities in V.

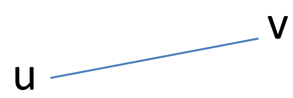
d determined by edge length

$$\ell_d: E \to R_{>0}$$

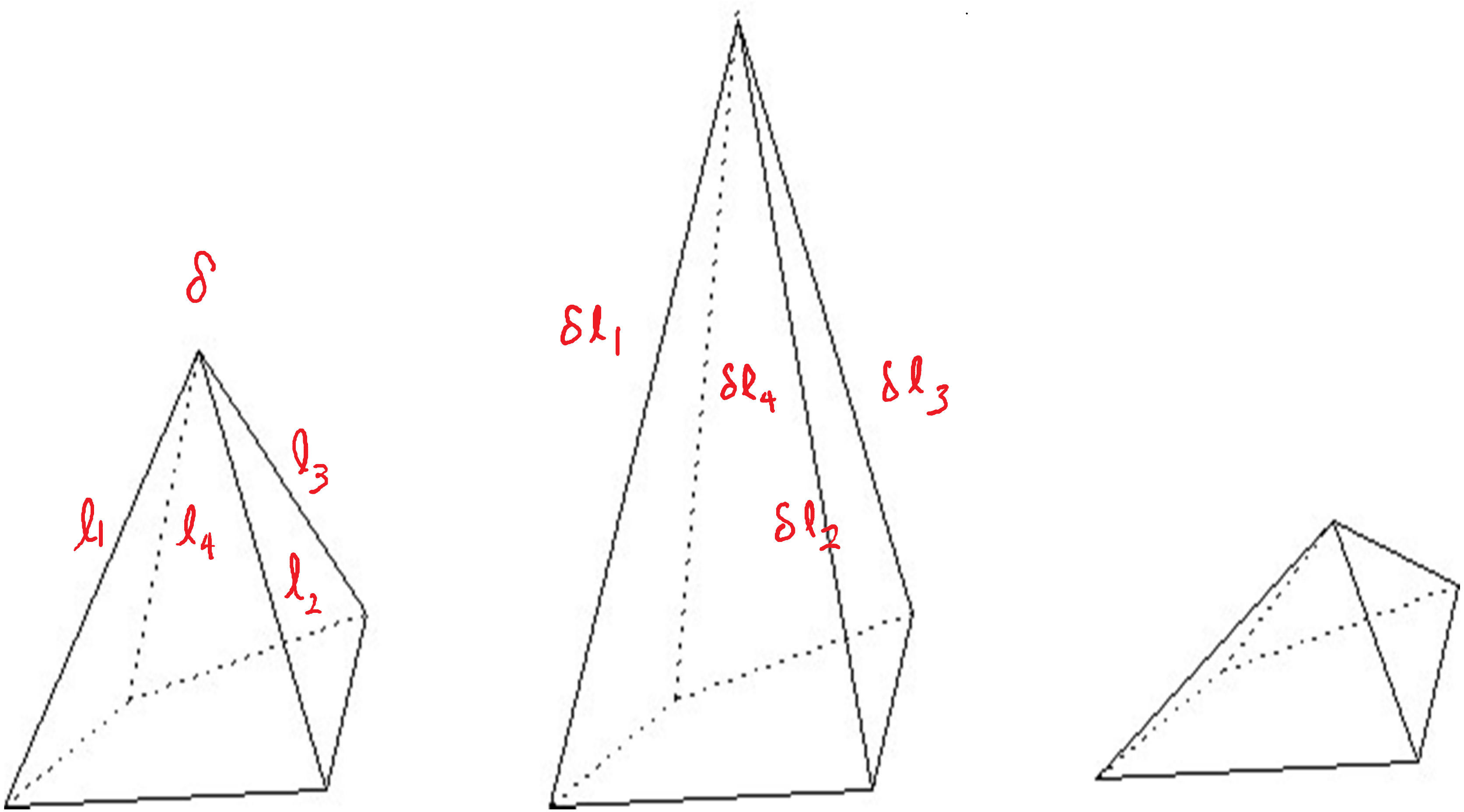
Discrete conformality I: vertex scaling

Def. (Vertex scaling) Given $\lambda: V \to R$ and $\ell : E \to R$,

$$\lambda * \ell(uv) = e^{\lambda(u)+\lambda(v)} \ell(uv)$$



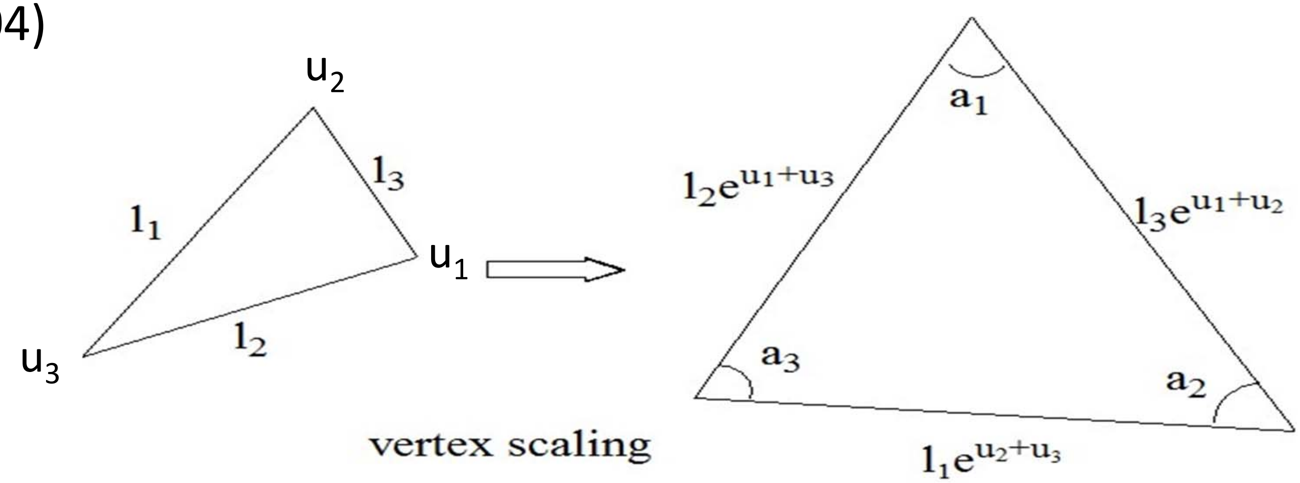
$$|d_{\lambda^4 g}(\mathrm{u}, \mathrm{v}) - \lambda(u)\, \lambda(v) d_g(u,v)| \le \quad C d_g(u,v)^3$$ (Gu-L-Wu, 2015)



Same triangulation, scale edge lengths from vertex weights

A variational principle

Prop (L, 2004)



Then $\frac{\partial a_i}{\partial u_j} = \frac{\partial a_j}{\partial u_i}$, $[\frac{\partial a_i}{\partial u_j}]_{3x3}$ semi-negative definite,

and there exists a locally concave function f(u) such that $\nabla f = (a_1, a_2, a_3)$.

Bobenko-Pinkahl-Springborn (2010).

f can be extended to a convex function on R^3 and is explicit.

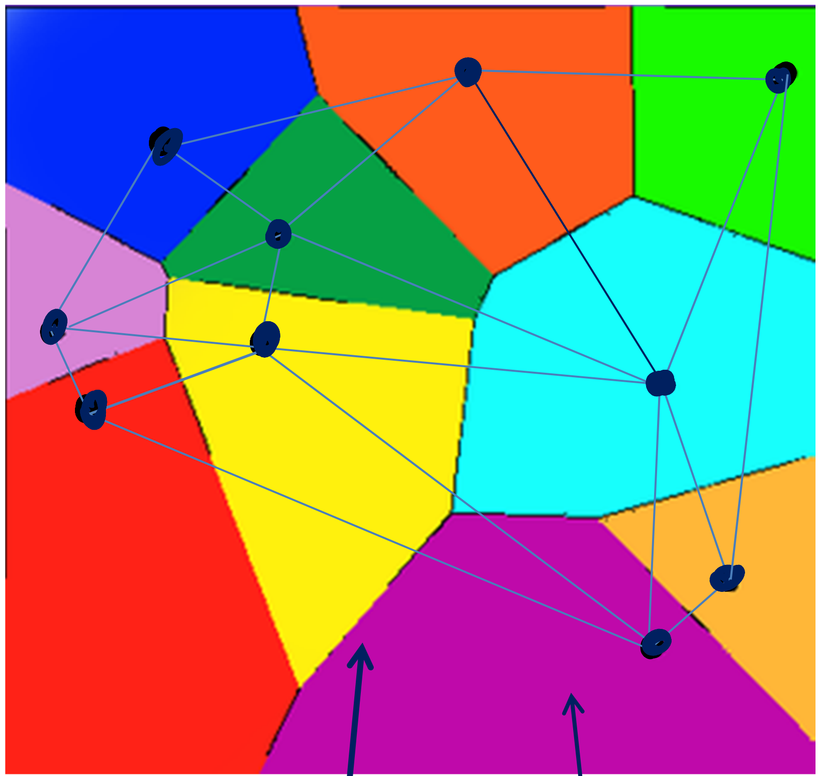
Corollary (BPS, 2010).

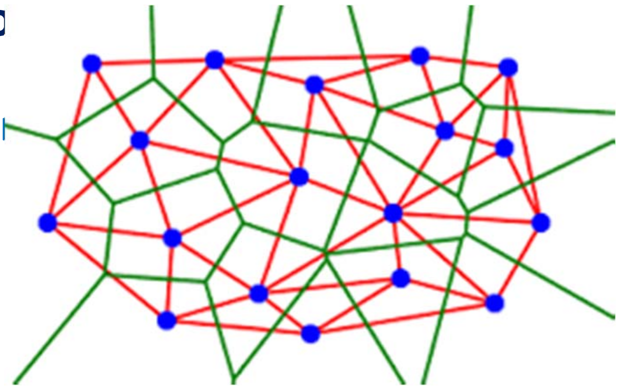
If ℓ and $u*\ell$ are two PL metrics on T with the same curvature, then $u \equiv c$.

However, given ℓ on T, there are in general no constant curvature metrics of the form $u*\ell$.

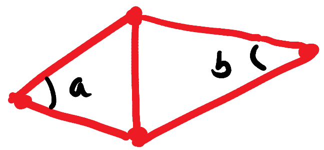
Discrete conformality, Part II: **Delaunay triangulatic**

A finite point set V produces a Delaunay triangulatio

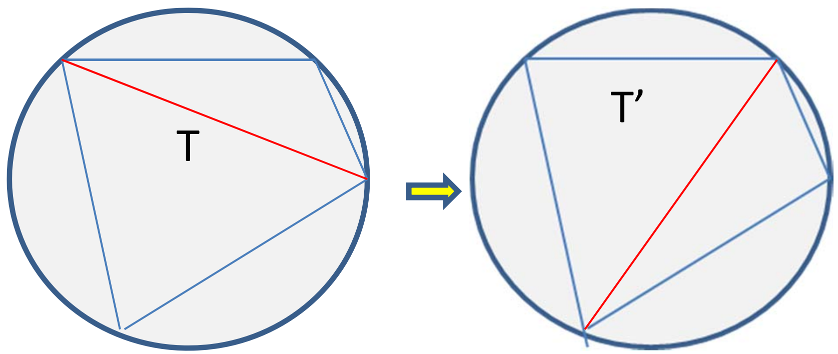




Delaunay : a+b $\leq \pi$ at each edge e



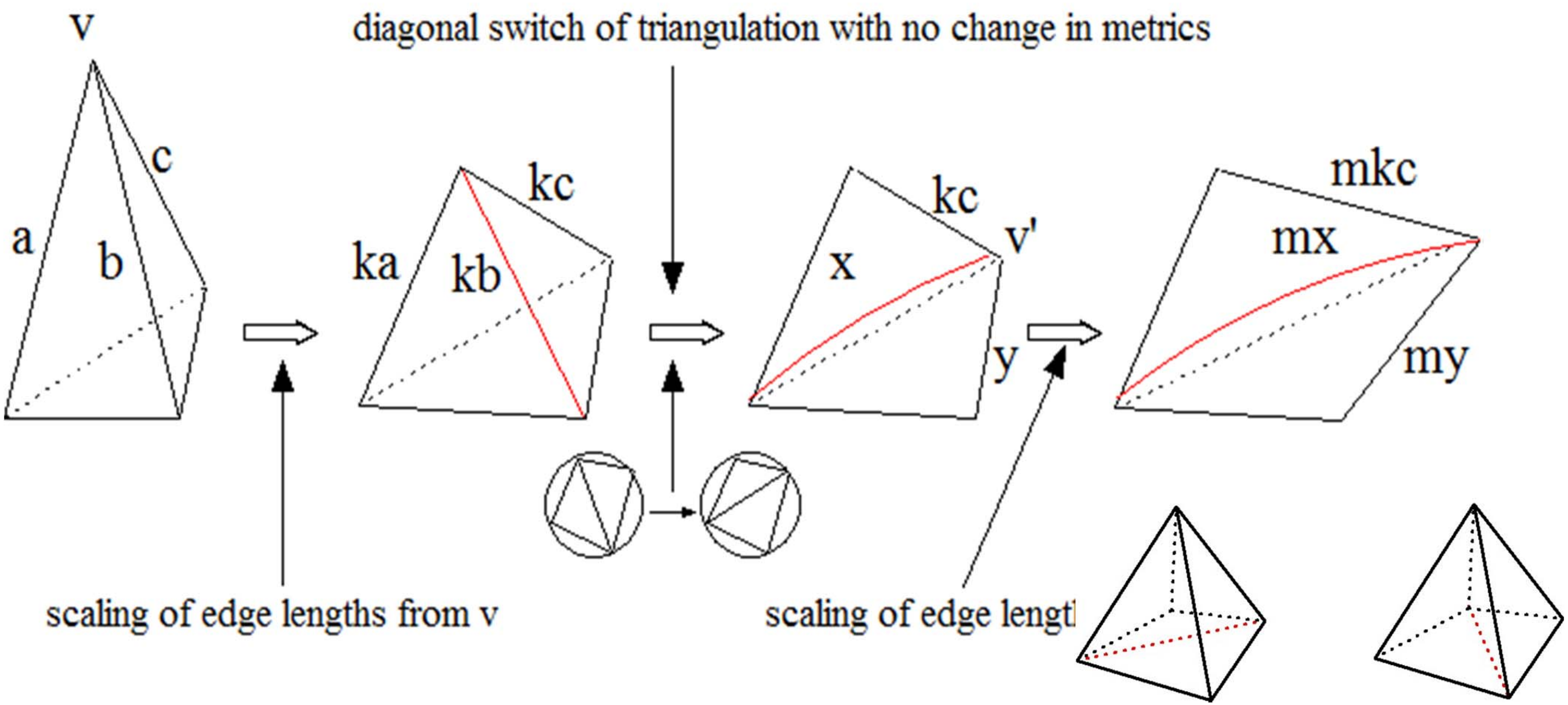
Different Delaunay triangulations of the same metric (S,V,d) are related by :



Voronoi cells

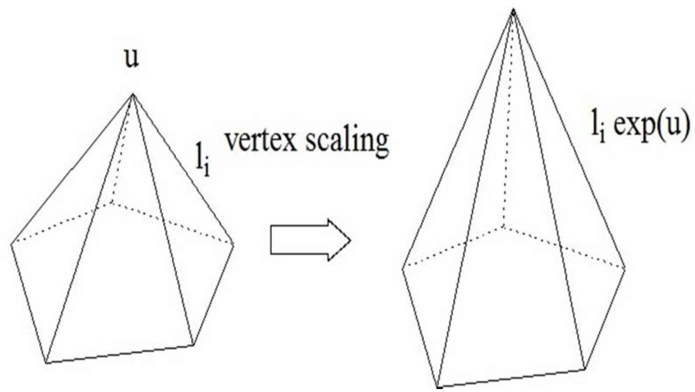
Edge e=vv’ in T

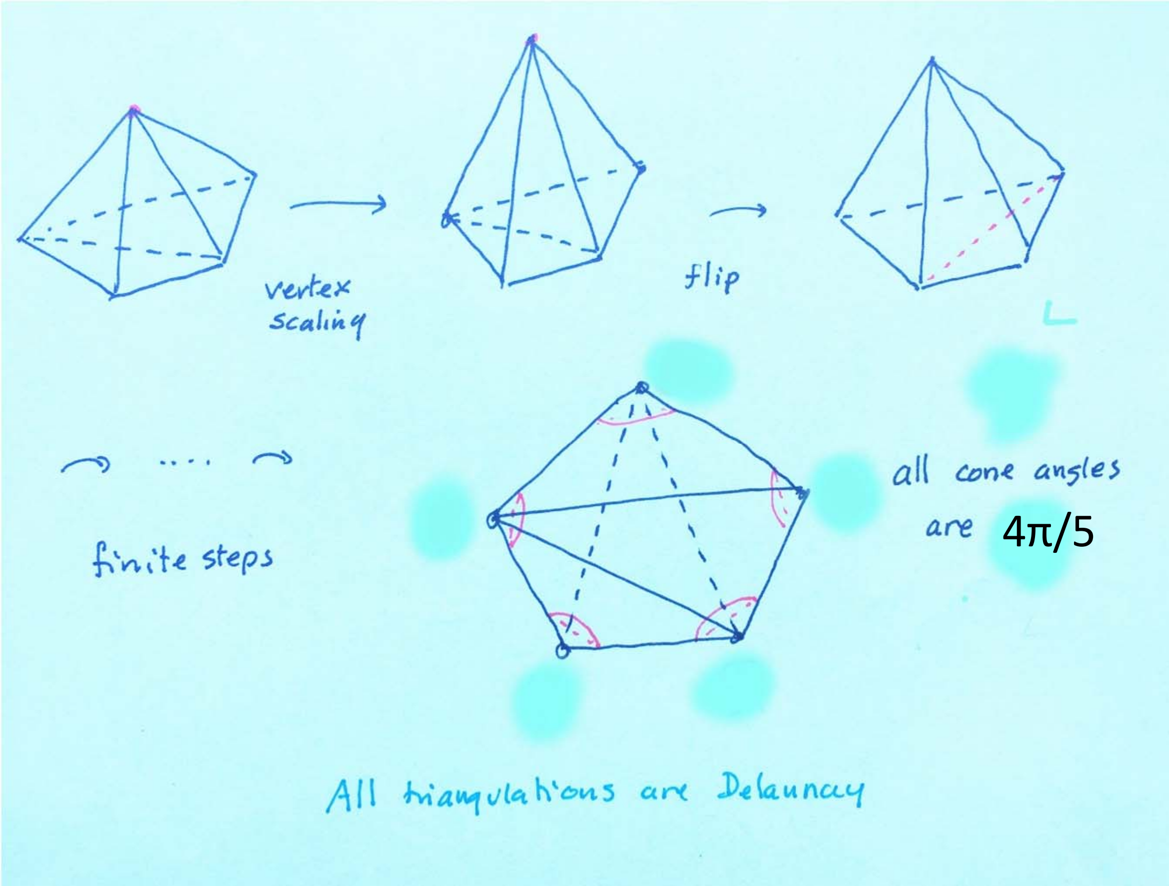
Diagonal switch from T to T’



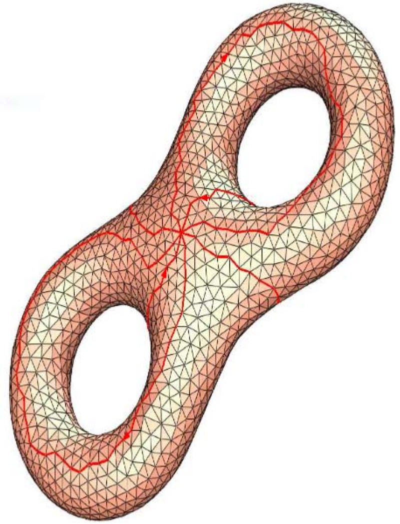
(b) If $T_i \neq T_{i+1}$, then $(S, d_i) \cong (S, d_{i+1})$ by an isometry homotopic to id,

(c) If $T_i = T_{i+1}$, $\exists\ \lambda_i : V \to \mathbf{R}$, s.t., $\ell_{d_{i+1}} = \lambda_{i*}\ell_{d_i}$



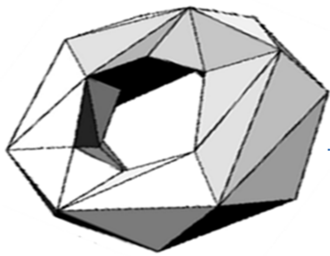


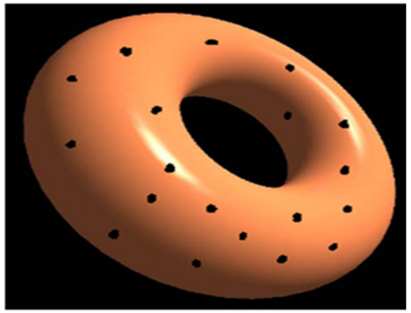
and ∀ K^*: V→ (-∞, 2π),



al principle.

$K^*=0$

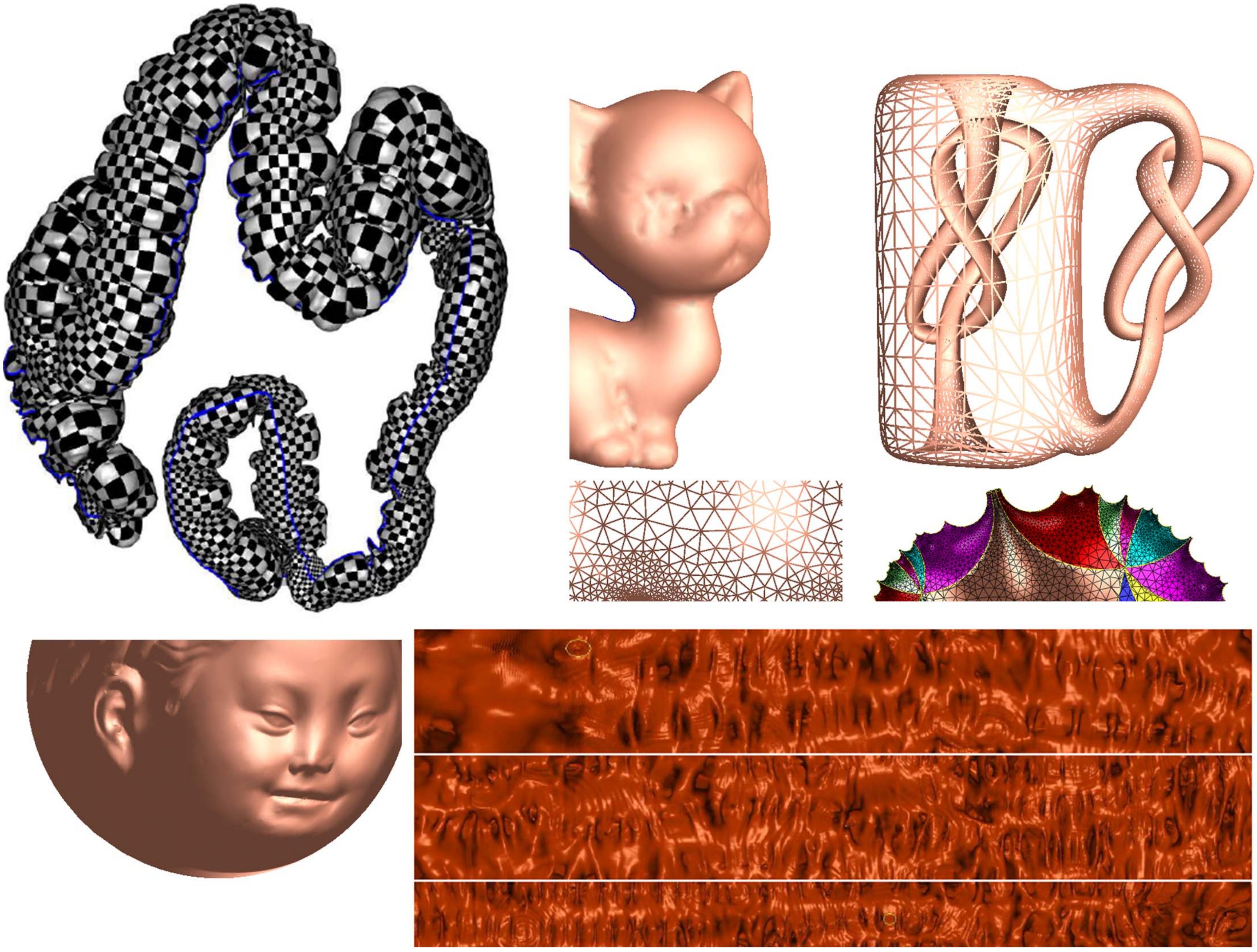




Thm(Fillastre 2008) Every cusped hyperbolic puncture torus is isometric the boundary of a convex hull of a finite set of points in a Fuschian hyperbolic 3-manifolds.

There exists a hyperbolic version of thm 1

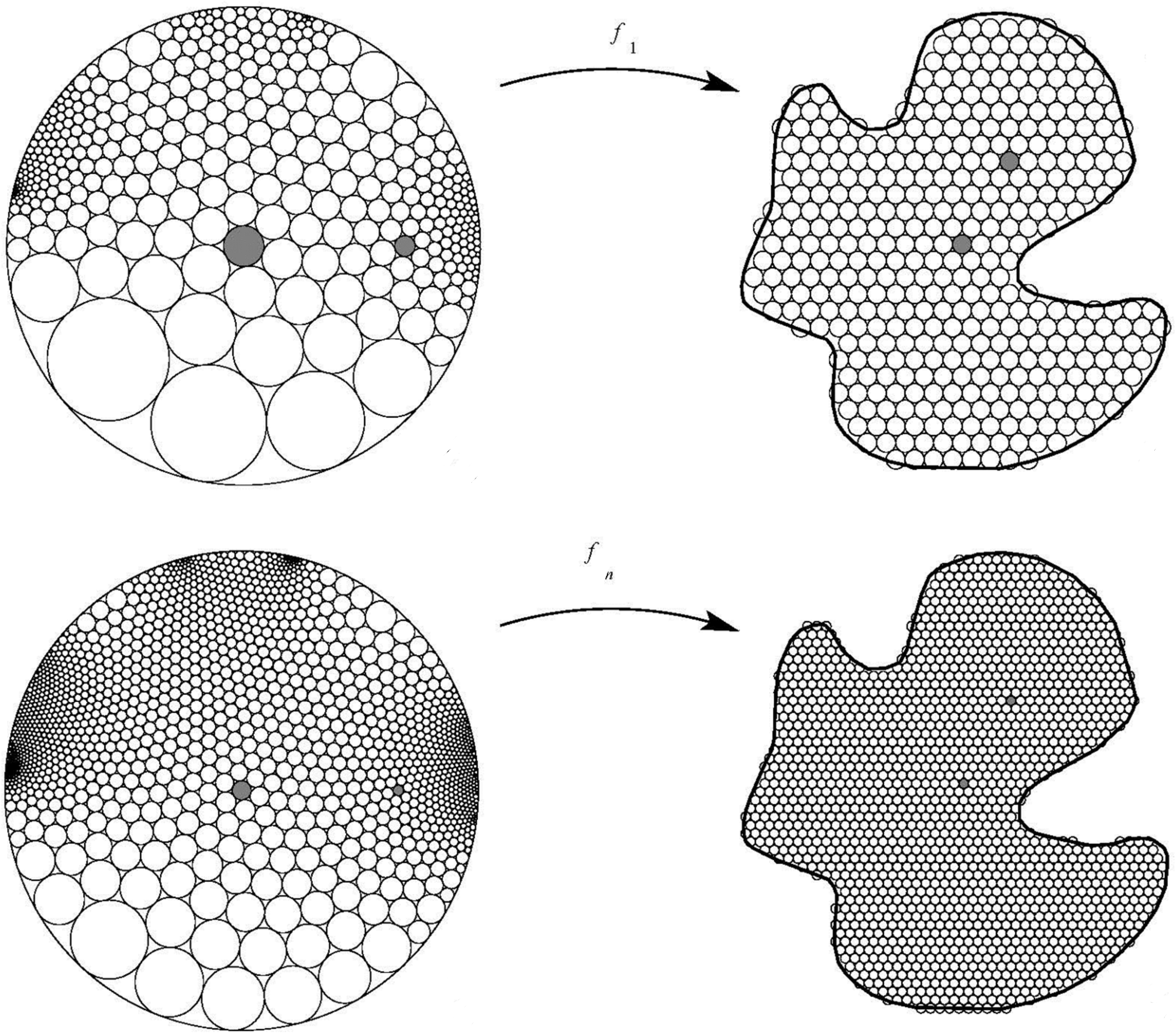
For $K^* = \frac{2\pi\cdot\chi(S)}{|V|}$, d^* is a discrete uniformization metric.



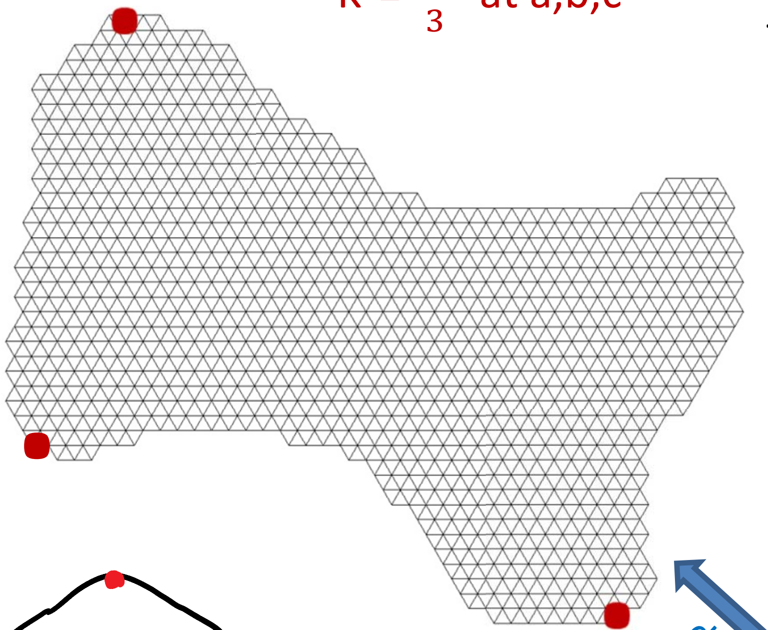
Convergence

Thurston's discrete Riemann mapping conjecture,

Rodin-Sullivan's theorm: $f_n \rightarrow$ the Riemann mapping



$K^* = \frac{4\pi}{3}$ at a,b,c



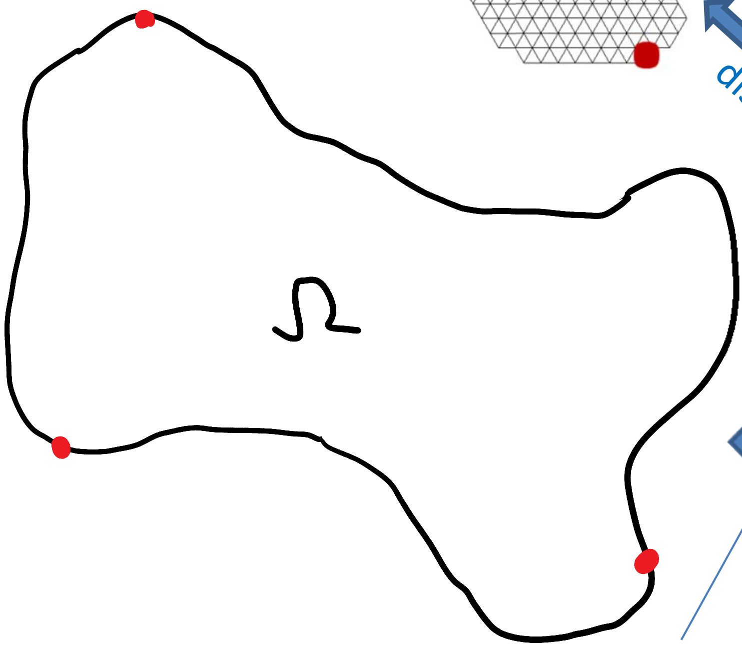
Cor. A polygonal disk (D, V; a,b,c) in **C** is d.c. to the equilateral triangle (ΔABC, V', {A,B,C})

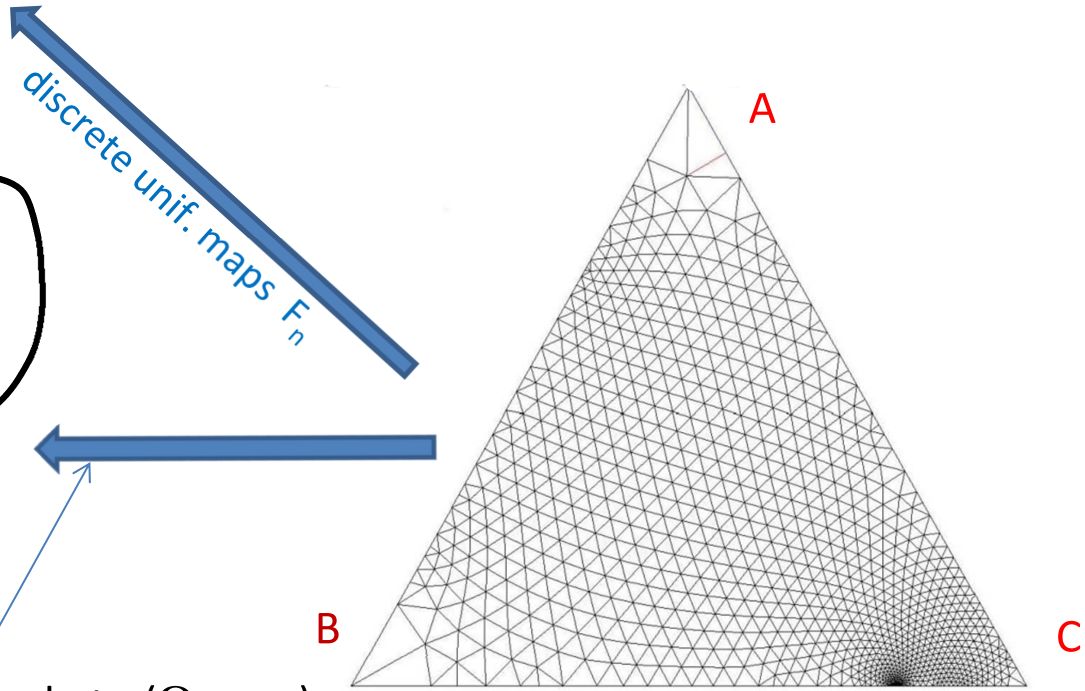
Thm 2 (L-Sun-Wu)

$f_n \to$ Riemann mapping for (Ω;p,q,r).

Counterpart of Thurston's circle packing conjecture:

F_n converges to the Riemann mapping.





Riemann mapping sending the triangle to (Ω;p,q,r).

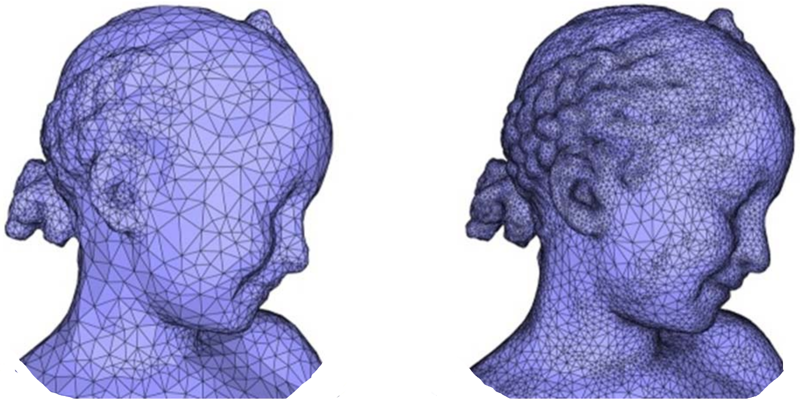
Convergence

(∑,d) is a disk with a Riemannian metric d, and p,q,r three boundary points.

A sequence of PL triangulations (∑, T_n) is *regular* if there exist δ>0, C >0 s.t.

(1) all angles in T_n are in (δ, $\frac{\pi}{2}$ - δ),

(2) all lengths of edges in T_n are in ($\frac{1}{C \cdot n}$, $\frac{C}{n}$)



Thm 3(Gu-Wu-L). If (∑, T_n) is a regular sequence of triangulations of a Riemannian disk (∑, d, p,q,r) and f_n: ∑ → Δ is the associated discrete uniformization map, then f_n converges uniformly to the uniformization map associated to (∑, d).

The same is true for a torus (S^1XS^1, g) with any Riemannian metric.

Cor. The uniformization map for simply connected surface and torus is computable.

Thank you.