#### A metric on the space of genus-zero surfaces



or

#### The harmony of the spheres The harmonic maps of the 2-spheres

Joint with Patrice Koehl

#### The Centrality of Surface Comparison

1. Every object we see is a surface (almost).

2. Cell phone software can already digitize surfaces. Scanning soon to be everywhere.

3. Can we use this digitized data?

4. Potential applications:
Diagnose disease or fracture
Drug design
Recognize bones and fossils
Compare teeth
Recognize bones and fossils



Need definitions: Shape, distance, alignment. Appropriate definition will vary.

#### Uniformization



All maps are conformal

## GOAL 1: A metric on Genus-Zero surfaces



1. $d(C_1, C_2) = 0$   $C_1$  is isometric to  $C_2$  (isometry) 2.  $d(C_1, C_2) = d(C_2, C_2)$  (symmetry) 3.  $d(C_1, C_3) \le d(C_1, C_3) + d(C_1, C_3)$  (triangle inequality) Why a metric?

Each property plays an important role in applications

#### **Isometry:** $d(C_1, C_2) = 0 \iff C_1$ is isometric to $C_2$

Allows for identifying different views of the same object.



We probably want to consider these to be the same object. If so, our distance measure should not change if one shape is moved by a Euclidean Isometry.

It also should not depend on a parametrization.

#### **Symmetry:** $d(C_1, C_2) = d(C_2, C_2)$

The distance between two objects does not depend on the order in which we find them.



If I own the square, and you own the circle, we can agree on the distance between them.



This means that noise, or a small error, does not affect distance measurement very much.

#### **Comparing Surfaces**

Problem: How to compare two surfaces?

1. What is the distance between a pair of surfaces?

2. What is a good alignment between two surfaces?

When  $d(F_1, F_2)$  is small, find a "good" correspondence

$$f:F_1 \longrightarrow F_2$$



#### Landmarks?

Landmark free: Alignment determined completely by geometry.

*Landmarks*: Key feature points are (somehow) chosen and used to align.



Problem: Choosing landmarks can be hard and expensive and is error prone. Focus first on *landmark free methods*.

## Why Intrinsic geometry?

1. Captures similarity between flexible surfaces.



2. Captures similarity between (seemingly) rigid surfaces.

metatarsal bones:



## Searching among diffeomorphisms

How can we search the vast space of diffeomorphisms for a map closest to an isometry?

Choosing the best  $f:F_1 \rightarrow F_2$  from this infinite-dimensional space is hard.

Idea: Restrict our search to conformal maps

$$C:F_1 \longrightarrow F_2$$

C is chosen from the much smaller space of conformal maps. This is still a big space, but not too big to work with.

**Conformal maps exist in genus 0** We can't always find an isometry between  $F_1$  and  $F_2$ .

**Probably Not** 

But for genus zero surfaces  $F_1$  and  $F_2$ , we can *always* find a map that preserves angles.





# Conformal maps don't always exist in genus > 0



A close to conformal map of genus-two surfaces. (Amenta)

## From smooth to discrete

The theory of conformal maps is well developed for smooth surfaces.

Yes



What can we say about discrete surfaces?



## What is a discrete conformal map?

Many definitions and algorithms exist:

- 1. Discrete Ricci Flow
- 2. Discrete Yamabe Flow
- 3. Conformal Mean Curvature Flow
- 4. Harmonic Maps
- 5. Finite Elements
- 6. Optimize a cost function
- 7. Discrete Differential Equation
- 8. Wilmore Flow
- 9. Circle Packings

### Good and Bad triangulations

Warning: Working with discrete surfaces often requires special types of "nice" triangulations. eg Delaunay triangulations.





Good triangulations

Bad triangulations

## Computing a conformal map Finding a conformal map $f: F_1 \longrightarrow F_2$



What does it mean to say that a map  $f: F_1 \longrightarrow F_2$  is conformal when the surfaces are triangulated rather than smooth? How do we compute f? How unique is f?



- a. Compute discrete conformal map to round sphere.
- b. Choose any Mobius transformation *m*.
- c. Take  $f = C_1 m C_2^{-1}$

This gives all possible conformal maps.

## Implementing Conformal Mappings

Uniformization: Any genus-zero surface can be mapped conformally to a round sphere.



A variety of discrete conformal mapping algorithms exist. (Circle packing, Ricci flow, energy minimization ...)



Conformal map by Keenan Crane



We choose *m* to make *f* close to an isometry?

*f* is conformal. At each point *x* of  $F_1$ , *f* stretches lengths by a conformal factor  $\lambda_f(x)$ . If  $\lambda_f(x)=1$  then *f* is an isometry. Idea: Measure how  $\lambda_f(x)$  differs from 1.

#### Definitions:

Symmetric Distortion Energy;

$$E_{sd}(f) = \sqrt{\int_{F_1} (1 - \lambda_f(z))^2 \, dA_1} + \sqrt{\int_{F_2} (1 - \lambda_{f^{-1}}(z))^2 \, dA_2}$$

The smallest energy among all conformal maps from  $F_1$  to  $F_2$  defines a distance:

Symmetric Distortion distance:

$$d_{sd}(F_1, F_2) := \inf_{f \in \mathcal{C}} E_{sd}(f)$$

#### Properties of $d_{sd}$

$$d_{sd} = \inf_{f \in \mathcal{C}} \sqrt{\int_{F_1} (1 - \lambda(f))^2 \ dA_1} + \sqrt{\int_{F_2} (1 - \lambda(f^{-1}))^2 \ dA_2}$$

#### Theorem (H-Koehl)

Given two genus-zero surfaces  $F_1, F_2$ ,

- 1. There is a conformal diffeomorphism  $f : F_1 \to F_2$  with  $E_{sd}(f) = d_{sd}$ .
- 2. d<sub>sd</sub> gives a metric on the space of Riemannian genus-zero surfaces.

**Idea of proof:** Show that  $E_{sd}$  is a proper map on the space of conformal diffeomorphisms ( $PSL(2, \mathbb{C})$ ).

**Question:** What does d<sub>sd</sub> tell us about the resemblance of two shapes? **The hope:** Sensitive to change in shape. Not sensitive to noise.

### Discrete Version

#### Optimizing the conformal map









Distance  $d(S_1, S_2)$  where  $S_1$  is a sphere with 1000 uniformly distributed points and  $S_2$  has N vertices, distributed uniformly (blue) or randomly (red).

#### How do shapes align?



Rotate one of three bumps on a sphere



## Practice Problem: How Round is an object?

Perhaps the simplest shape question: How round is an object?

#### or

How close is an object to a round sphere.



We measure the distance from objects to the round sphere.

#### How Round is a Platonic Solid?



#### How round are the Platonic Solids?





#### **Protein Surfaces**

Proteins are complex molecules whose function in biology is largely determined by their shape.



Proteins can be flexible, like the calmodulin protein above. We would like to compare the "surfaces" of two proteins.

#### From Protein to Surface



Define a surface that envelops the protein.

#### Triangulated Surface from Protein





Two representations of a protein, a stick model and a molecular surface model.

#### Experiment - Roundness of 533 Proteins



## Experiment - A non-rigid Protein



#### Alignments of Brain Cortices



Neuroscientists want to understand the alignment of sulci.

#### Computing distance between Brain Cortices





Find conformal maps to the sphere for each cortex

### Step 1: Brain Cortex to Sphere



#### Computing distance between Brain Cortices



Chose *M* so that *C* minimizes stretching energy among all conformal maps.

#### Computing distance between Brain Cortices



This gives a distance  $d_{sd}$  between the two brain surfaces.

#### How well does this work?



This alignment minimizes E<sub>sd</sub>. It was produced with no human input.

## Results





#### What if we do want landmarks?



#### Find cortex correspondence matching marked points.

## Alignment with Hyperbolic Orbifolds





# Place index-2 cone points with cone angle $\pi$ at endpoints of 8 sulci (4 on each half of the cortex).

Globally Optimal Cortical Surface Matching with Exact Landmark Correspondence IPMI 2013: Information Processing in Medical Imaging, 487-498 Alex Tsui, Devin Fenton, Phong Vuong, Joel Hass, Patrice Koehl, Nina Amenta, David Coeurjolly, Charles DeCarli, Owen Carmichael

## Hyperbolic Orbifolds





(2,2,2,2,2,2) orbifold Covers a (2,3,8) triangle

Octahedron

(2,2,2,2,2,2,2,2,2,2,2,2) orbifold Covers a (2,3,10) triangle

Icosohedron

**Pictures by Gerard Westendorp** 

#### Uniformize



 $B_1$ 



Compute hyperbolic orbifold metric for each brain. (We used Bobenko-Pinkall-Springborn).



Minimize the Dirichlet energy of  $f: O_1 \rightarrow O_2$ There is a *unique* harmonic diffeomorphism in each isotopy class (Eells-Sampson). This gives a *canonical alignment*.



We get a *canonical* alignment that exactly matches chosen landmark points. Conformality is comparable to other methods. (But other methods are not canonical)

#### Acknowledgements

Mobius Voting for Surface Correspondence Yaron Lipman, Thomas Funkhouser ACM Transactions on Graphics (Proc. SIGGRAPH), August 2009

Algorithms to Automatically Quantify the Geometric Similarity of Anatomical Surfaces Doug M. Boyer\*, Yaron Lipman\*, Elizabeth St. Clair, Jesus Puente, Biren A. Patel, Thomas Funkhouser, Jukka Jernvall, and Ingrid Daubechies *PNAS, vol. 108 no. 45, November 8, 2011* 

*Computing General Geometric Structures on Surfaces Using Ricci Flow* Miao Jin, Feng Luo, and Xianfeng Gu. Computer-Aided Design (CAGD), 39(8):663-675, August 2007.

## Thanks for coming!



#### THANK YOU FOR COMING!



#### Morphometrices

## A traditional approach to measuring distance between shapes.





#### Requires human expertise. Expensive and slow and error prone.

#### **Triangulated Surfaces from Bones**



Metatarsal (toe) bones from 23 old and new world monkeys and 38 prosimians (Boyer, Debauchies, Lipman, et al}

#### Distance between Teeth



#### Distance between Arm Bones



#### Proximal radius turned from a disk to a sphere.

#### Distance between Toe Bones



Metatarsal bones of primates

### Comparing results

Evolutionary tree based only on metatarsal bone shapes.



#### Some results



ROC tests on 61 metatarsals, 45 radius surfaces, 99 teeth. Red is optimal diffeomorphism, Blue is optimal transport, Black and Purple are expert observers.

#### Distal radius

#### Analysis of anatomical data

ROC (Receiver Operating Characteristic) Analysis



#### Metatarsal

#### Analysis of anatomical data

ROC (Receiver Operating Characteristic) Analysis





#### Phylogenetic Tree from Toe Bones



This tree is generated from a distance between metatarsal bones.