On the Applicability of Convex Relaxations for Matching Symmetric Shapes

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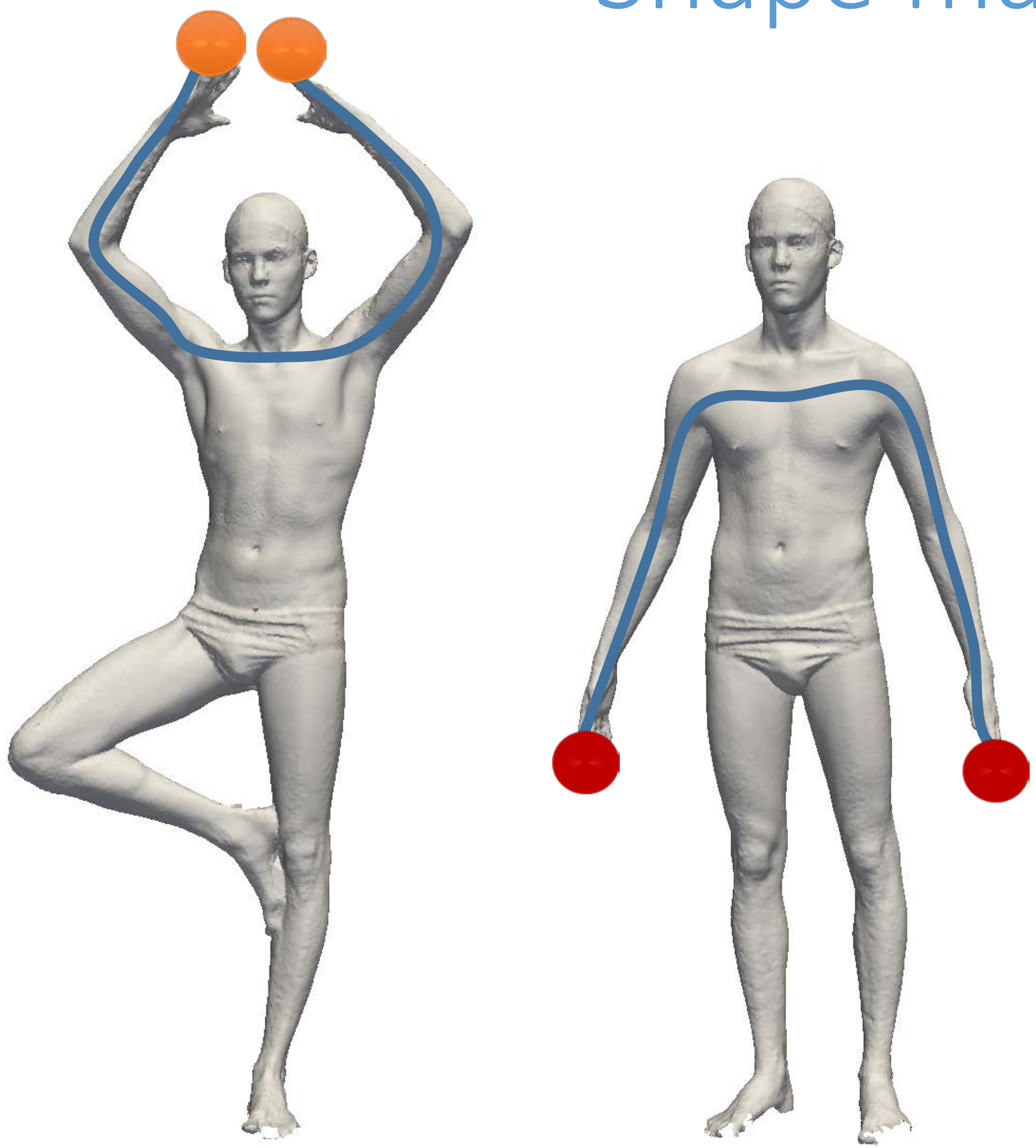
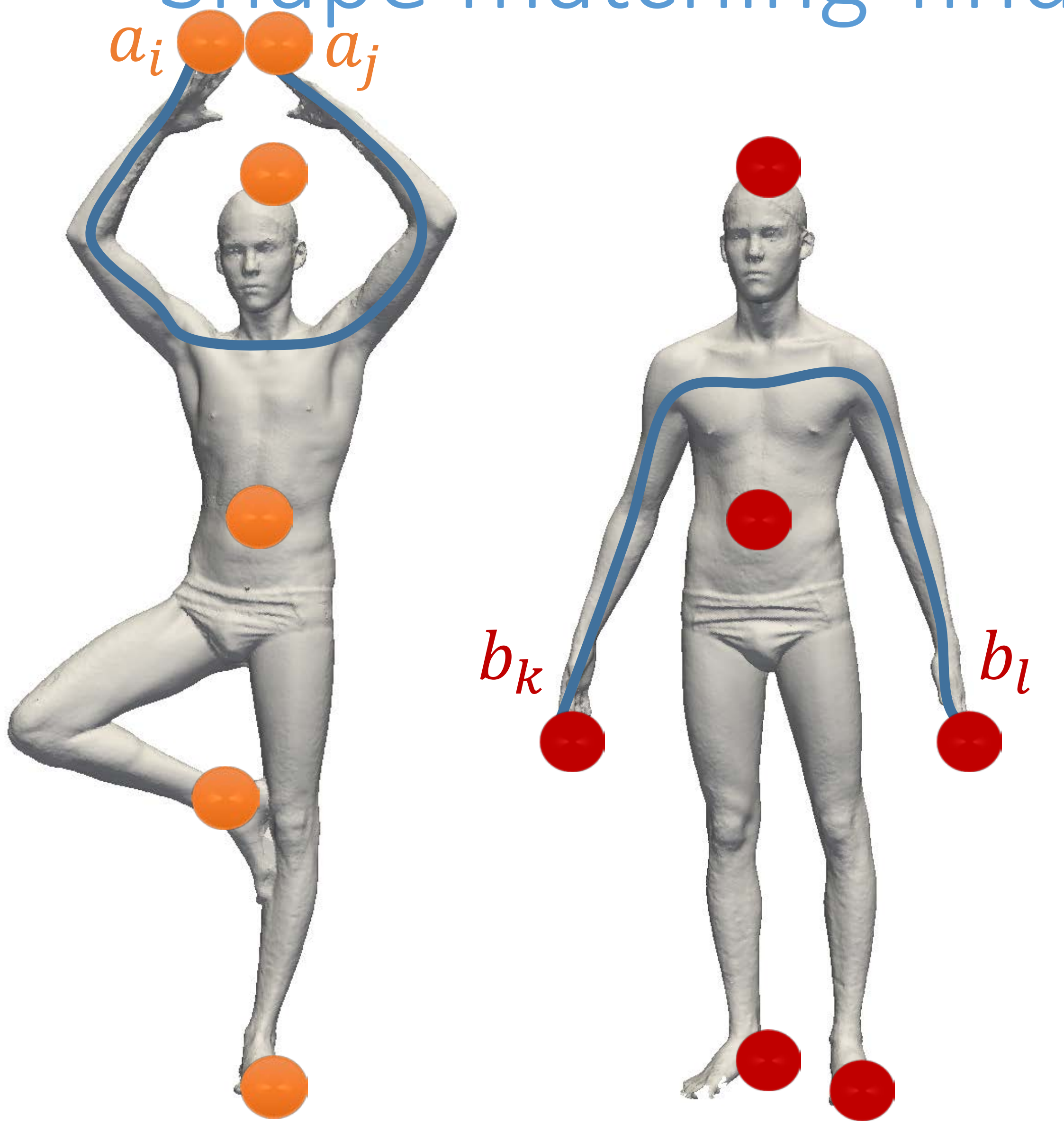
Shape matching



Finding isometries

Shape matching-finding isometries



$$A_{ij} = d(a_i, a_j)$$

$$B_{kl} = d(b_k, b_l)$$

Goal: find mapping (permutation)
$\sigma: \{a_1, \ldots, a_n\} \rightarrow \{b_1, \ldots, b_n\}$
such that

$$A_{ij} = B_{\sigma(i),\sigma(j)}$$

Graph isomorphism

Input: $A = A^T, B = B^T$

Goal: find permutation (if exists) σ such that

$$A_{ij} = B_{\sigma(i),\sigma(j)}$$

$$\Longleftrightarrow$$

Goal: find permutation matrix $P \in \Pi_n$ such that

$$A = P^T B P$$

Graph matching/quadratic assignment

Input: $A = A^T, B = B^T$

Output: $P \in \Pi_n$ such that $A \approx P^T B P$:

$$P^* = argmin_{P \in \Pi_n} ||A - P^T B P||_F$$

$$P^* = argmin_{P \in \Pi_n} ||PA - BP||_F$$

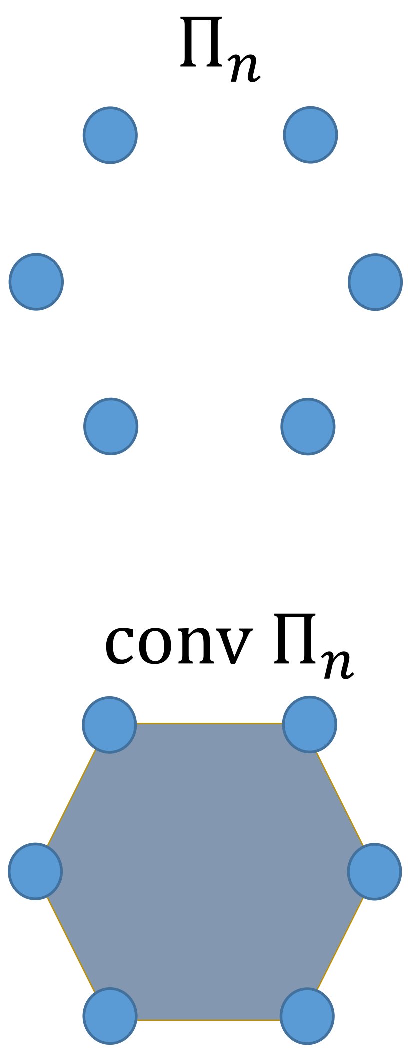
Graph matching

DS relaxation

$$P^* = argmin_{P \in \Pi_n} ||PA - BP||_F$$

NP Hard!

$$S^* = argmin_{S \in conv\ \Pi_n} ||SA - BS||_F$$



DS relaxation

conv Π_n is the Birkhorff polytope:

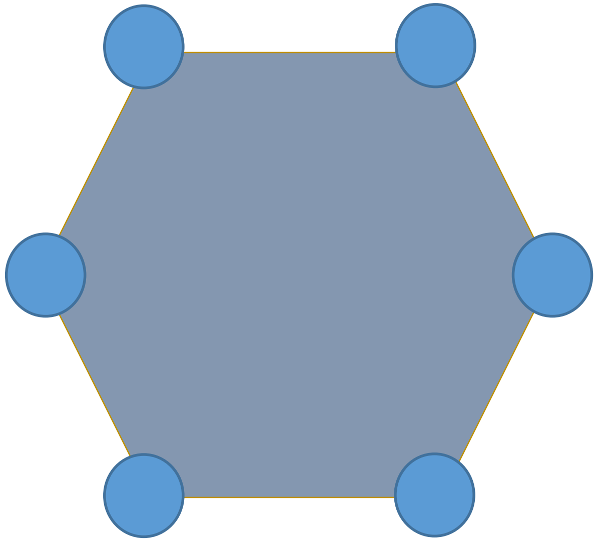
$$DS = \{S |\ S \geq 0, S1 = 1, S^T 1 = 1\}$$

DS relaxation:

$$S^* = argmin_{S \in DS} ||SA - BS||_F$$

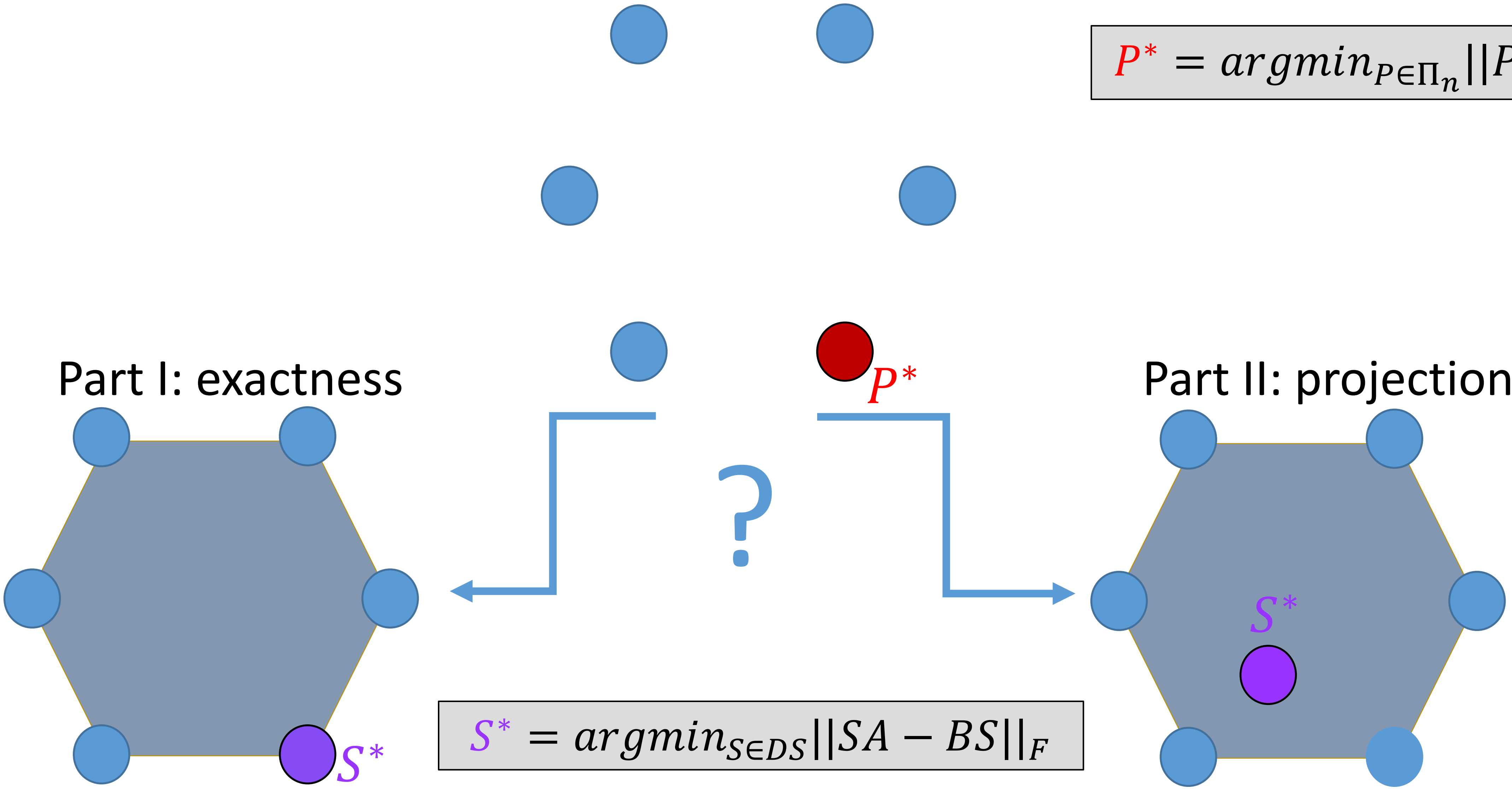
$$GM_{DS}(A, B)$$

conv Π_n



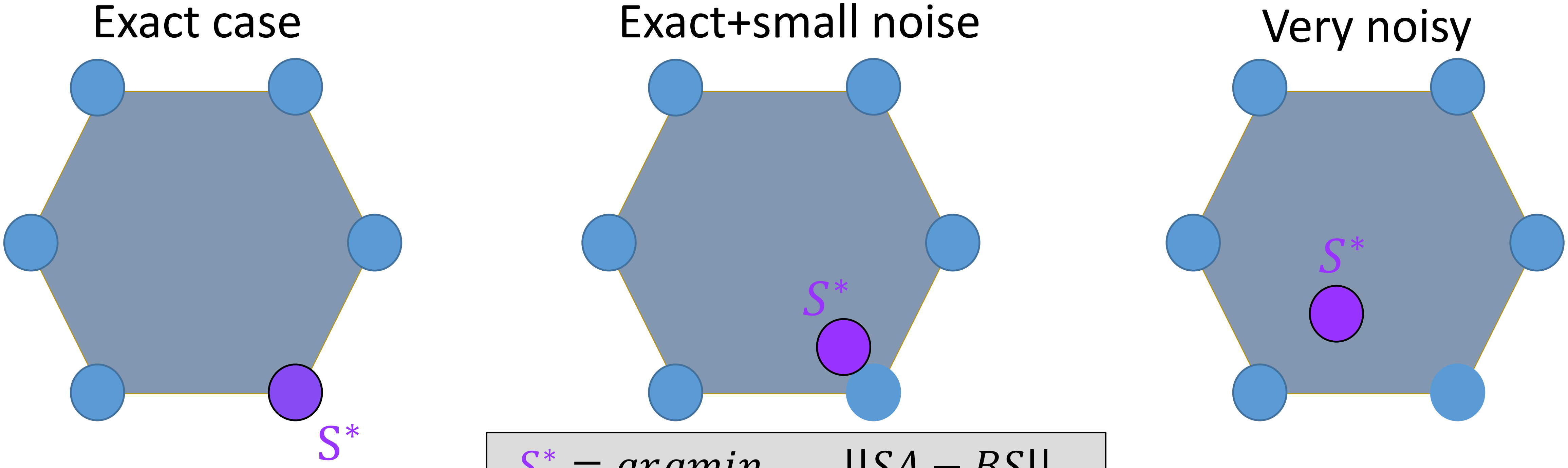
DS relaxation-does it work?

$$P^* = argmin_{P \in \Pi_n} ||PA - BP||_F$$



$$S^* = argmin_{S \in DS} ||SA - BS||_F$$

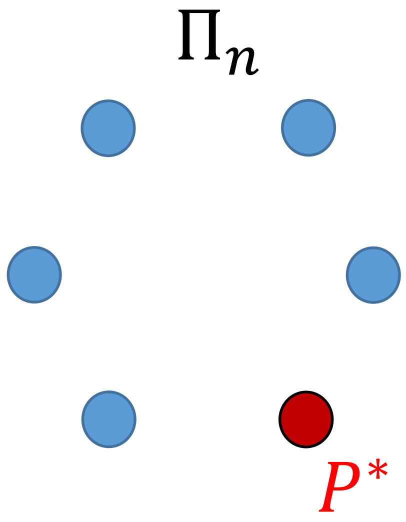
Exactness affects projection efficiency



$$S^* = argmin_{S \in DS} ||SA - BS||_F$$

Part I: Exactness

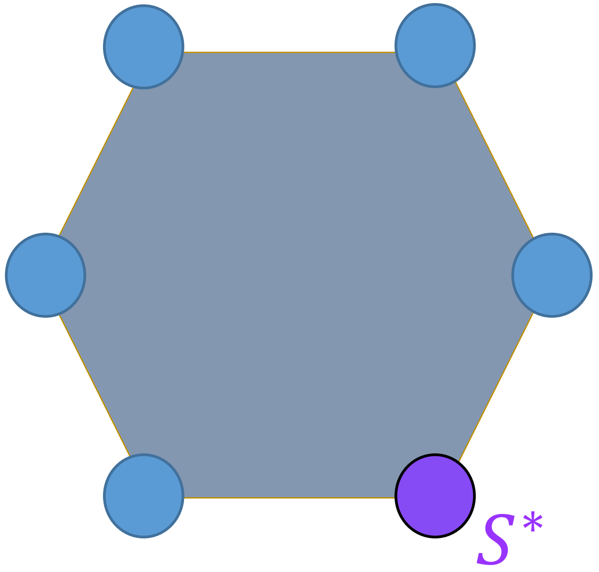
[Aflalo et al 2015], [Fiori and Sapiro 2015]



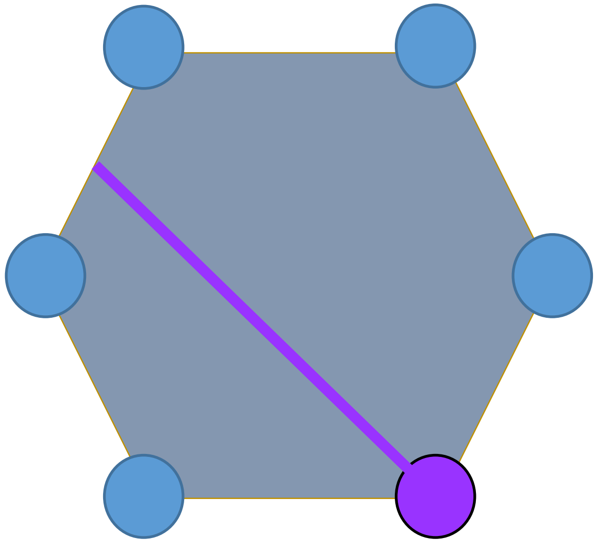
Assume: (i) $A \cong B$ (ii) Unique isomorphism P^*

$$\min_{S \in DS} ||AS - SB||_F = 0$$

Usually

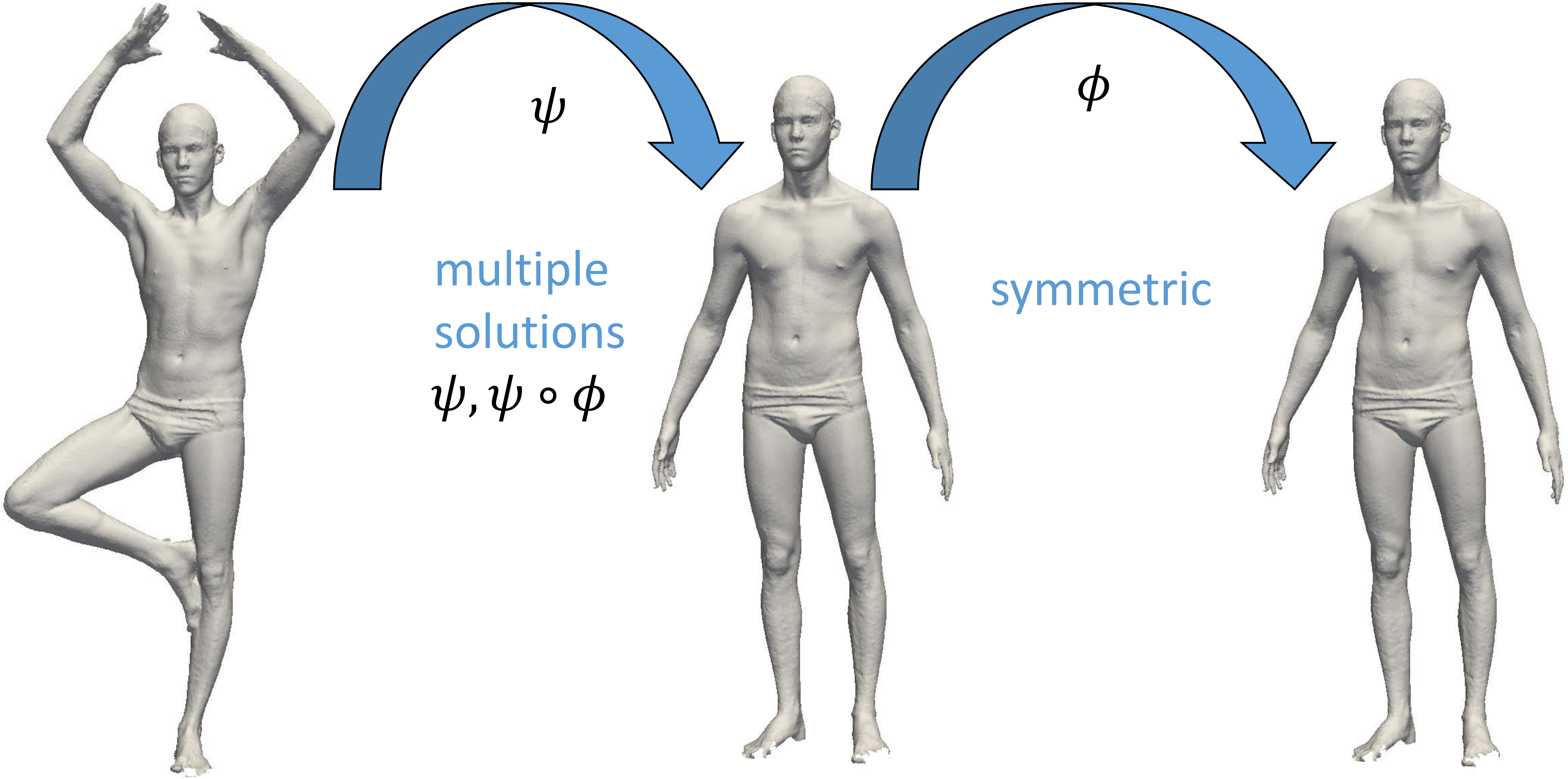


Usually not



$S^* = argmin_{S \in DS} ||SA - BS||_F$

Problem: Unique solution assumption



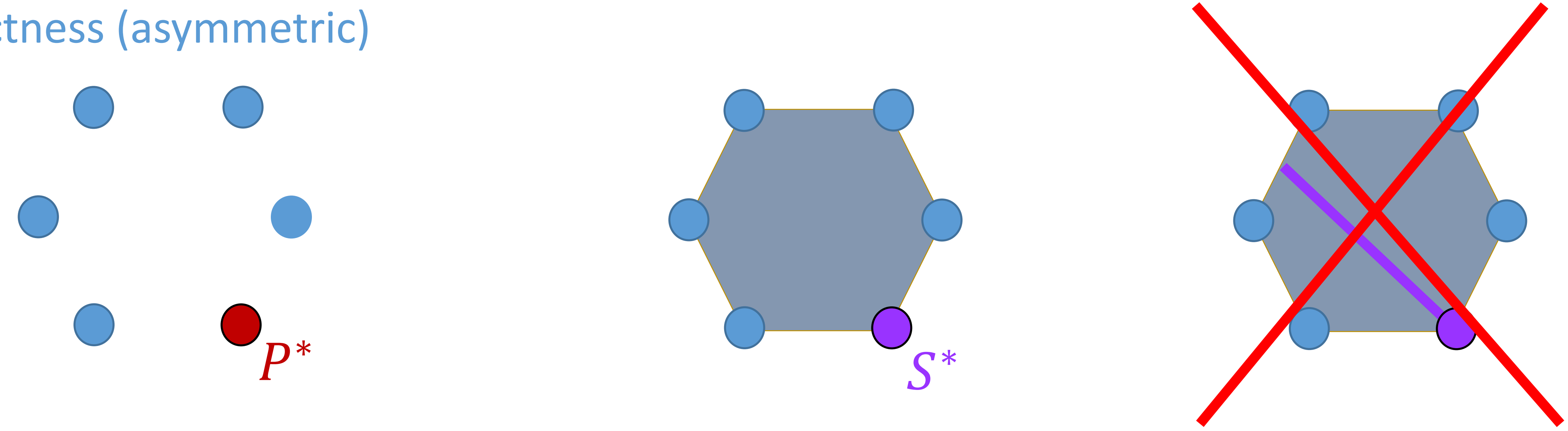
Symmetries of natural shapes

Bilateral symmetry: [SHREC]

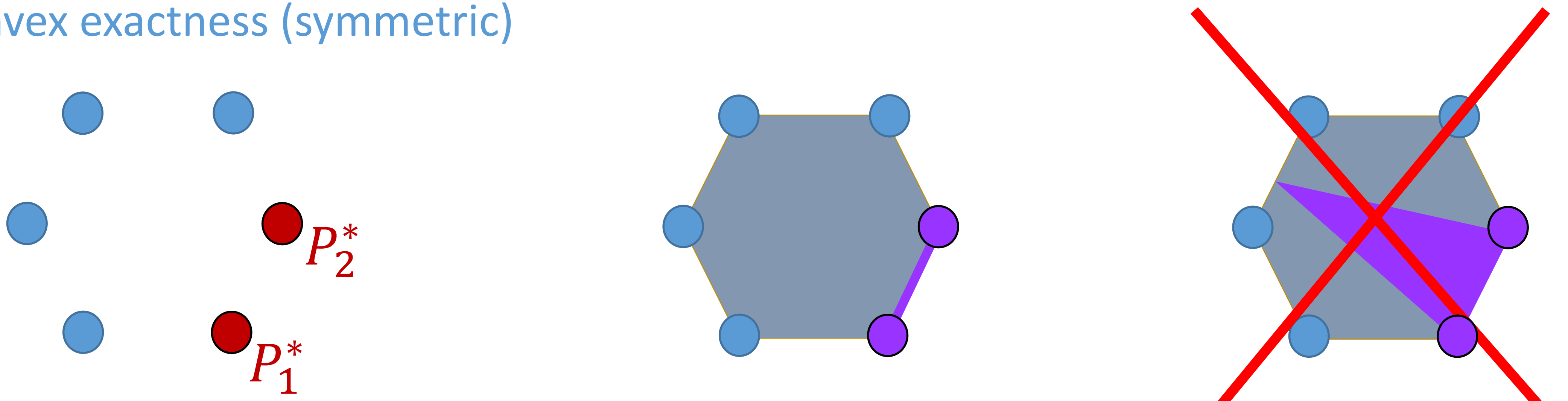
[SCAPE, FAUST, TOSCA]

Exactness vs. convex exactness

Exactness (asymmetric)



Convex exactness (symmetric)

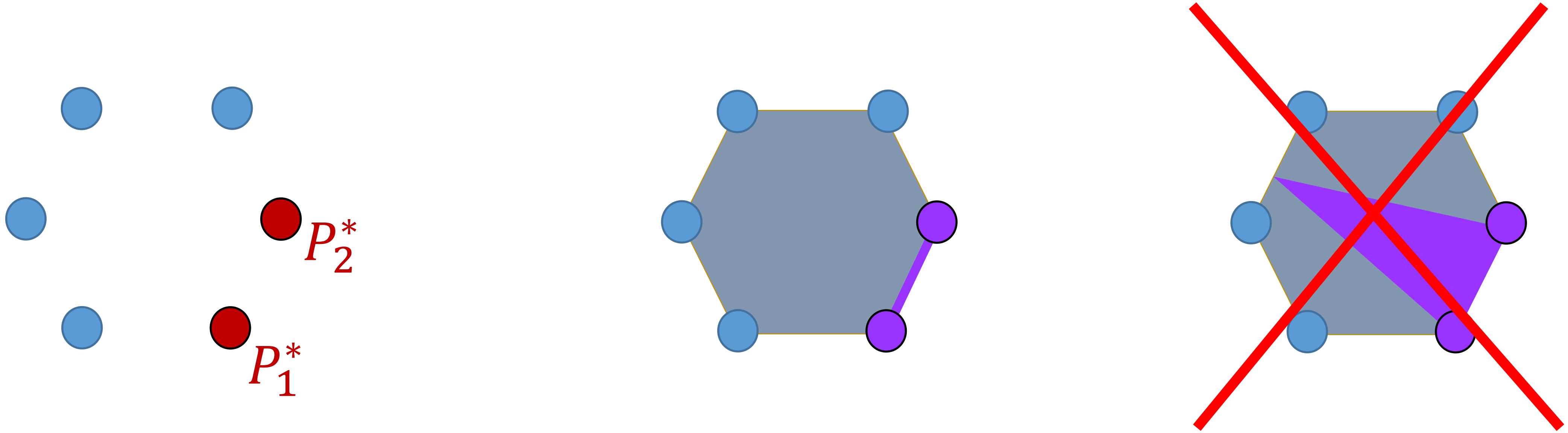


Convex exactness-definition

$$Iso(A,B) = \{P \in \Pi_n |\ AP = PB\}, \qquad Iso_{conv}(A,B) = \{S \in DS |\ AS = SB\}$$

$GM_{DS}(A,B)$ is convex exact if

$$\boxed{Iso_{conv}(A,B) = conv(Iso(A,B))}$$



A convenient reduction (B=A)

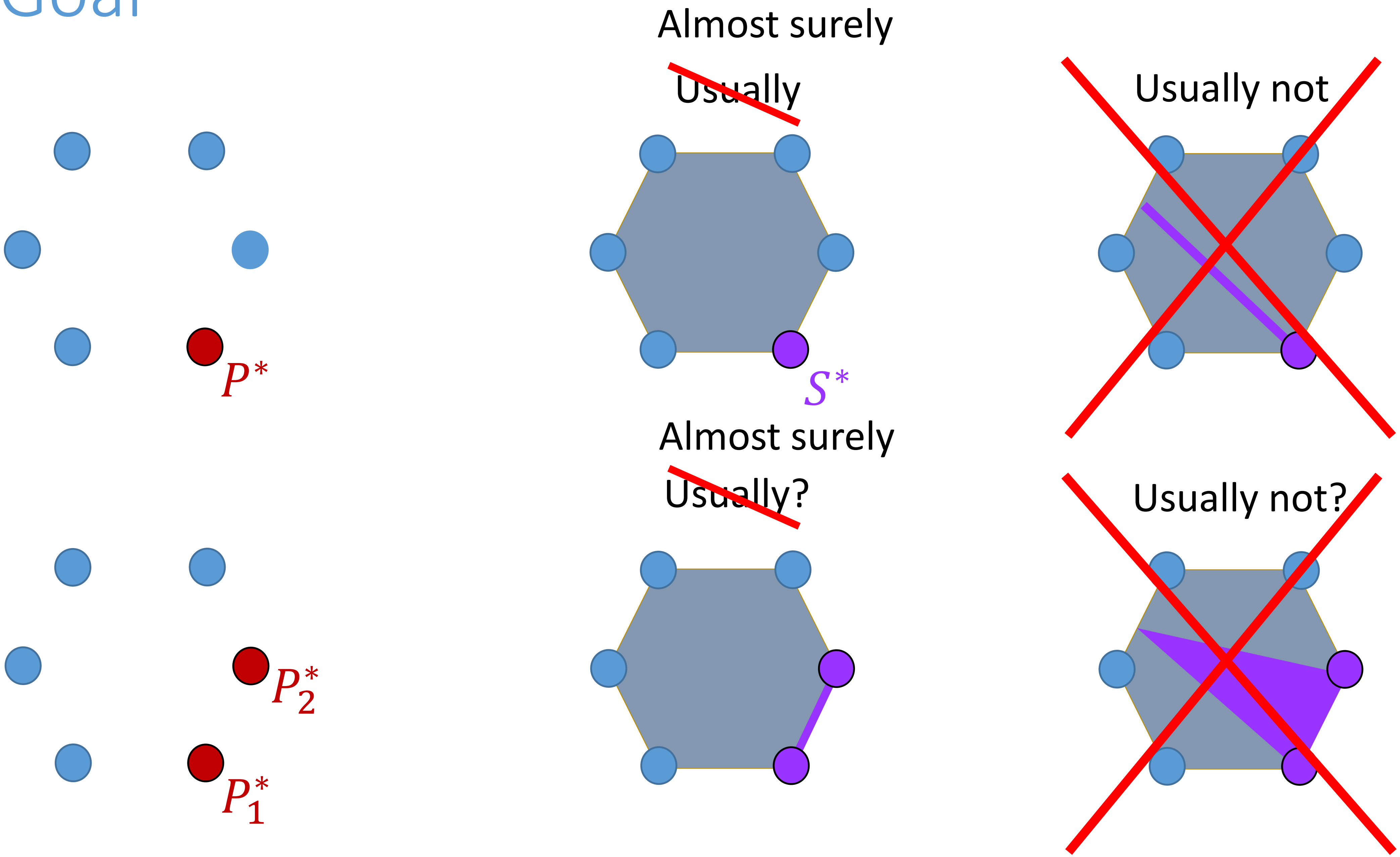
<u>(easy) Lemma:</u>

$GM_{DS}(\mathrm{A}, \mathrm{A})$ is convex exact

$\Leftrightarrow$

For any B s.t. $B \cong A$,
$GM_{DS}(\mathrm{A}, \mathrm{B})$ is convex exact

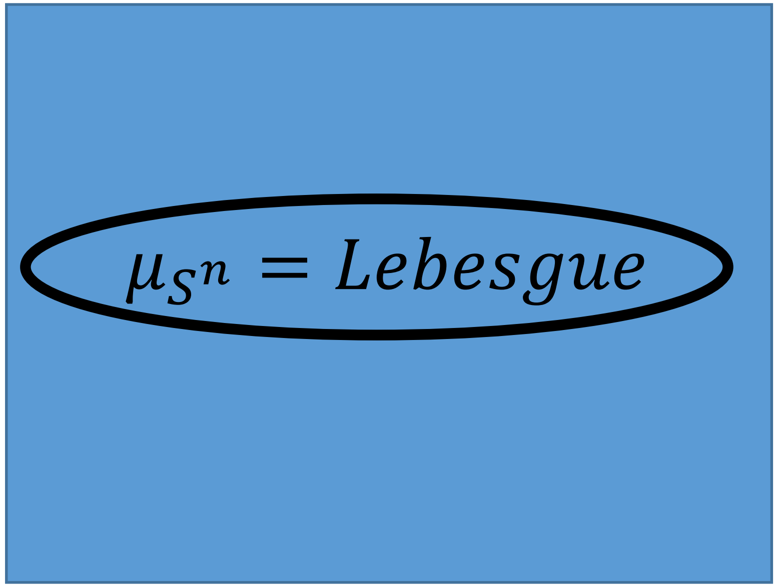
Goal


Almost surely
~~Usually~~
Usually not
P^*
S^*
Almost surely
~~Usually?~~
Usually not?
P_2^*
P_1^*

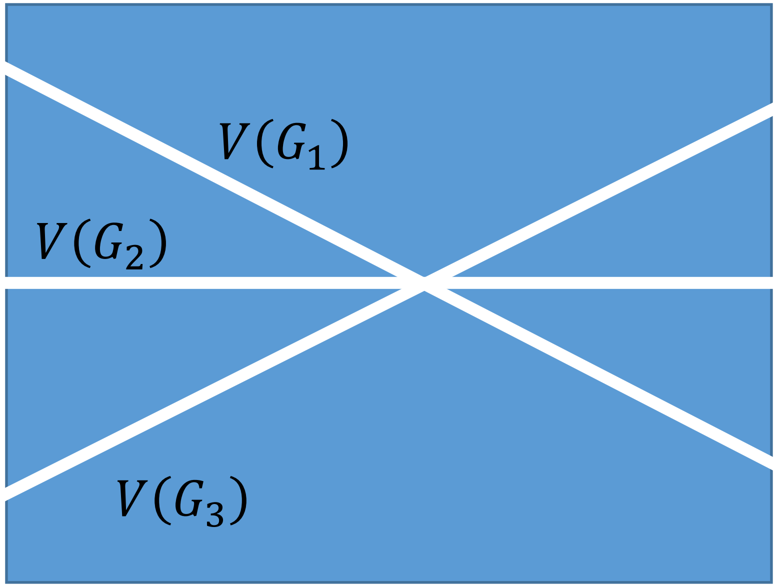
Measure for the space of asymmetric graphs

Asymmetric graphs: $\{A = A^T | A$ has no non-trivial automorphisms$\}$

$S^n = \{A | A = A^T\}$



Asymmetric graphs



$G_i > \{I_n\}$

$$V(G) = \{A \in S^n | \, PA = AP, \forall P \in G\}$$

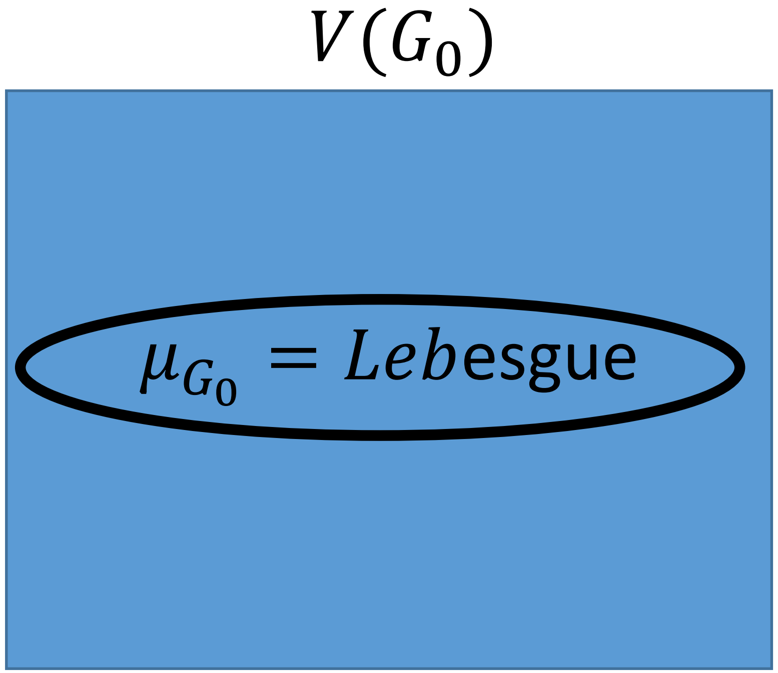
Measure for Graphs with prescribed symmetry group G_0

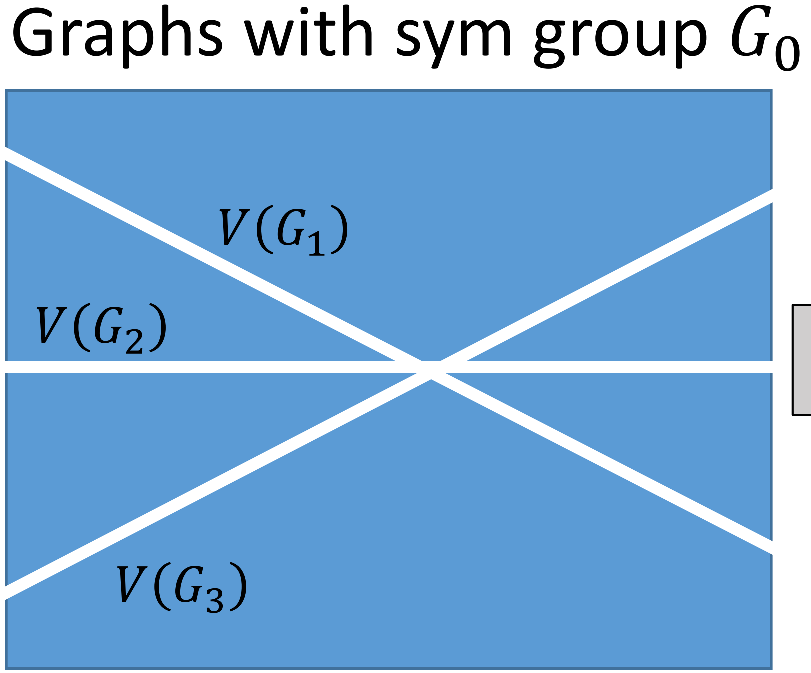
Graphs with sym group G_0: Graphs whose automorphism group is G_0

$$\cap$$

$$V(G_0) = \{A \in S^n \mid PA = AP, \forall P \in G_0\}$$

Measure for Graphs with prescribed symmetry group G_0





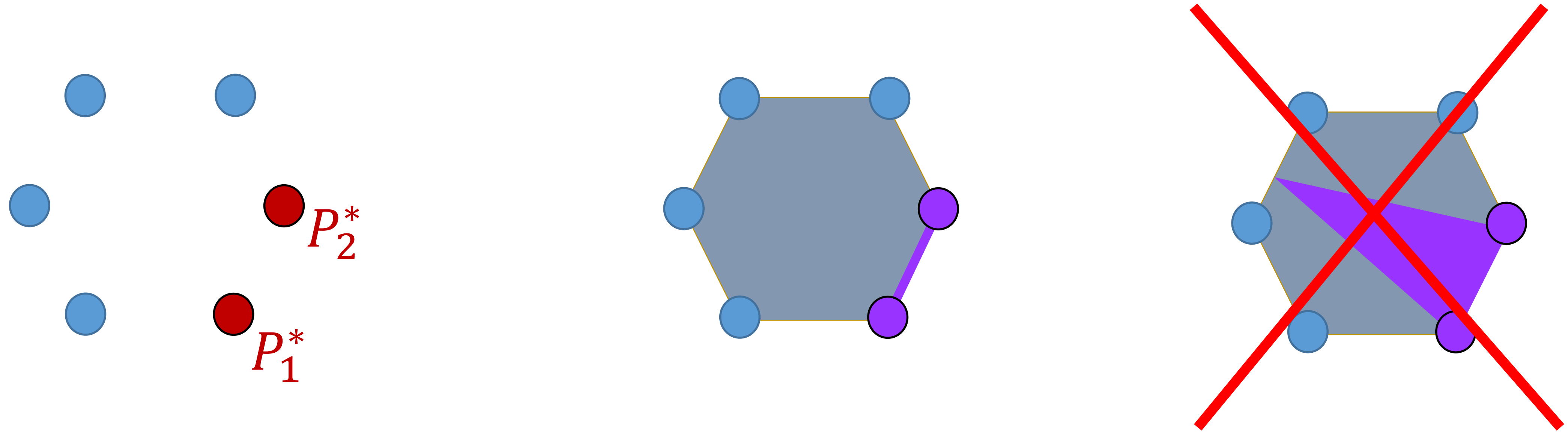
$G_i > G_0$

$$V(G) = \{A \in S^n \mid PA = AP, \forall P \in G\}$$

Convex exactness for reflective groups

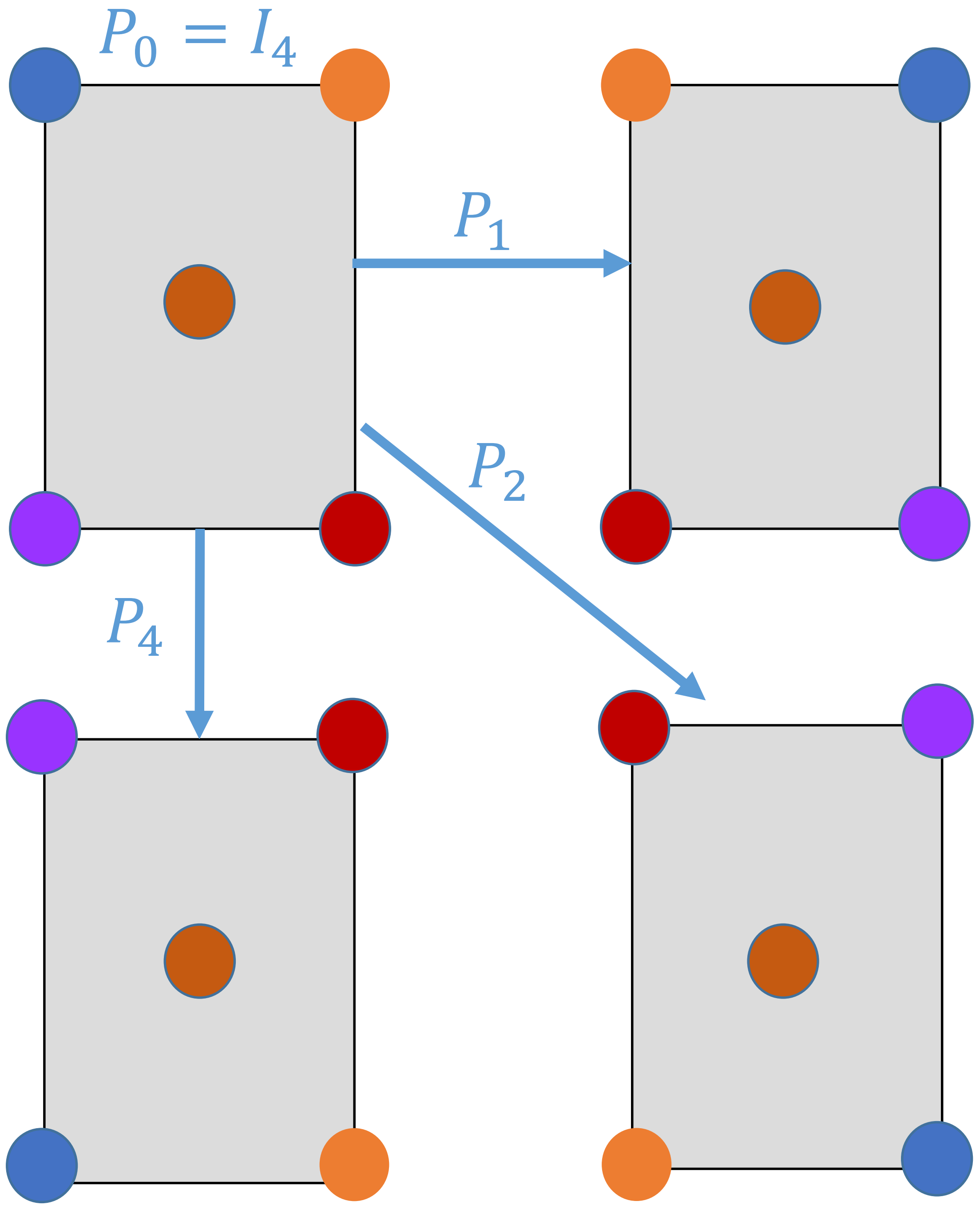
Theorem 1: If $G \cong Z_2$, then for μ_G almost every A, $GM_{DS}(A, A)$ is convex exact.

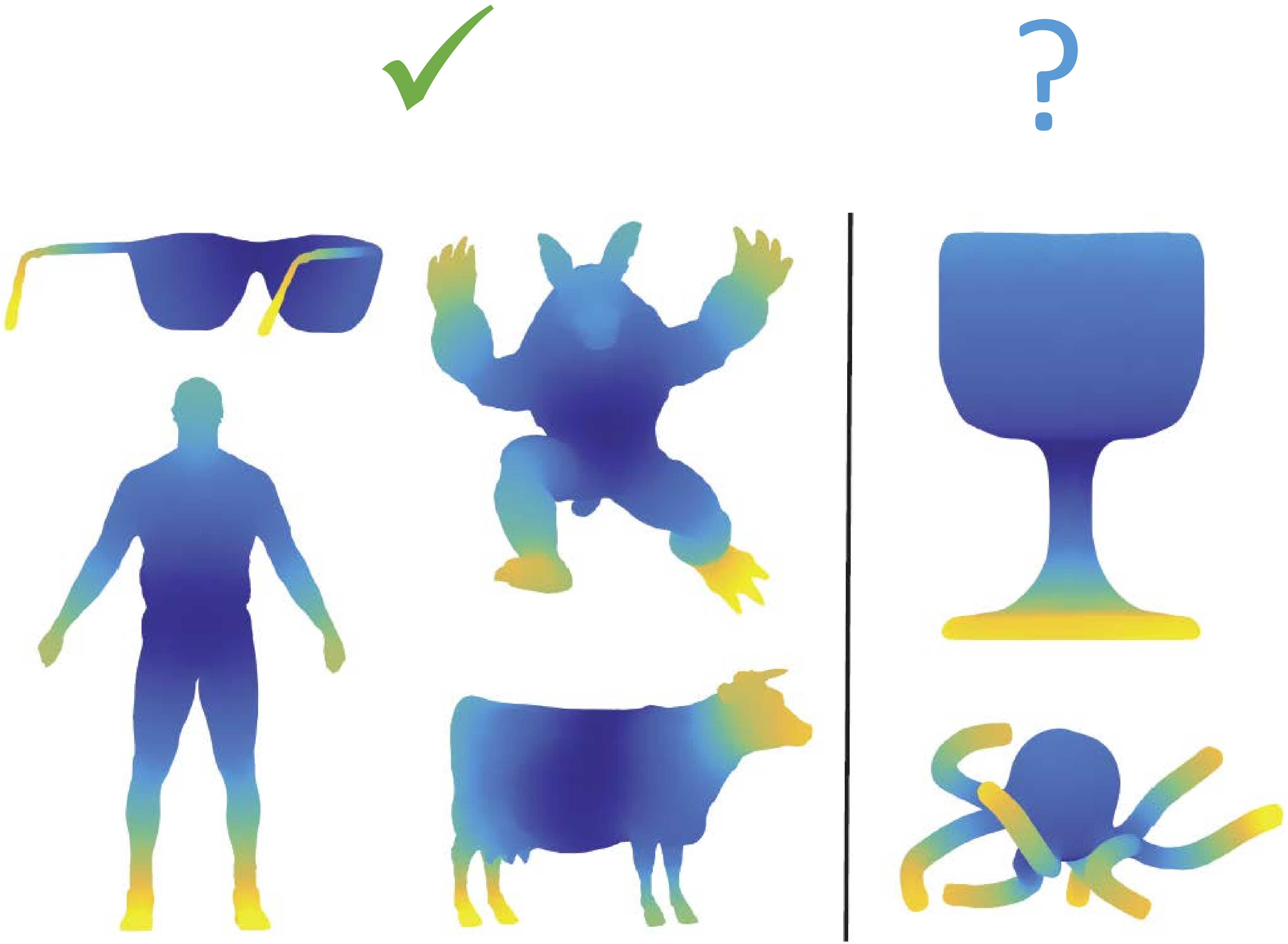
Also true for $G = \{I_n\}$



In general Theorem 1 holds if

- G is reflective ($P^2 = I_n, \forall P \in G$).
- G's action on the vertices has a full orbit.





General groups: 0-1 probability

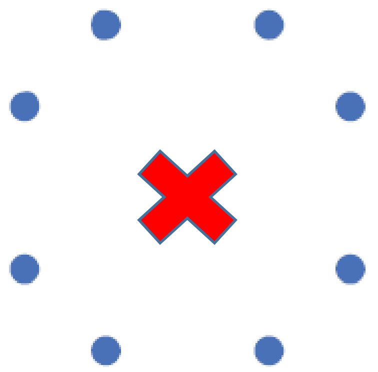
Theorem 2: For any $G \leq \Pi_n$, either

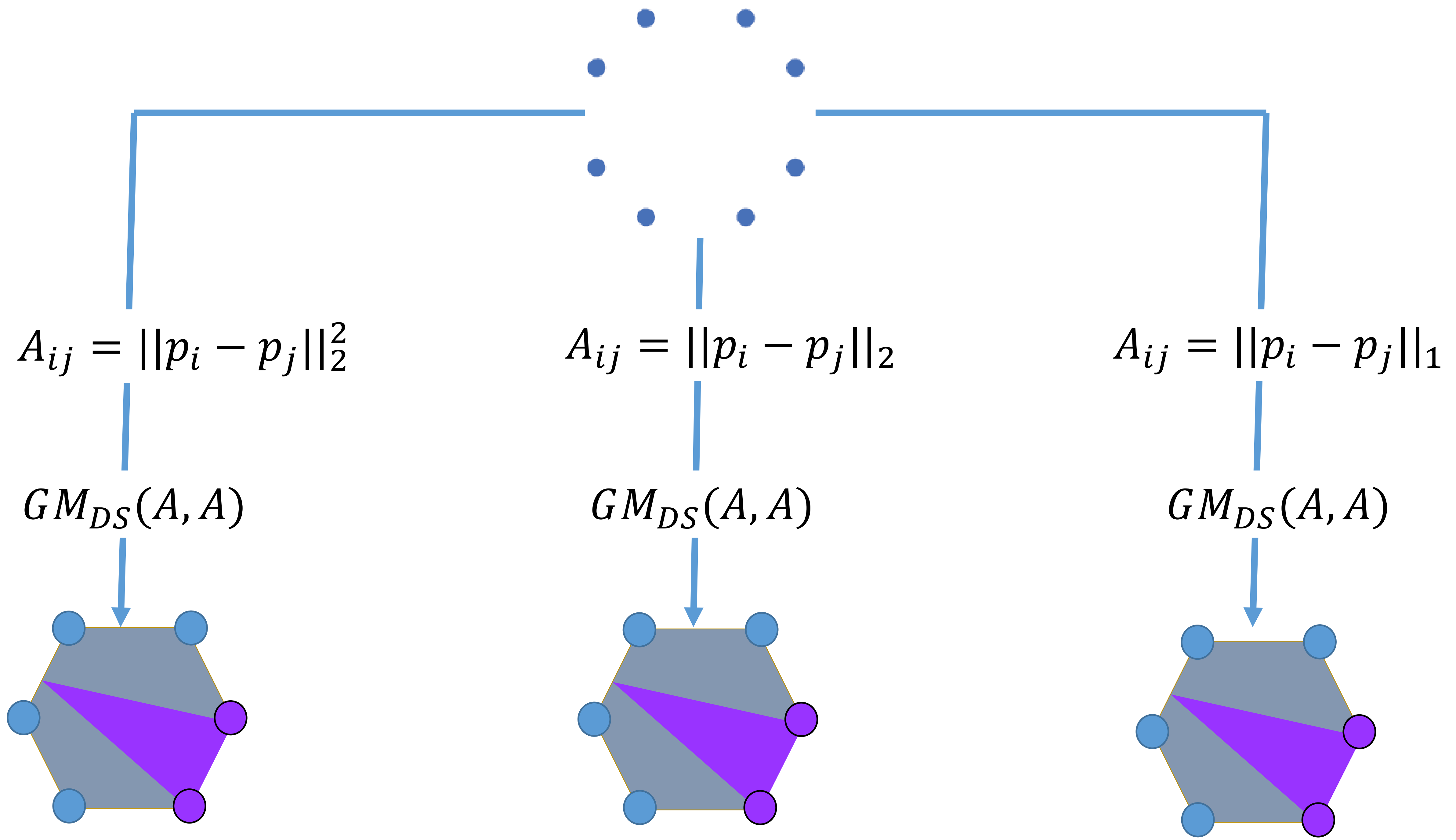
(i) For μ_G a.e. A, $GM_{DS}(A, A)$ is convex exact.

Or

(ii) For **all** A with sym group G, $GM_{DS}(A, A)$ is **not** convex exact.

Proof is constructive...





$A_{ij} = ||p_i - p_j||_2^2$
$A_{ij} = ||p_i - p_j||_2$
$A_{ij} = ||p_i - p_j||_1$
$GM_{DS}(A, A)$
$GM_{DS}(A, A)$
$GM_{DS}(A, A)$

General groups: 0-1 probability

Theorem 2: For any $G \leq \Pi_n$, either

(i) For μ_G a.e. A, $GM_{DS}(A, A)$ is convex exact.

Or

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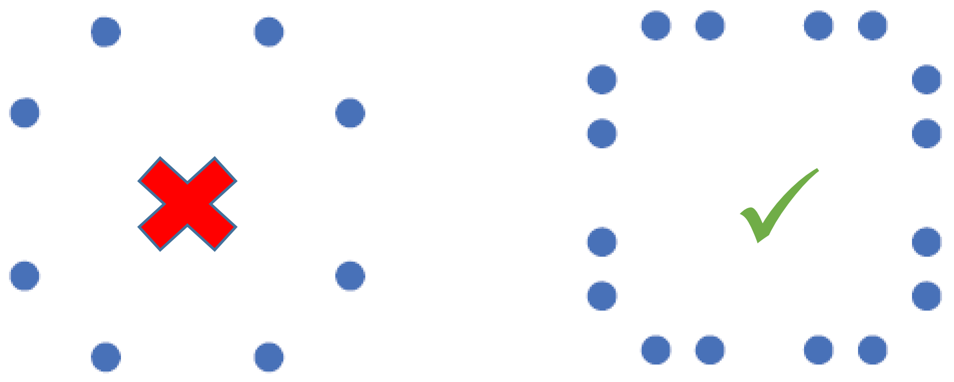
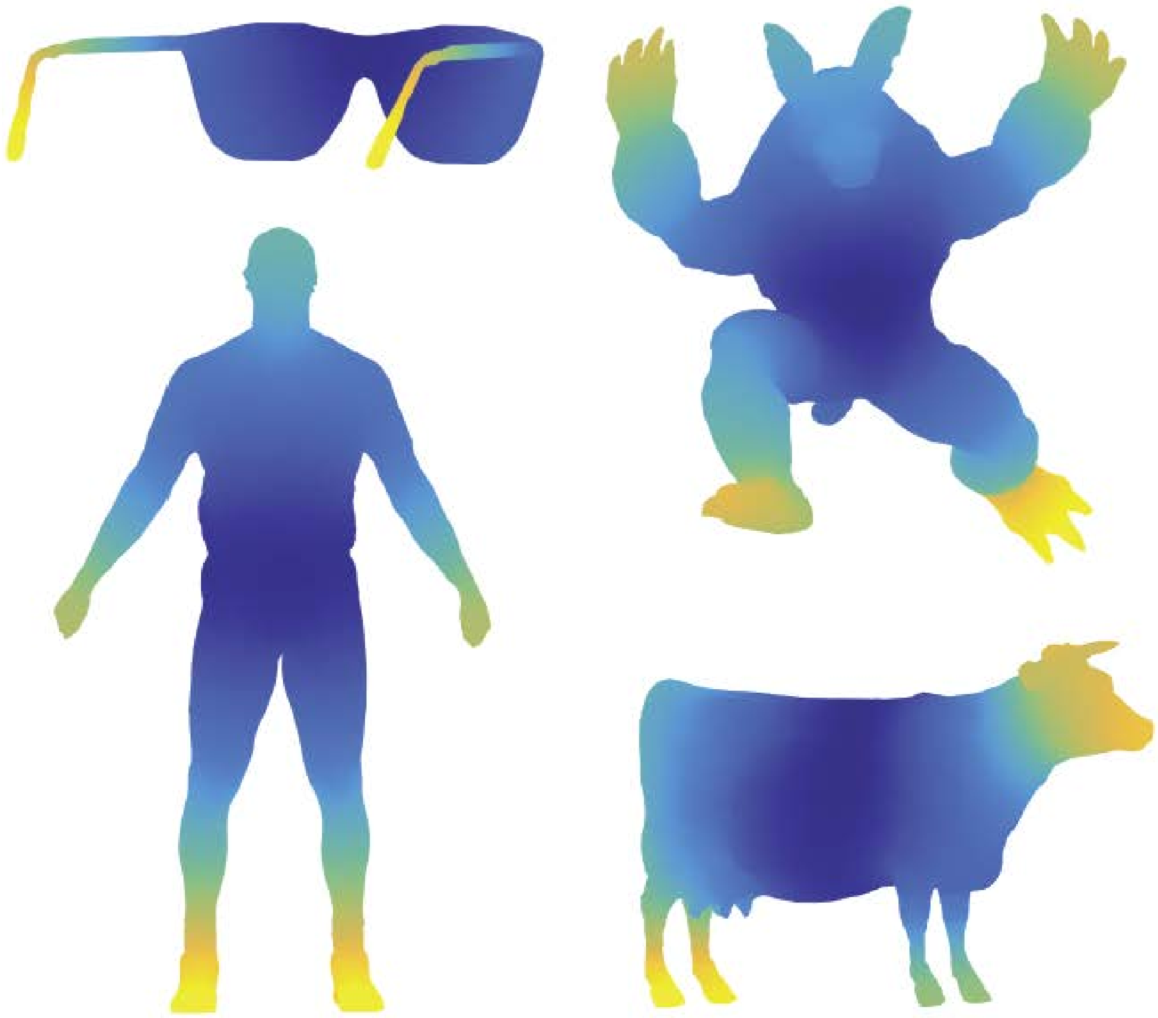


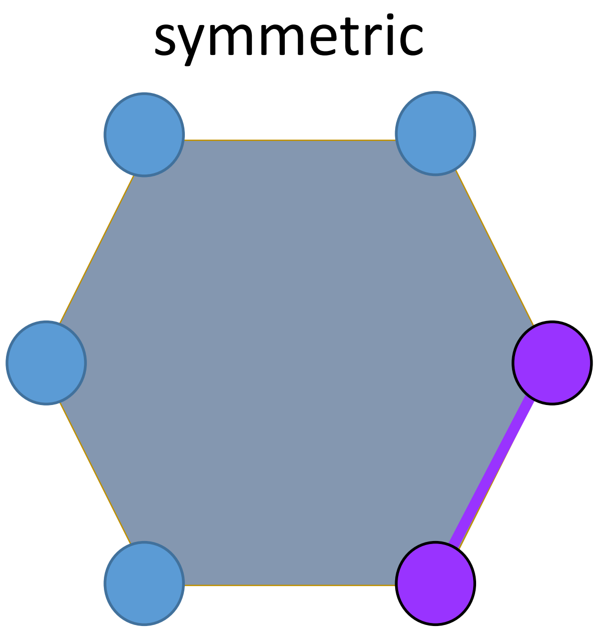
Summary- Part I convex exactness

✓ almost everywhere



? almost everywhere

Part II: Where's my permutation?

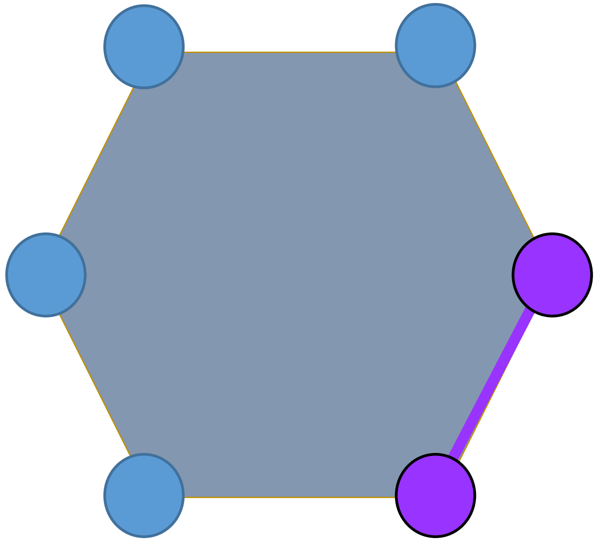


$$\textit{set of minimizers} = \{S \in DS | AS = SB\}$$

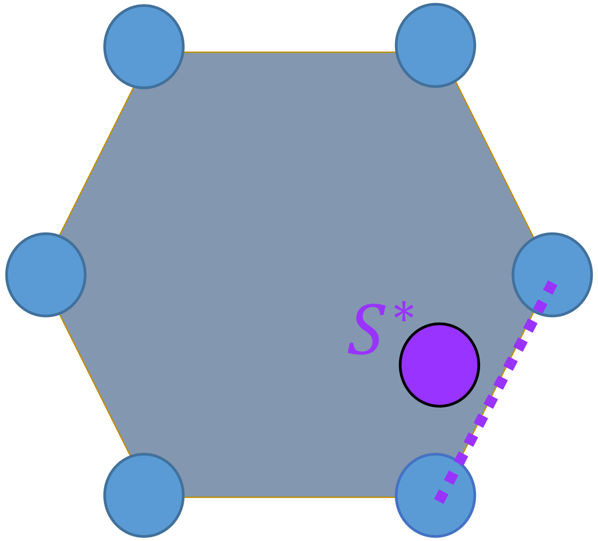
Simplex
algorithm

Part II: Where's my permutation?

symmetric



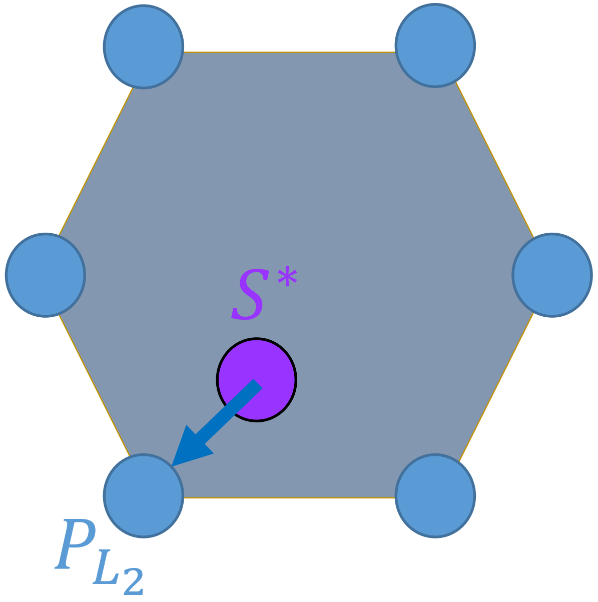
noise



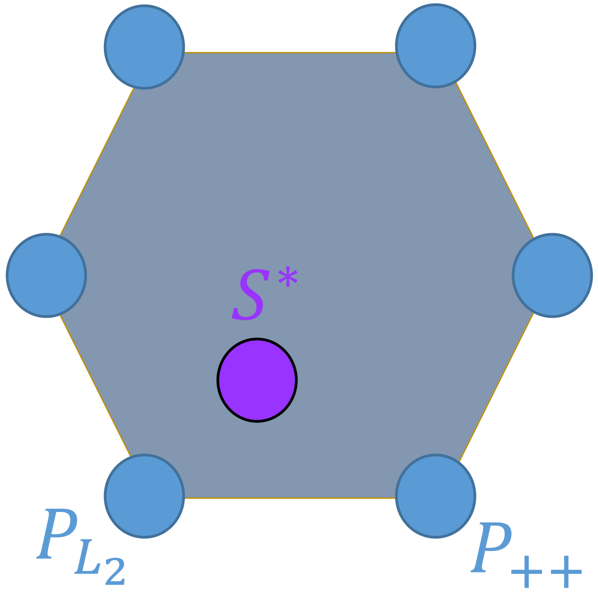
DS relaxation- L_2 projection

$$S^* = argmin_{S \in DS} ||SA - BS||_F$$

$$P_{L_2} = argmin_{P \in \Pi} ||S^* - P||_F$$



DS++: convex2concave projection



convex2concave projection

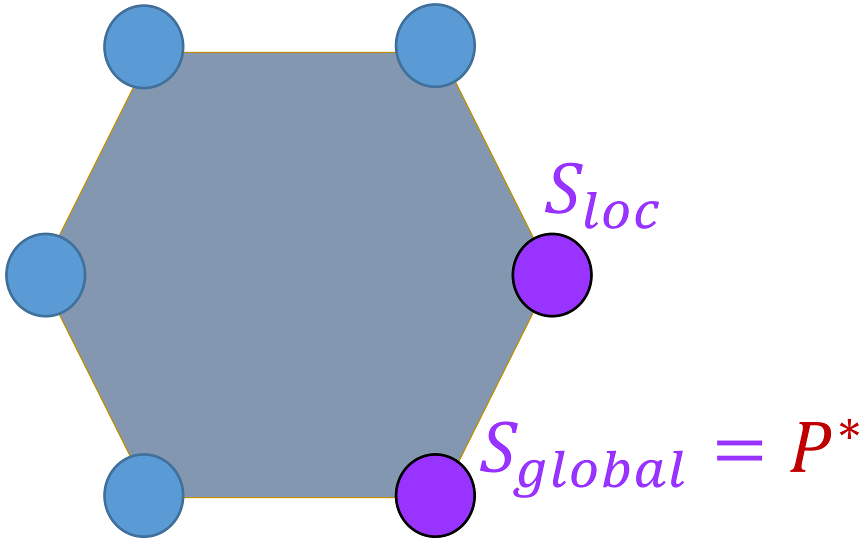
[Zaslavskiy, Bach and Vert 2009]

Observation: Convex energy E_0 is equivalent over Π_n to concave energy E_T.

Concave energy:

(i) Local/global minima are permutations!

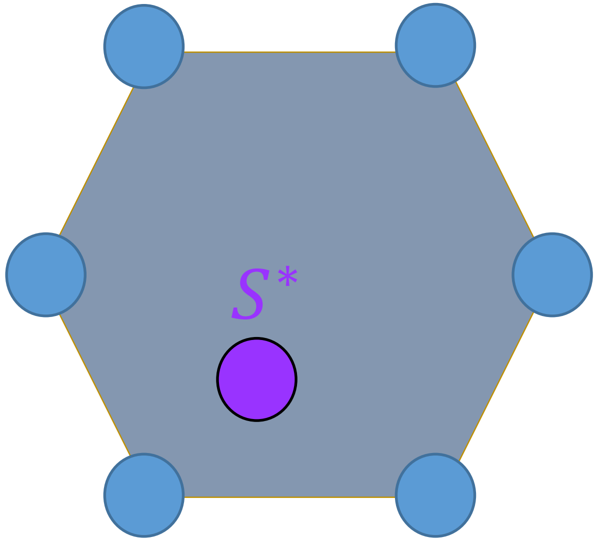
(ii) intractable



Convex energy:

(i) minima may not be permutations

(ii) tractable!



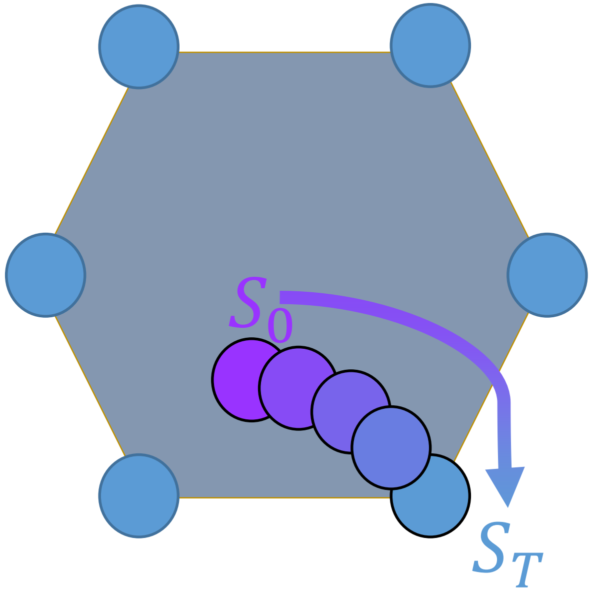
convex2concave projection

convex — concave

E_{t_0} E_{t_1} E_{t_2} … E_T

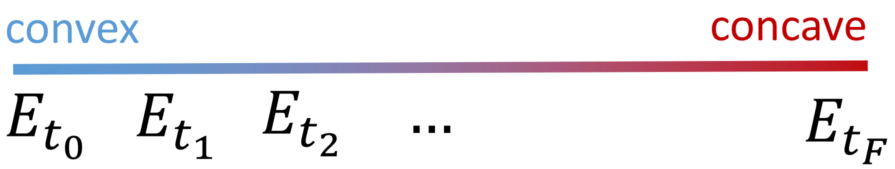
$S_k = \text{"}argmin\text{"}_{S \in DS} E_{t_k}(S)$

Warm start optimization from S_{k-1}.

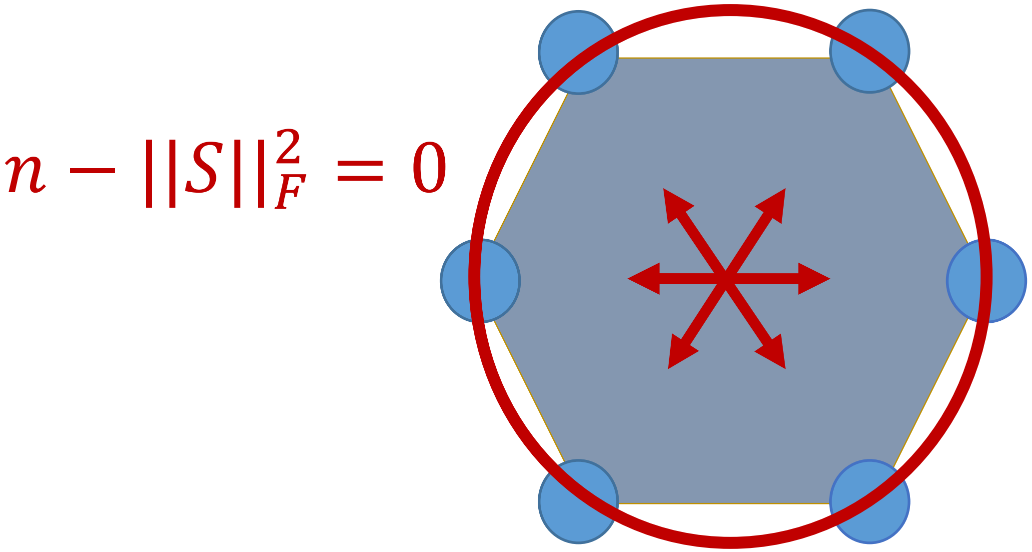


DS++: convex2concave projection

$$E_t(S) = ||SA - BS||_F^2 + t(n - ||S||_F^2)$$



- $E_0 = E$
- $E_t = E$ over Π_n
- E_{t_F} strictly concave for $t_F \gg 0$



Choosing $[t_0, t_F]$

$$E_t(S) = ||SA - BS||_F^2 + t(n - ||S||_F^2)$$

convex → concave

E_{t_0} E_{t_1} E_{t_2} … E_{t_F}

Best choice of t_F:

$t_F = \lambda_{max}$

$t_F = \lambda_{max}$ over $V_{DS} = \{S |\ S1 = 0, S^T 1 = 0\}$

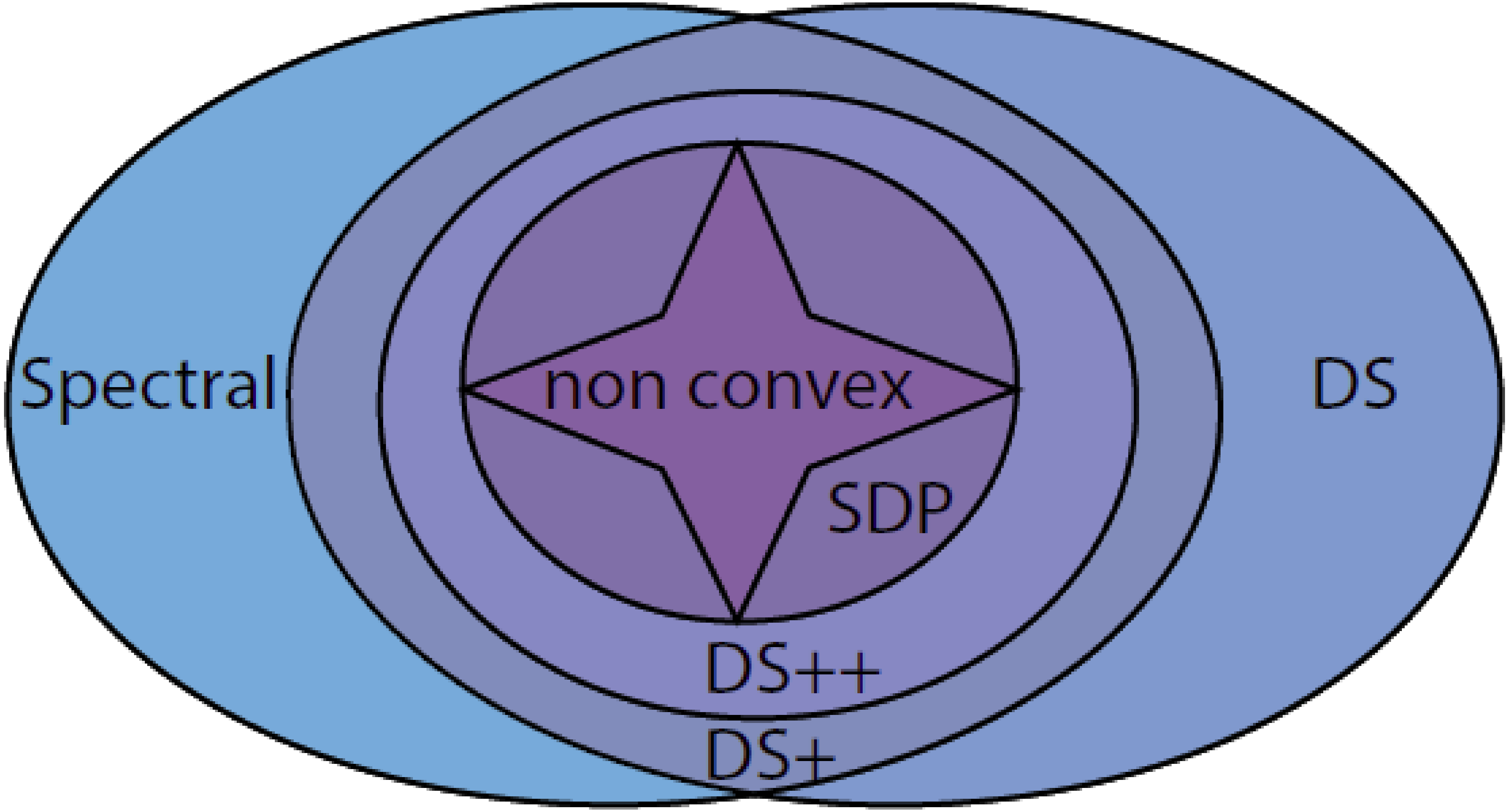
Best choice of t_0:

(DS) $t_0 = 0$ [Aflalo et al. 15]

(DS+) $t_0 = \lambda_{min}$ [Fogel et al. 13,15]

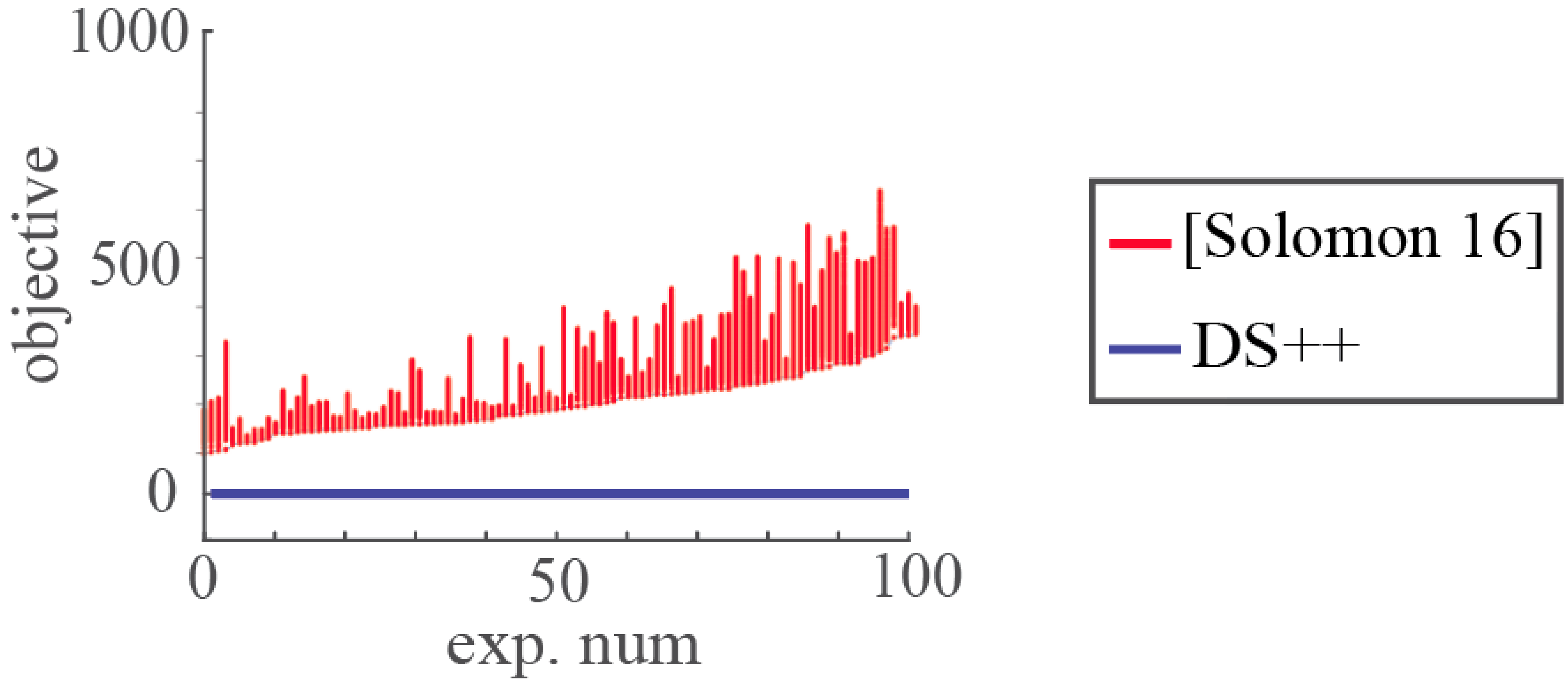
(DS++) $t_0 = \lambda_{min}$ over V_{DS}

Relaxation comparison

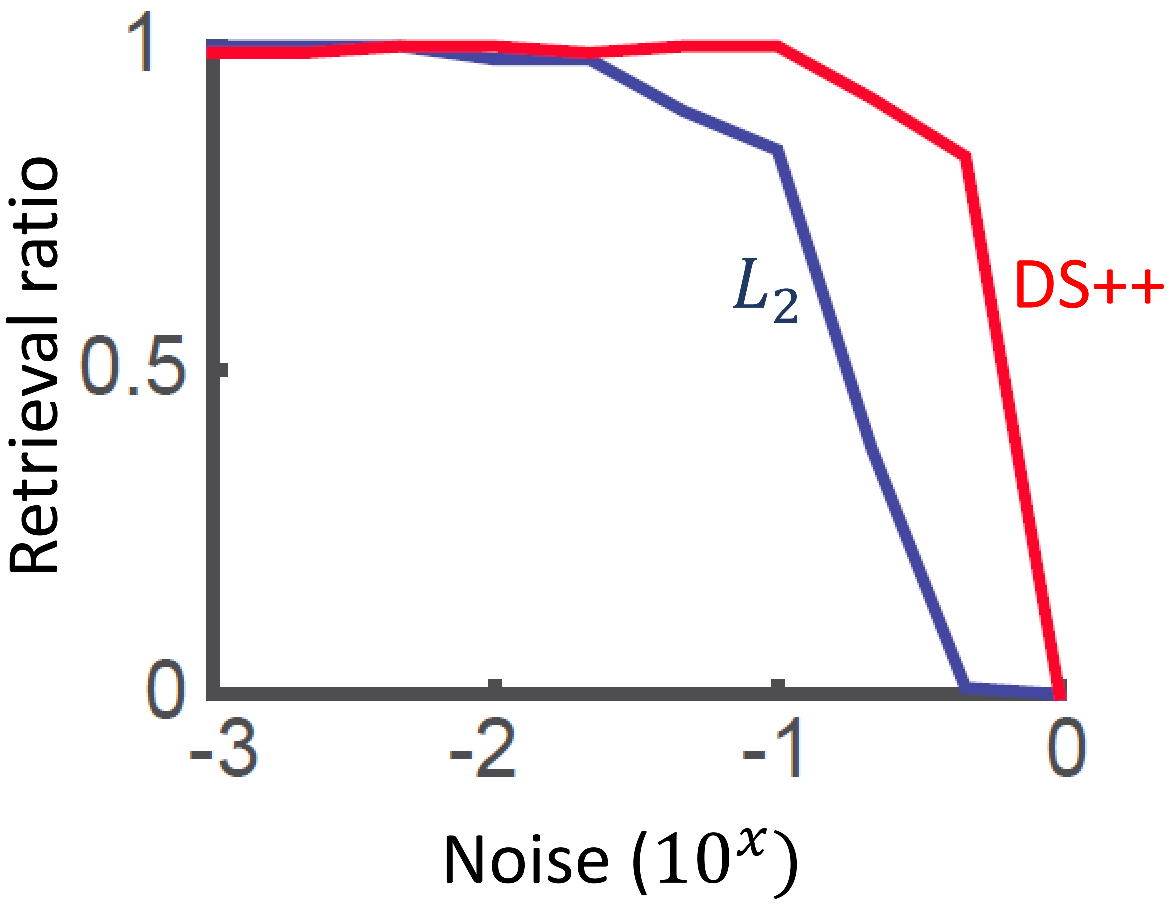


DS++ vs local minimization

DS++ vs local minimization with 1000 different initializations:



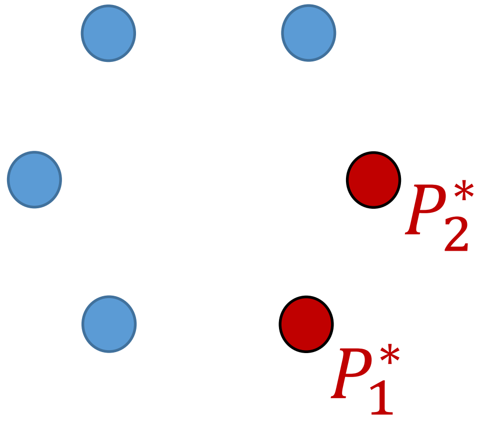
Projection comparison

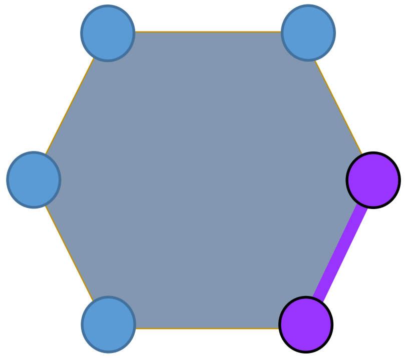


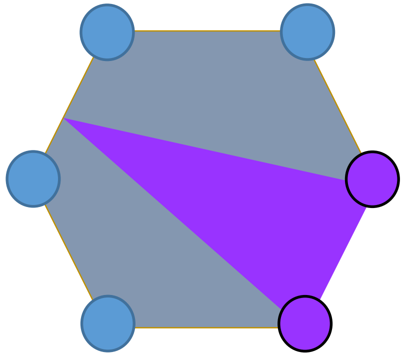
Symmetric, no noise

$$E_t(S) = ||SA - BS||_F^2 + t(n - ||S||_F^2)$$

For $t > 0, P_i^*$ are the only global minima!



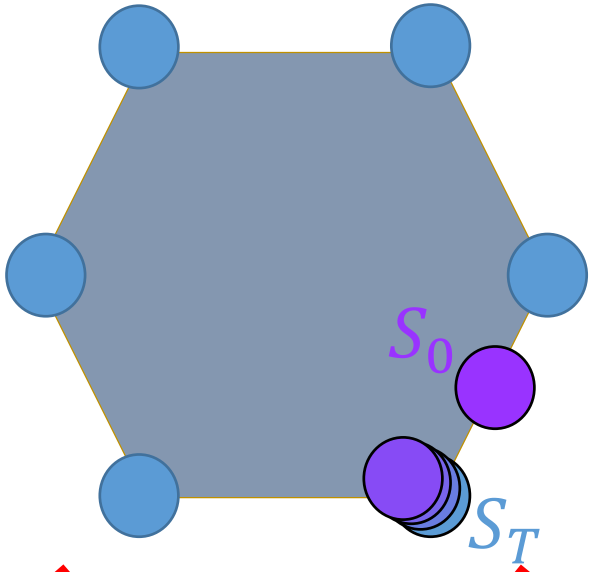


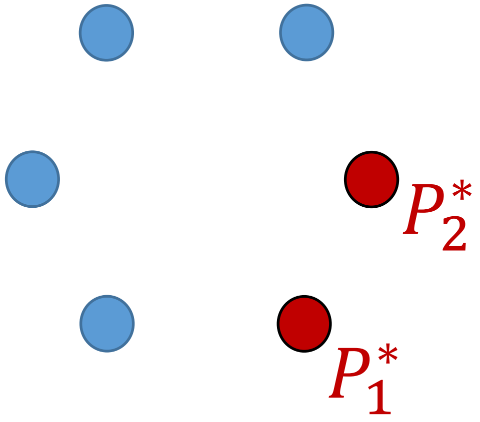


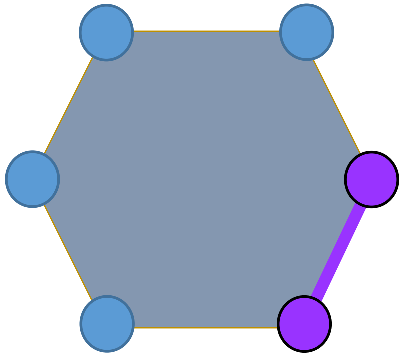
Symmetric, no noise

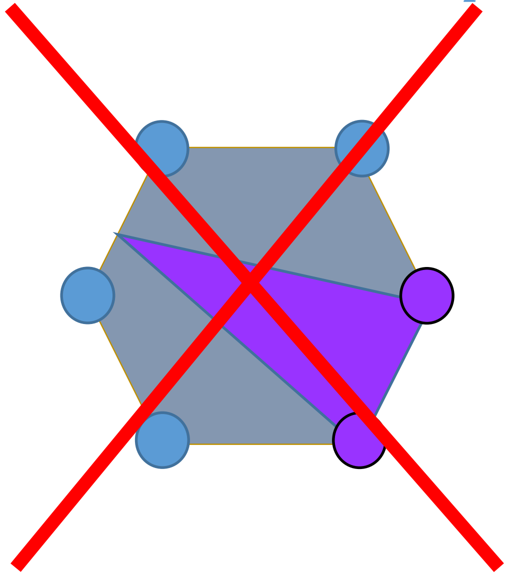
Theorem 3: If DS(A,B) is convex exact, then (under some conditions)

$S_1, S_2, \ldots, S_T = P_i^*$









Thank you!

For more details see:
"Exact Recovery with Symmetries for the Doubly-Stochastic Relaxation."
"DS++: A Flexible, Scalable and Provably Tight Relaxation for Matching Problems."

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