Geometric Optimization
for
Surface Parameterization

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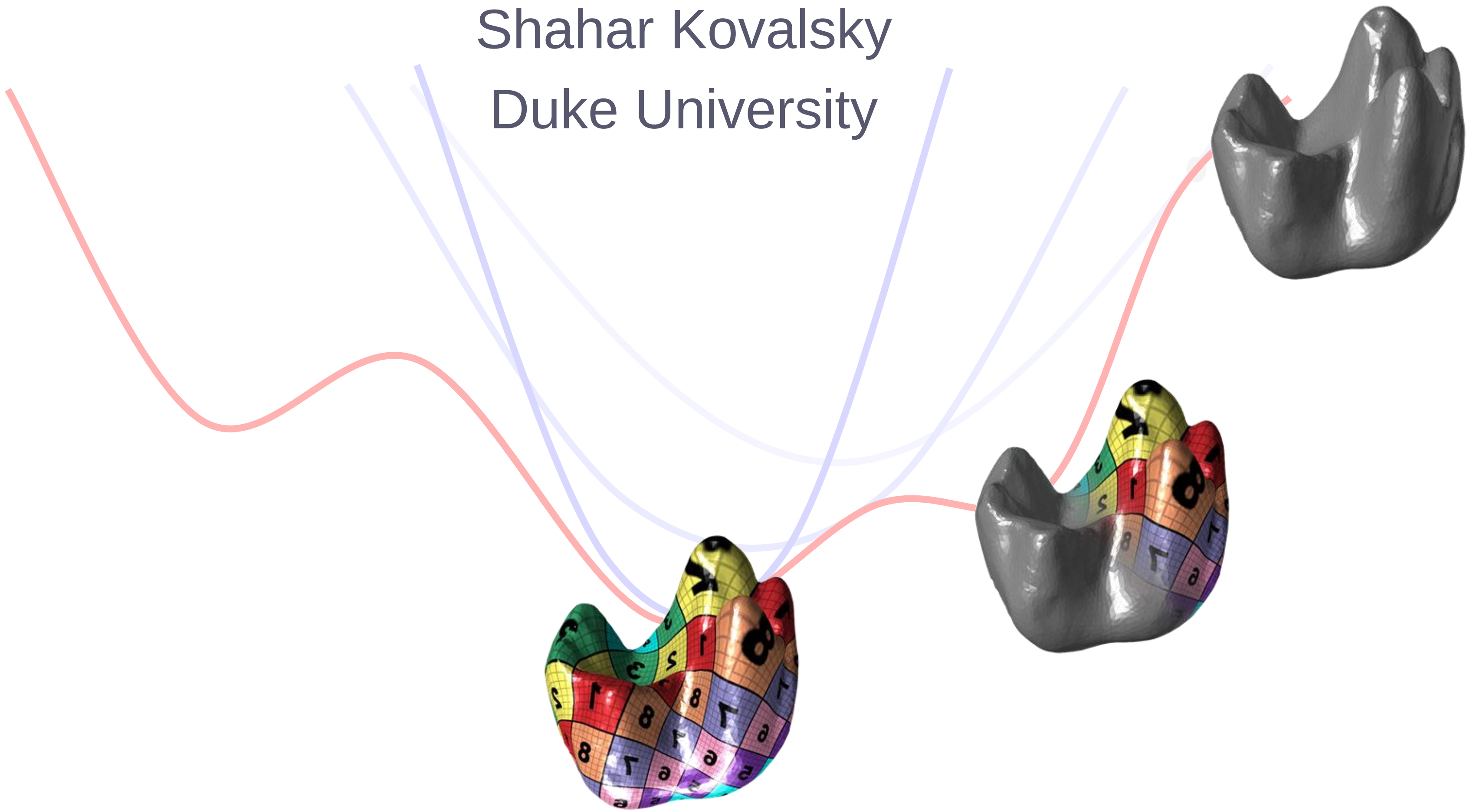
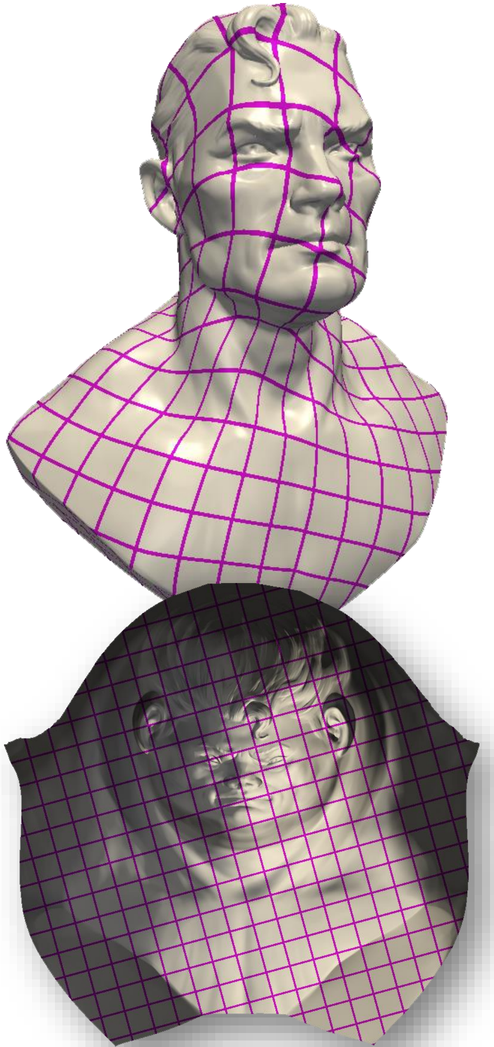
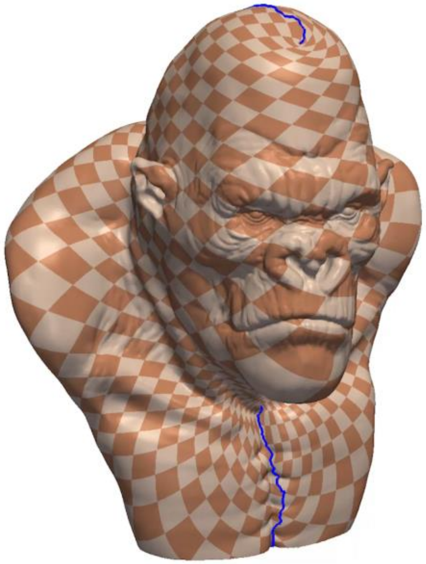
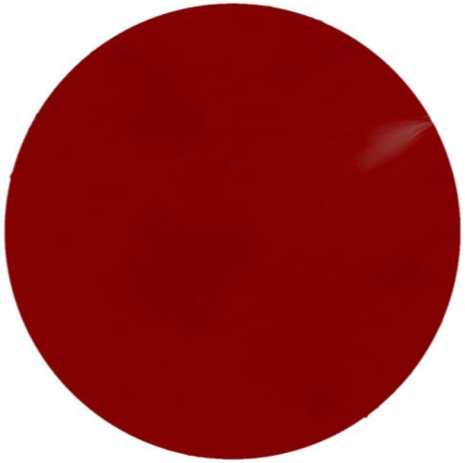




Surface Parameterization

Bio-Motivation





Optimization

Surface Parameterization

Surface Parameterization

- Tobler hyperelliptical

- Mollweide

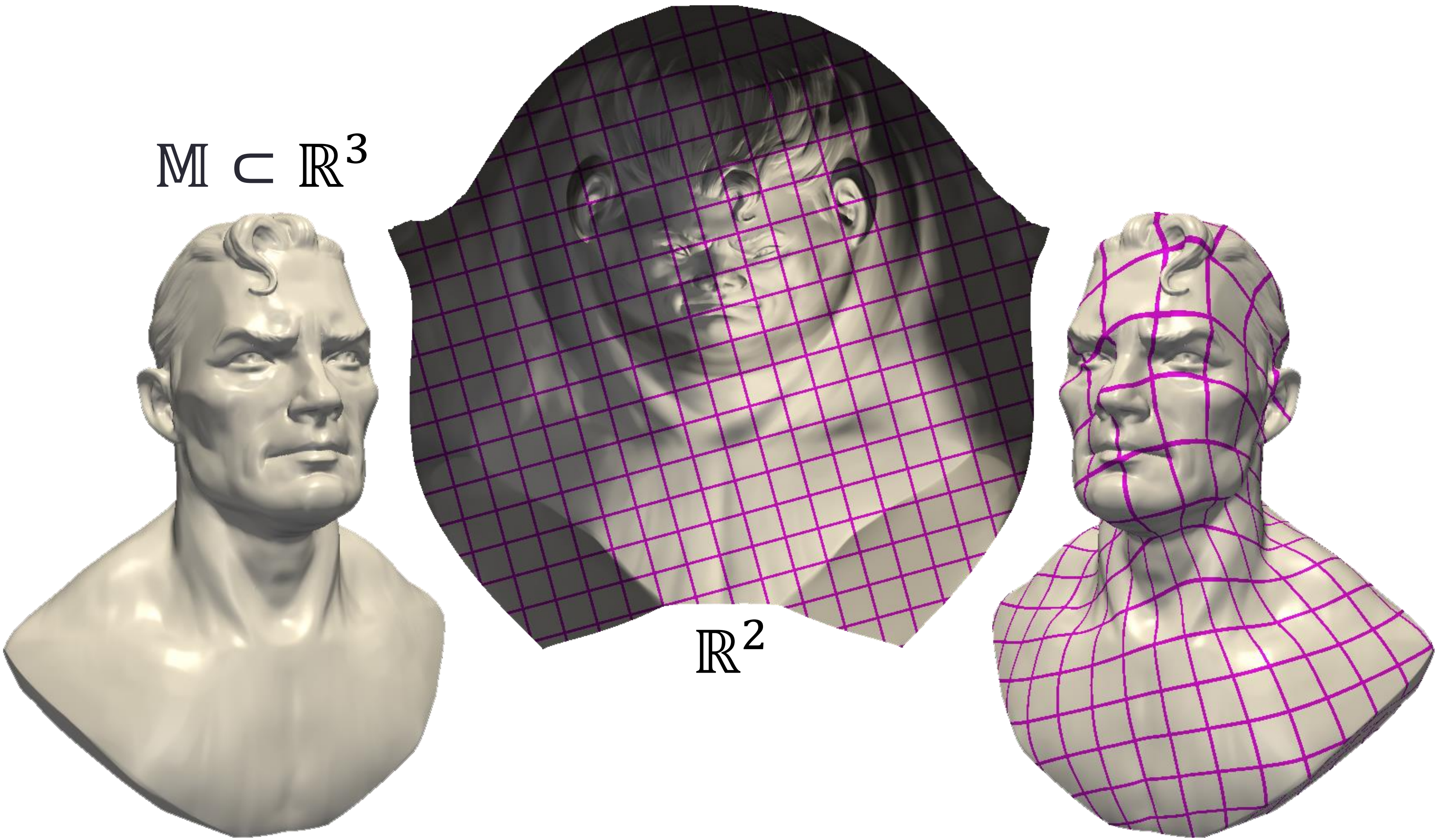
- Goode homolosine

- Eckert IV

- Eckert VI

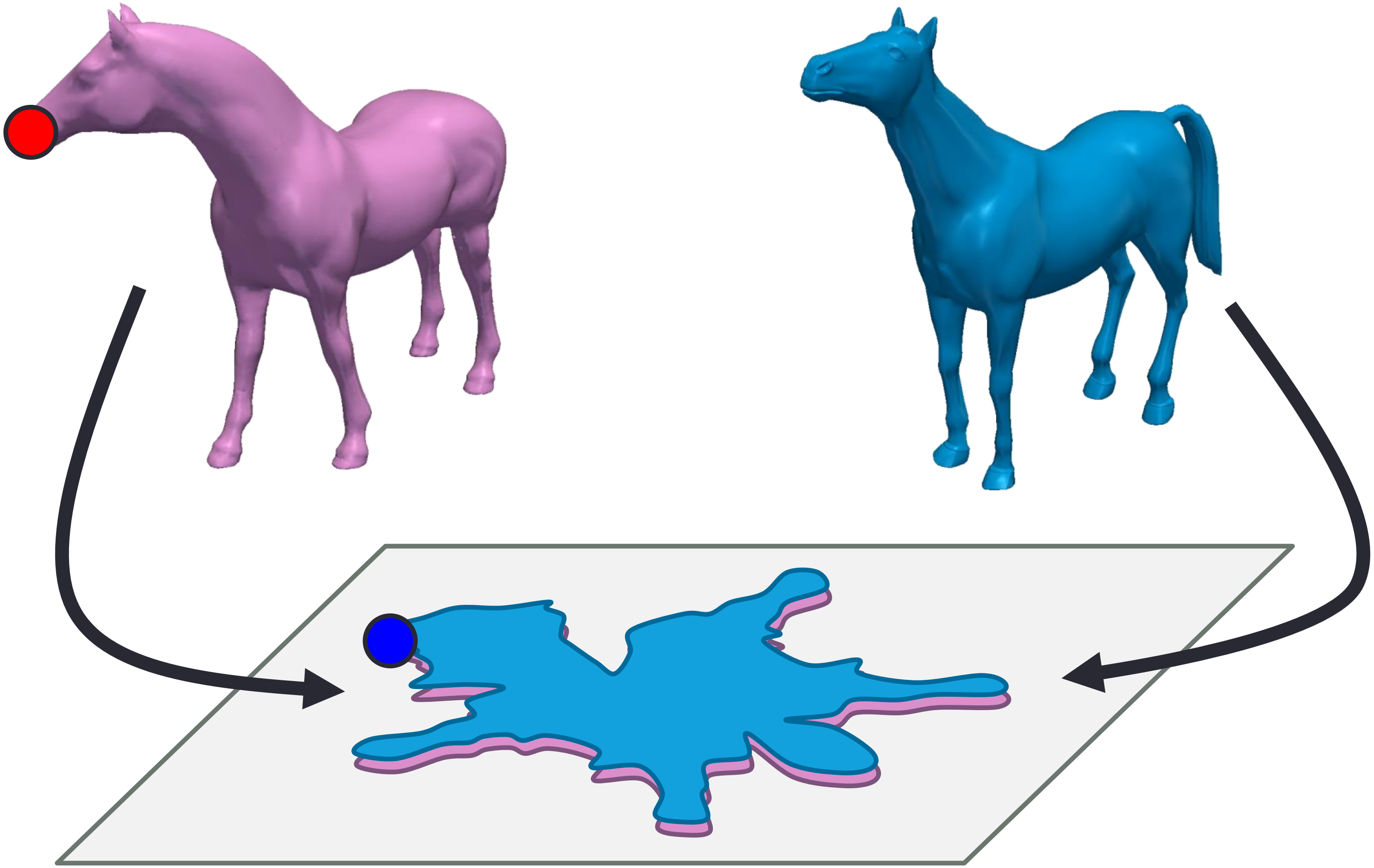
- Kavrayskiy VII

Surface Parameterization

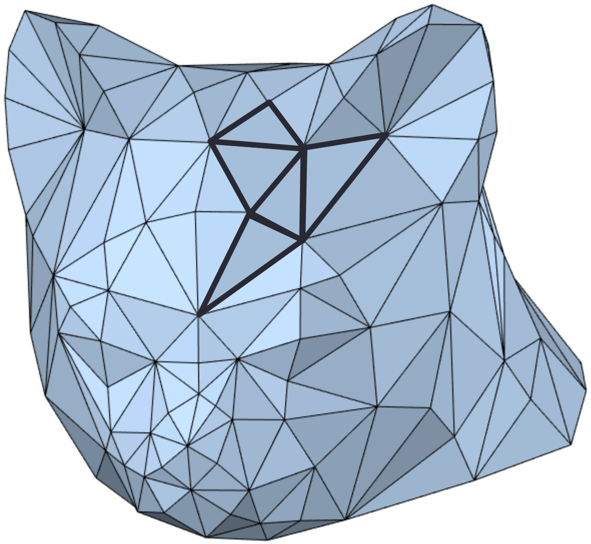


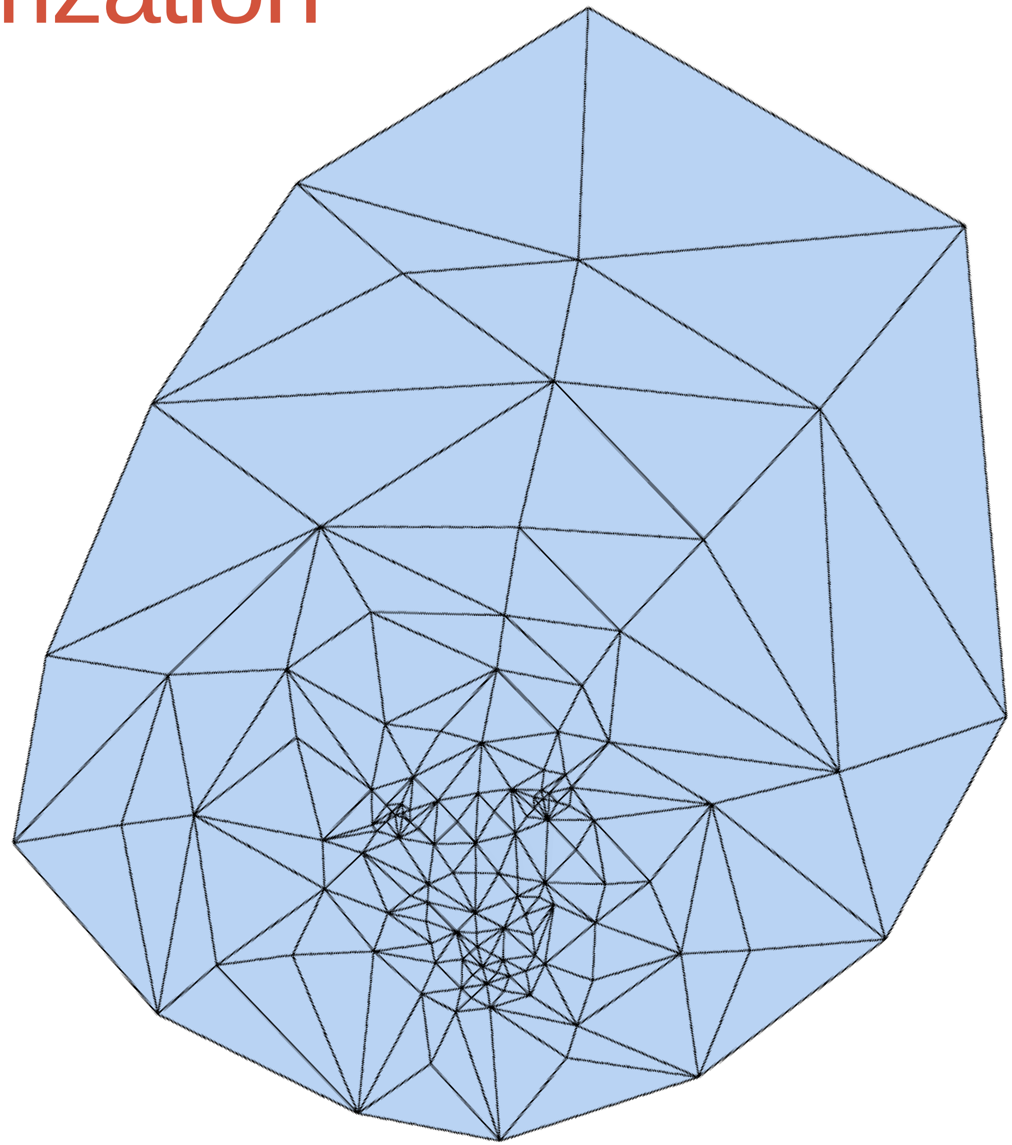
Texture Mapping

Inter-Surface Mapping

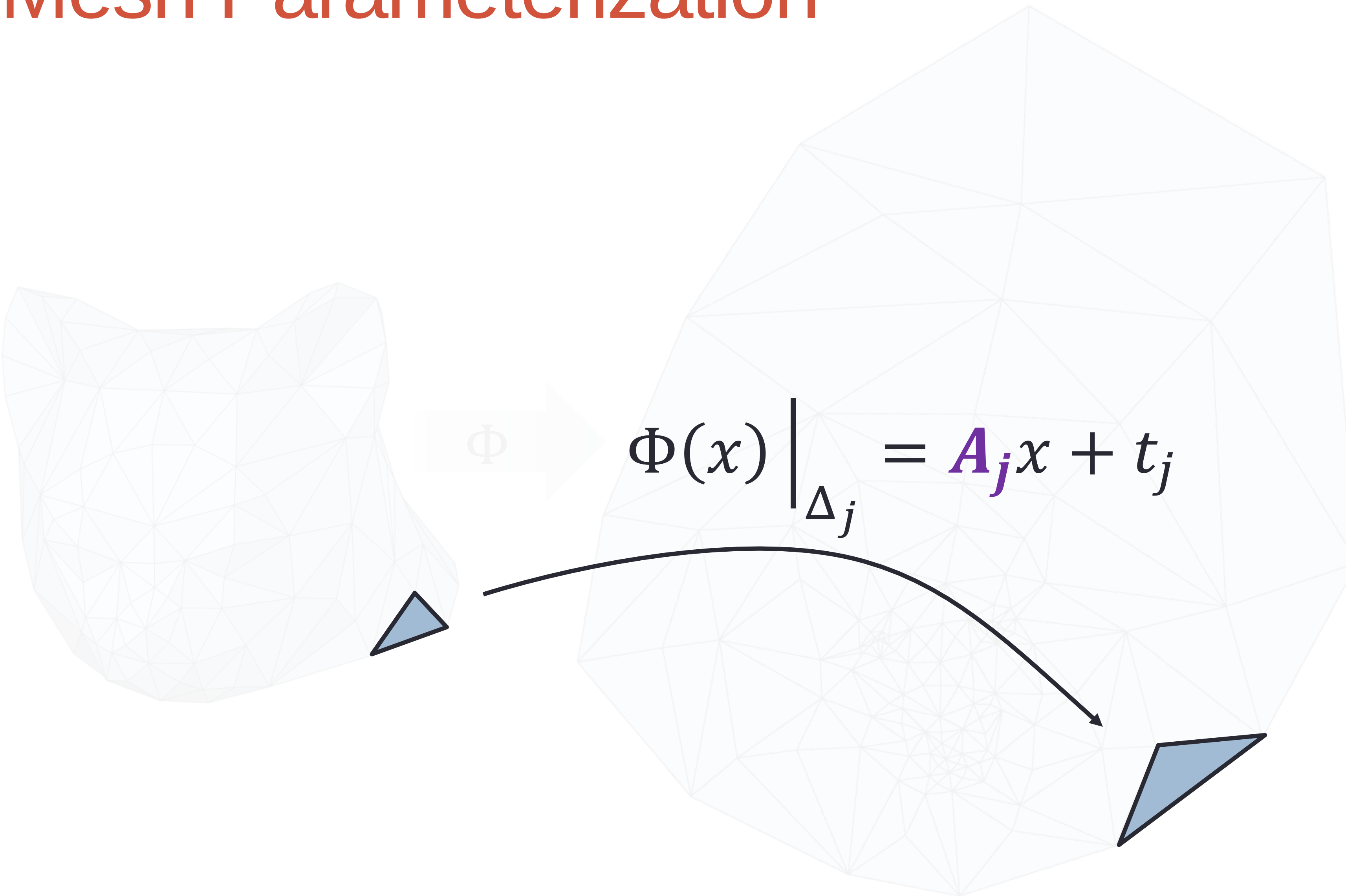


Mesh Parameterization





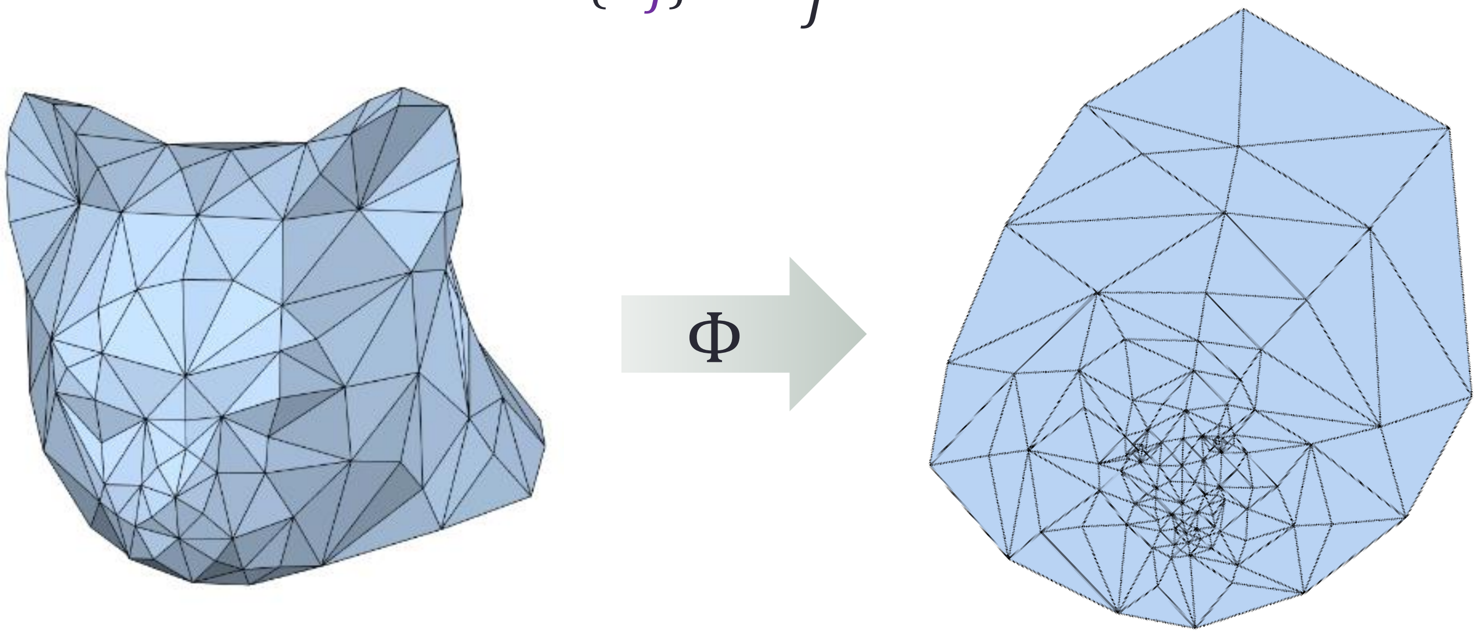
Mesh Parameterization



$$\Phi(x)\Big|_{\Delta_j} = \boldsymbol{A_j}x + t_j$$

Computation

$$\Phi = \underset{\{A_j\}}{\operatorname{argmin}} \sum_j E(A_j)$$



Computation

$$\Phi = \underset{\{A_j\}}{\operatorname{argmin}} \sum_j E(A_j)$$

$\|A_j\|_F^2$

Dirichlet

$d\left(A_j, O(n)\right)$

Isometric Distortion

$d\left(A_j, \mathbb{R}O(n)\right)$

Conformal Distortion

$\det(A_j) > 0$

Orientation Preservation

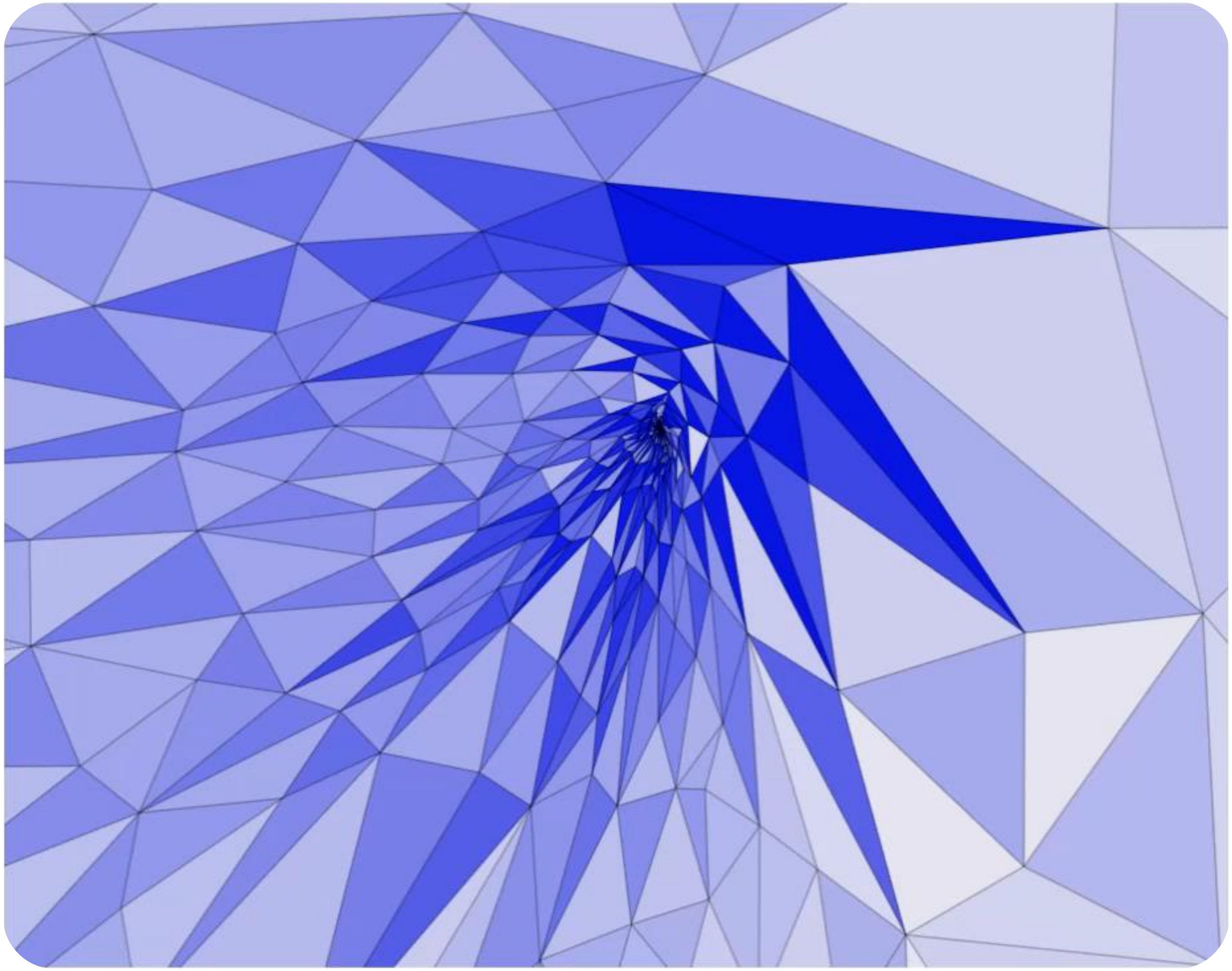
$\operatorname{cond}(A_j) < K$

Bounded Conformal Distortion

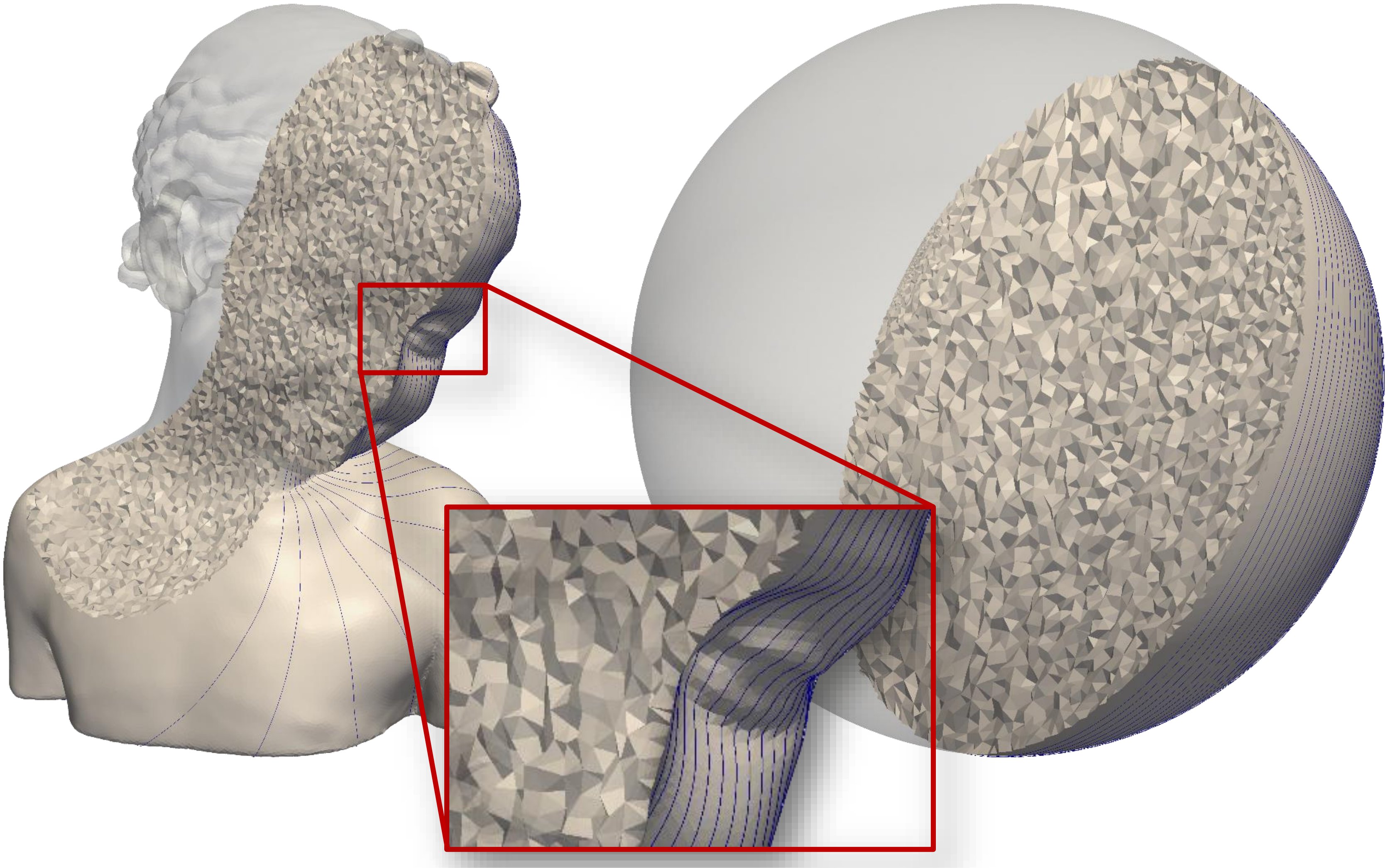
Computation

$$\Phi = \underset{\{A_j\}}{\operatorname{argmin}} \sum_j E(A_j)$$

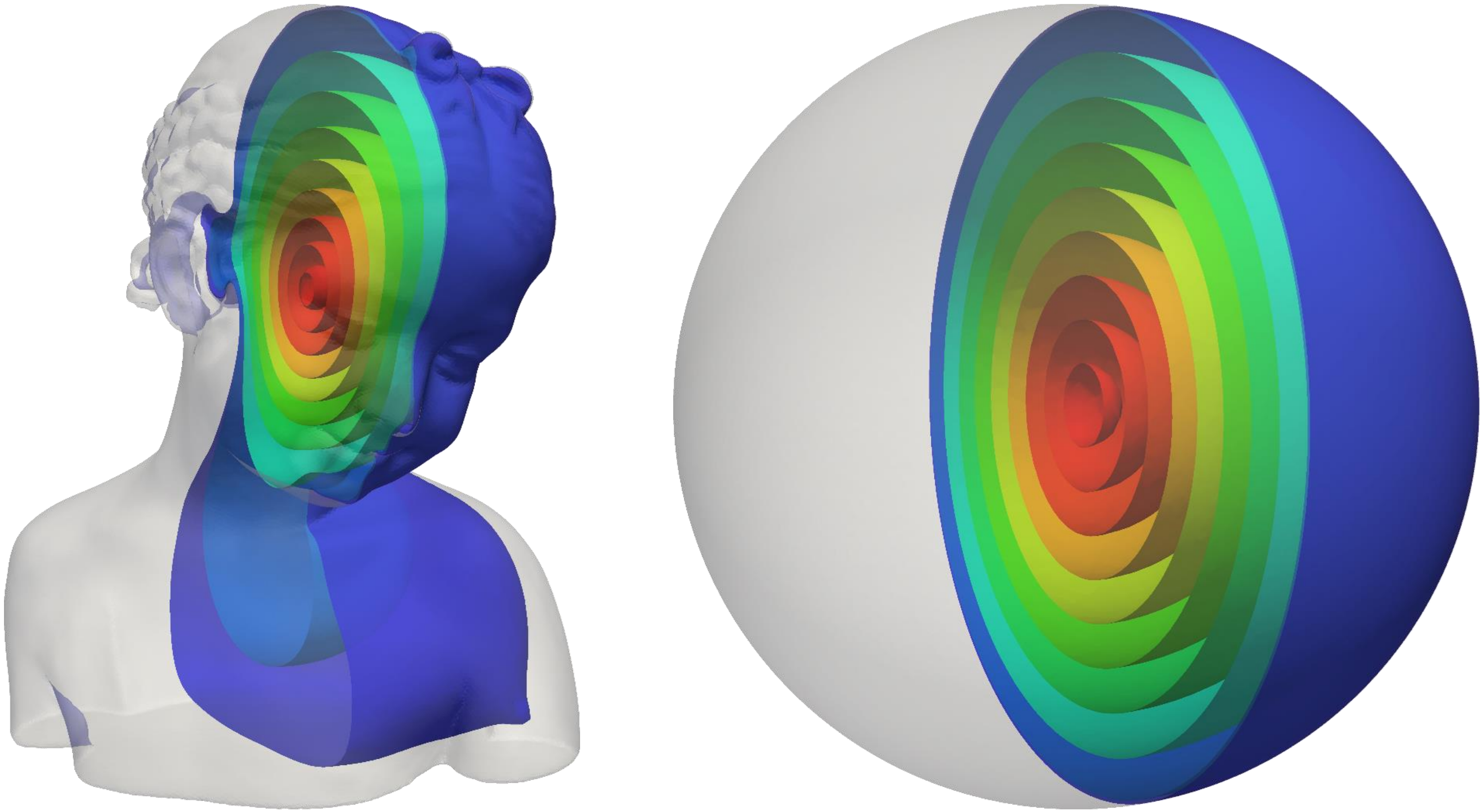
- Need **efficient** optimization!
- Why…?



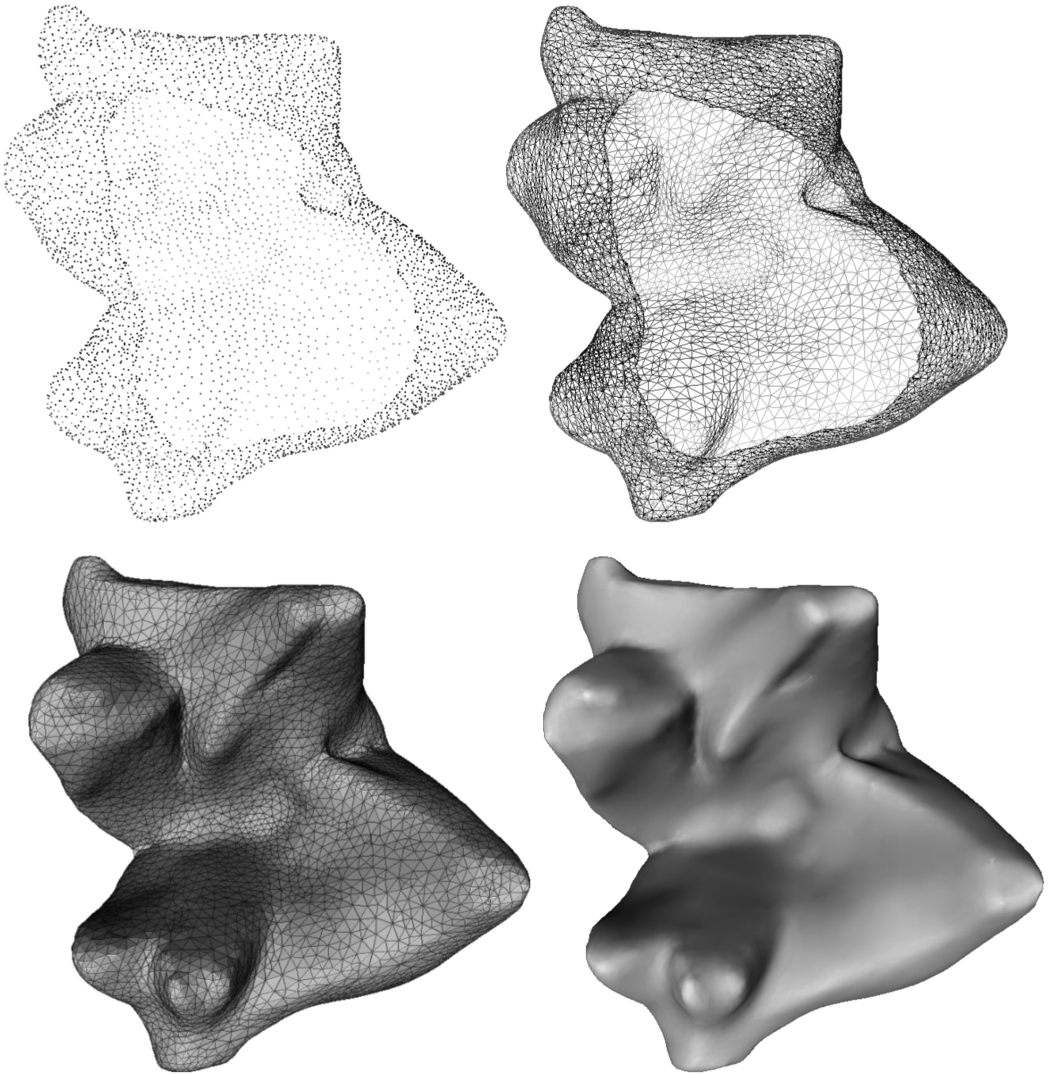
~1M Triangles

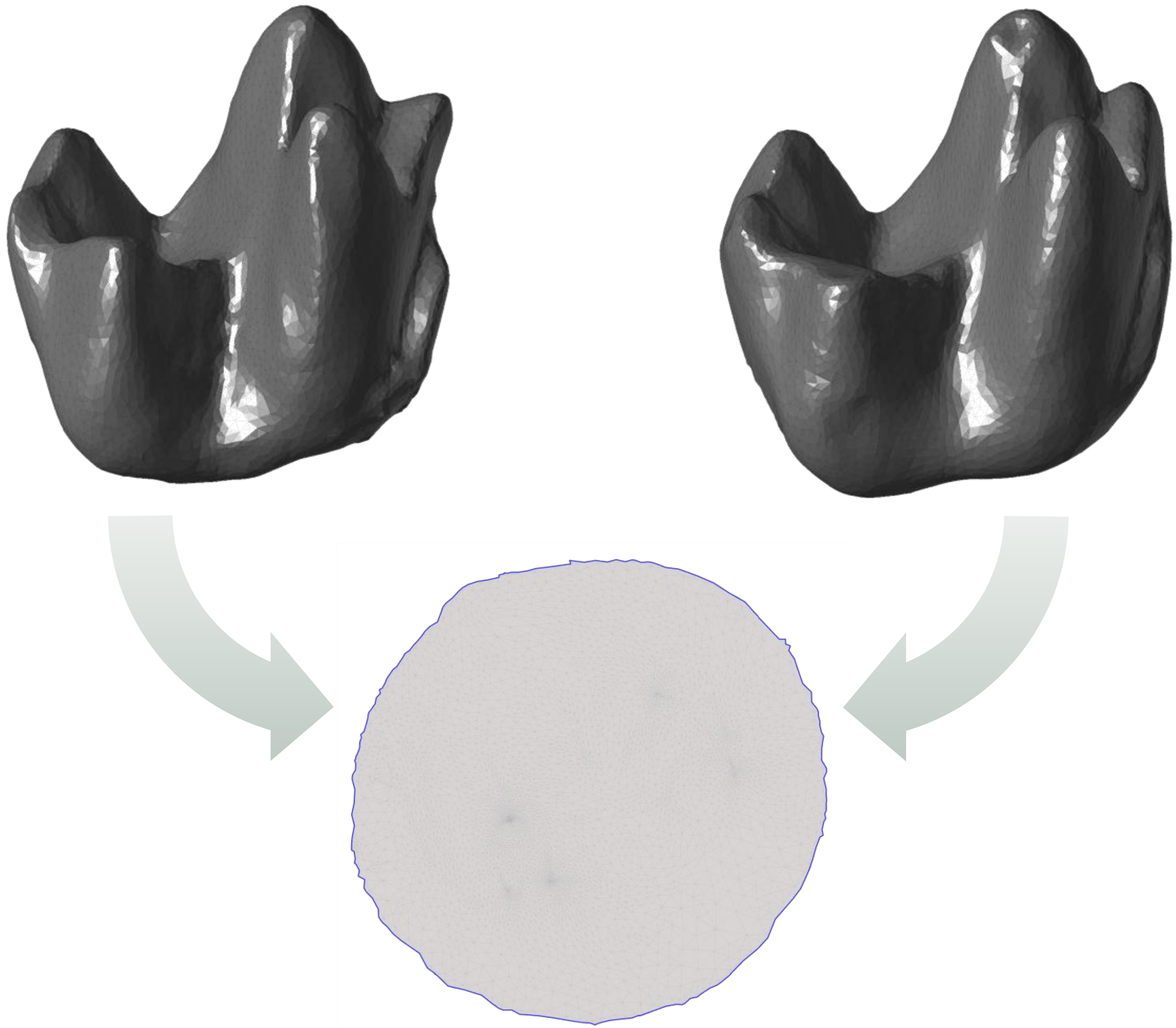


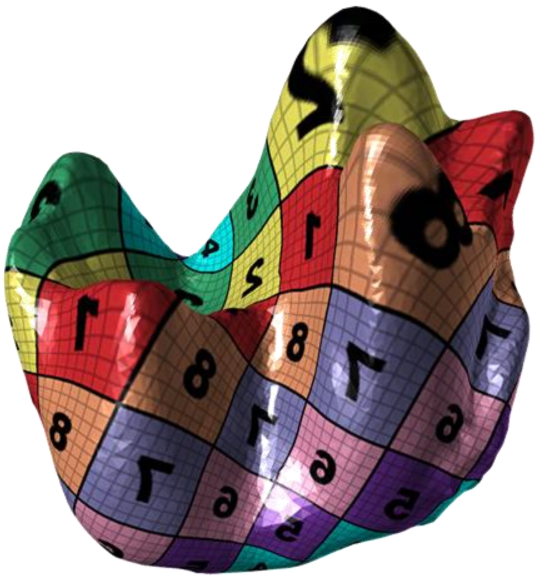
>1M Tetrahedra

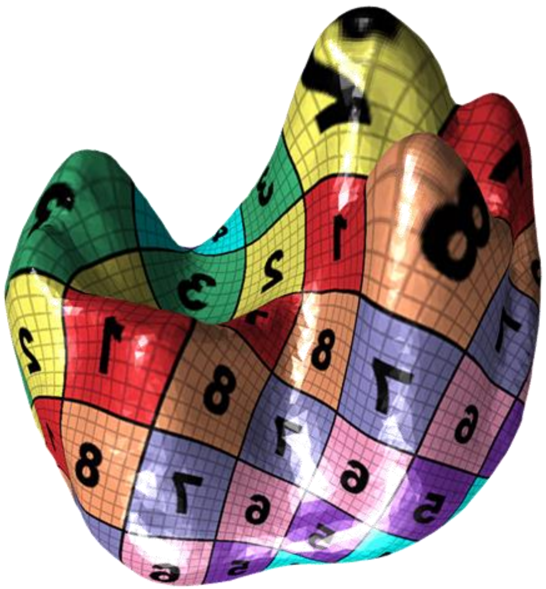




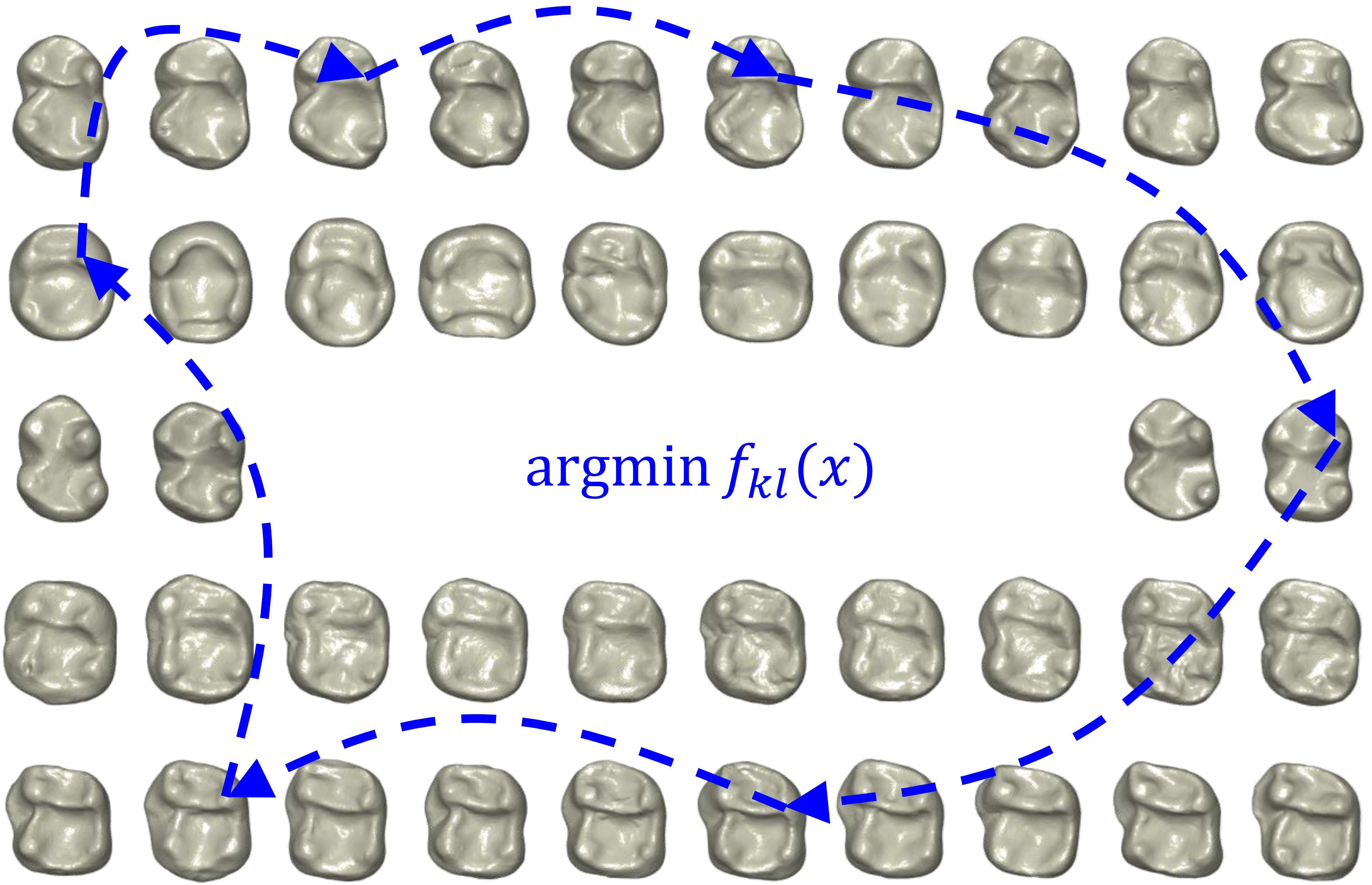








$$\operatorname{argmin} f(x)$$


argmin $f_{kl}(x)$

Learning Mappings

$$\Phi_{kl} = \operatorname{argmin} f_{kl}(x)$$

Learning Mappings

$$\Phi_{kl}^{\boldsymbol{\theta}} = \operatorname{argmin} f_{kl}(x; \boldsymbol{\theta})$$

$\boldsymbol{\theta}$ parameterizes the energy itself

Learning Mappings

$$\Phi_{kl}^{\boldsymbol{\theta}} = \operatorname{argmin} f_{kl}(x; \boldsymbol{\theta})$$

We want to

Learn $\boldsymbol{\theta}$

such that

$\{\Phi_{kl}^{\boldsymbol{\theta}}\}$ are **consistent**

Learning Mappings

$$\underset{\boldsymbol{\theta}}{\operatorname{argmin}}\, L_{consistency}(\{\Phi_{kl}^{\boldsymbol{\theta}}\})$$

where

$$\Phi_{kl}^{\boldsymbol{\theta}} \coloneqq \operatorname{argmin} f_{kl}(x; \boldsymbol{\theta})$$

requires ***efficient*** optimization

Majorization

Majorization Minimization (MM)

Def. $\overline{f}(x; x_0)$ is a ***convex majorizer*** of f at x_0 if

- $\overline{f}(x_0; x_0) = f(x_0)$
- $\overline{f}(x; x_0) \geq f(x)$

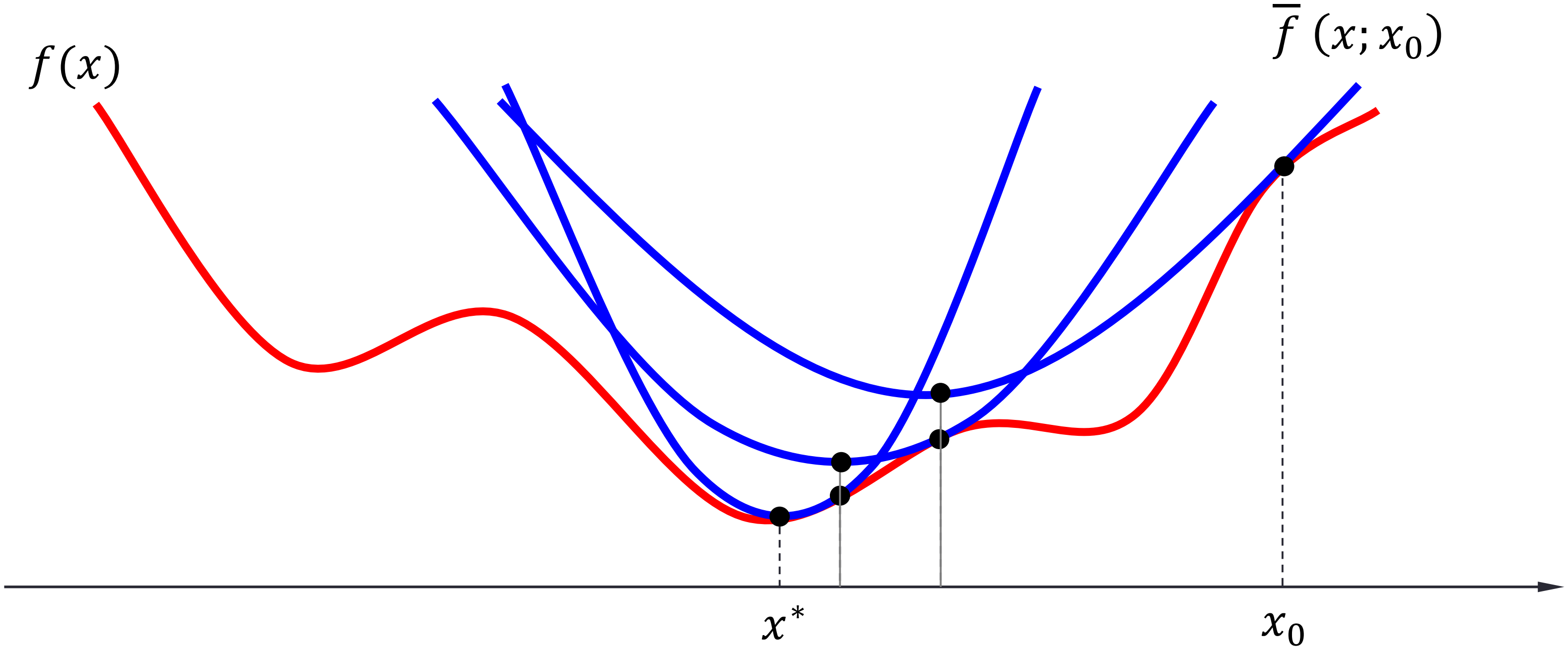


Fig. from [A. Bronstein, Course of Numerical Geometry Of Non-Rigid Shapes]

Majorization Minimization (MM)

Def. $\overline{f}(x; x_0)$ is a ***convex majorizer*** of f at x_0 if

- $\overline{f}(x_0; x_0) = f(x_0)$
- $\overline{f}(x; x_0) \geq f(x)$

$$\overline{f}(x; x_0) = ?$$

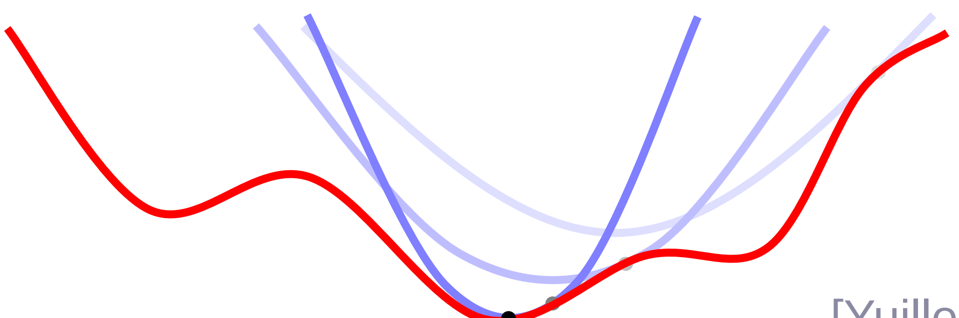
The Convex-Concave (CC) Procedure

$$f = f^+ + f^-$$

- f^+ is convex
- f^- is concave

Define a majorizer:

$$\bar{f}(x; x_0) = f^+(x) + f^-(x_0) + \nabla f^-(x_0)^T (x - x_0)$$

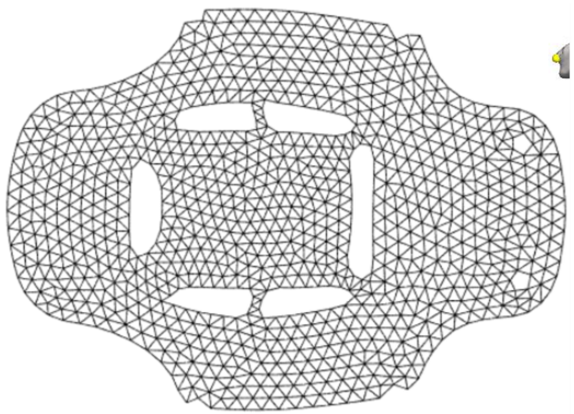


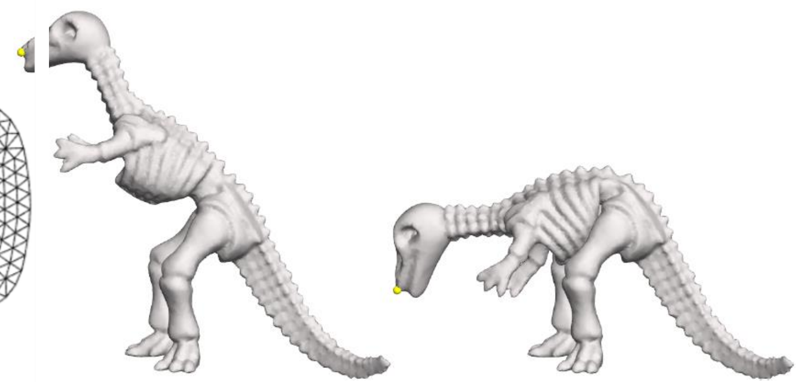
[Yuille and Rangarajan 2000]

Examples

As-Rigid-As-Possible (ARAP) Energy

[Sorkine and Alexa 2007, Liu et al. 2008]





$$\sum_j \|A_j - \mathcal{R}(A_j)\|_F^2 = \sum_j \|A_j\|_F^2 - 2\|A_j\|_* + n$$

Closest Rotation

Convex

Concave

The Convex-Concave (CC) Procedure

$$f = f^{+} + f^{-}$$

- Always* exists

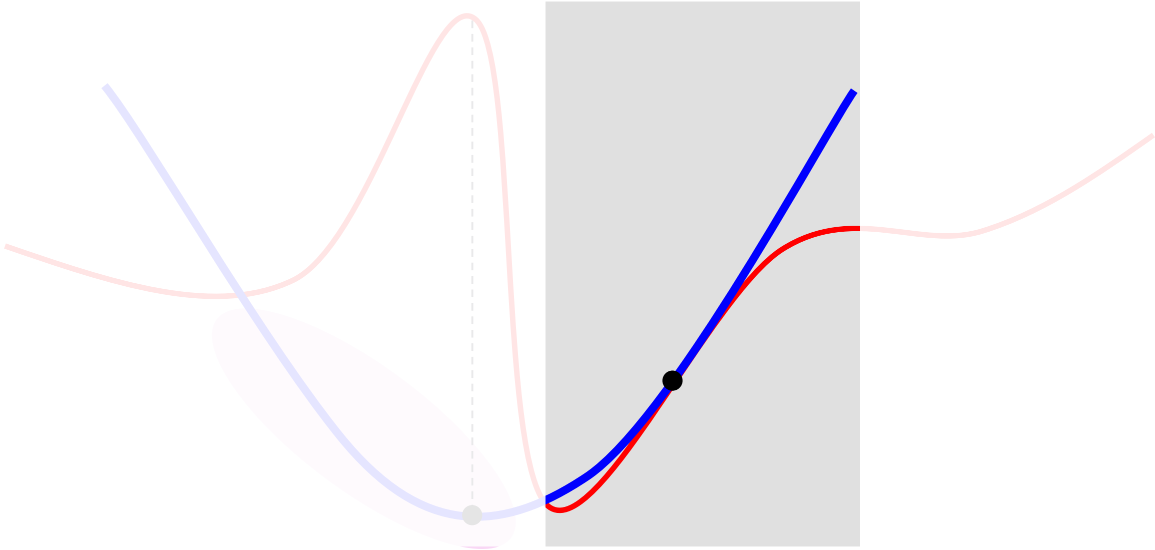
$$f(x) = f(x) + \lambda x^T x - \lambda x^T x$$

- Large λ $\Rightarrow$ $\nabla^2 f^{+} \approx \lambda I$

convex-concave decomposition is often unavailable…

CC-inspired quadratic proxy

- Use an *approximate/local majorizer*



- Take a *Newton* step

CC-inspired quadratic proxy

- **Iterate:**
  - *Locally* replace f at x_n with $\overline{f}$

  - Determine Newton's **descent** direction

$$p_n = -\left(\nabla^2 \overline{f}(x_n)\right)^{-1} \nabla \overline{f}(x_n)$$

  - Update

$$x_{n+1} = x_n + t_n p_n$$

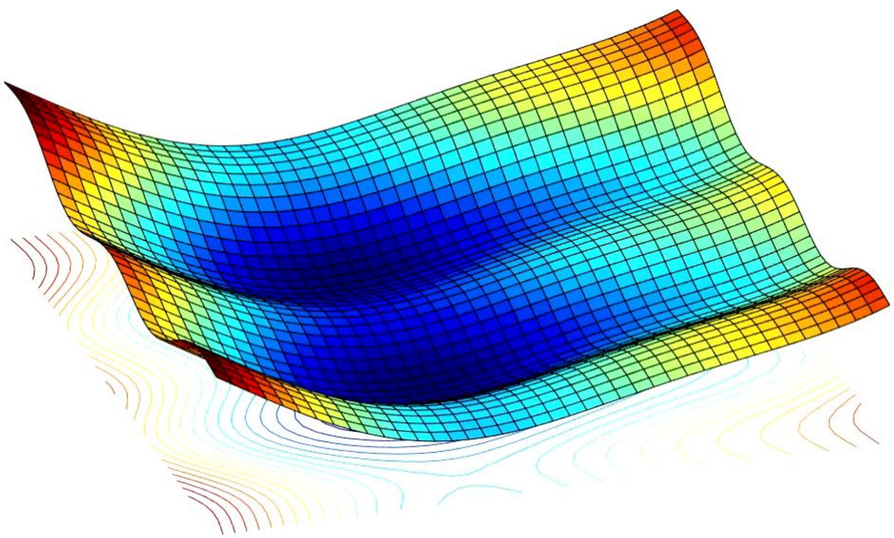
Observation I

[with Meirav Galun and Yaron Lipman; 2016]

- Often

$$f(x) = x^T L x + g(x)$$

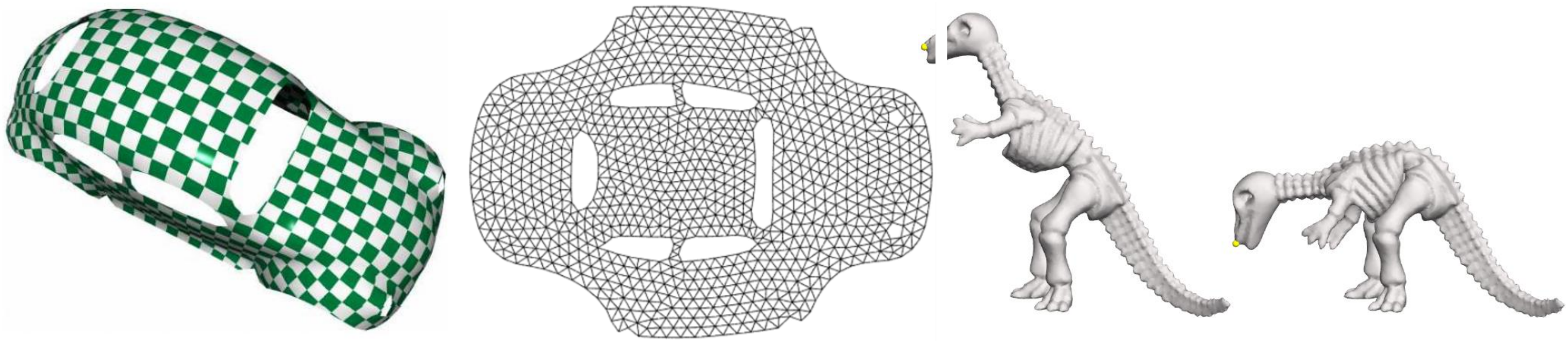
Laplacian
(cotangent)



Examples

As-Rigid-As-Possible (ARAP) Energy

[Sorkine and Alexa 2007, Liu et al. 2008]



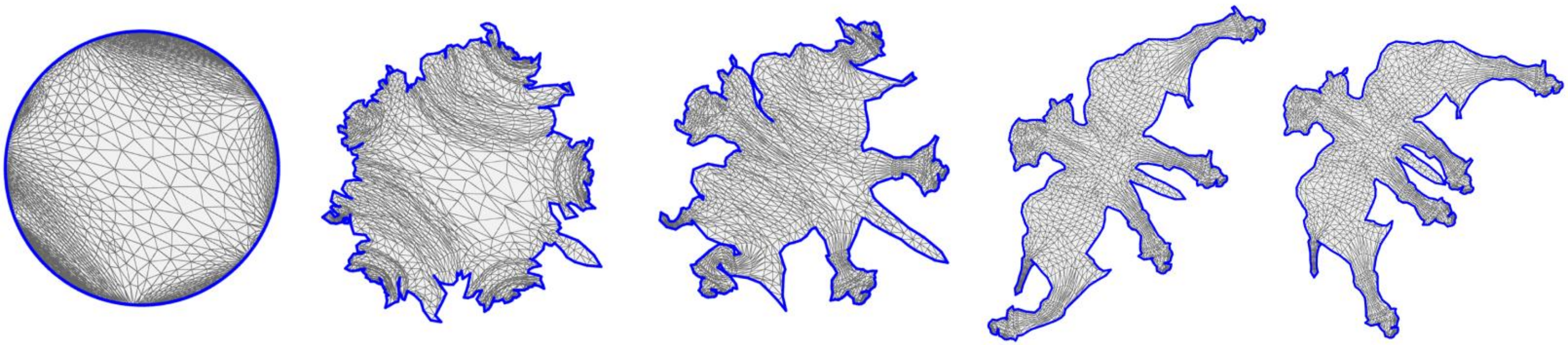
$$\sum_j \|A_j - \mathcal{R}(A_j)\|_F^2 = \sum_j \|A_j\|_F^2 - 2\|A_j\|_* + n$$

$$= x^T L x$$

Examples

Symmetric Dirichlet Energy

[Smith and Schaefer 2015]



$$\sum_j \|A_j\|_F^2 + \|A_j^{-1}\|_F^2$$

$$= x^T L x$$

“Laplacian-Dominant” Objective

$$f(x) = x^T L x + g(x)$$

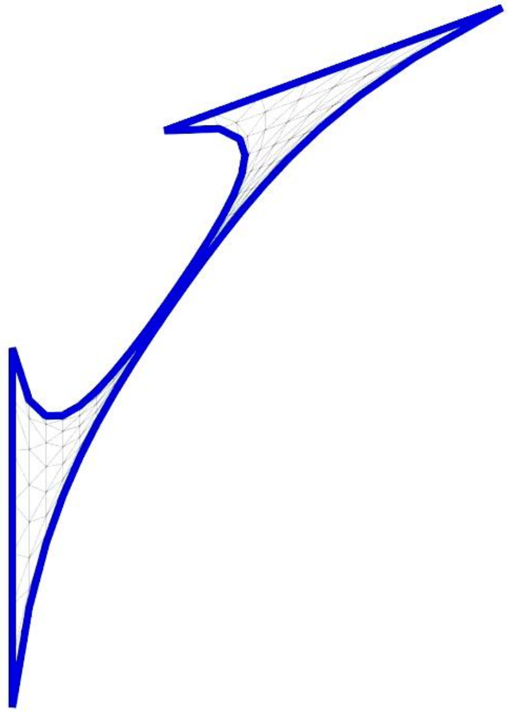
- Our approach reduces to

$$p_n = -L^{-1} \nabla f(x_n)$$

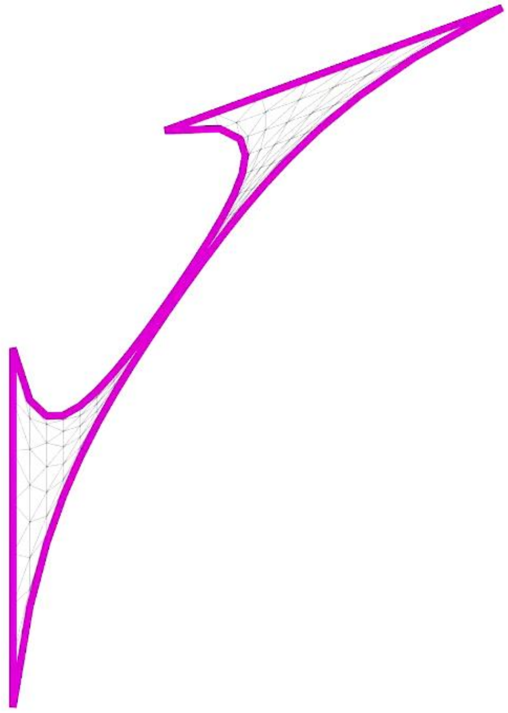
- **Laplacian-preconditioned gradient descent**

- Efficient + Effective!
- Super simple
- + accelerated & L-BFGS variants

Isometric Distortion Energy



Accelerated Quadratic Proxy



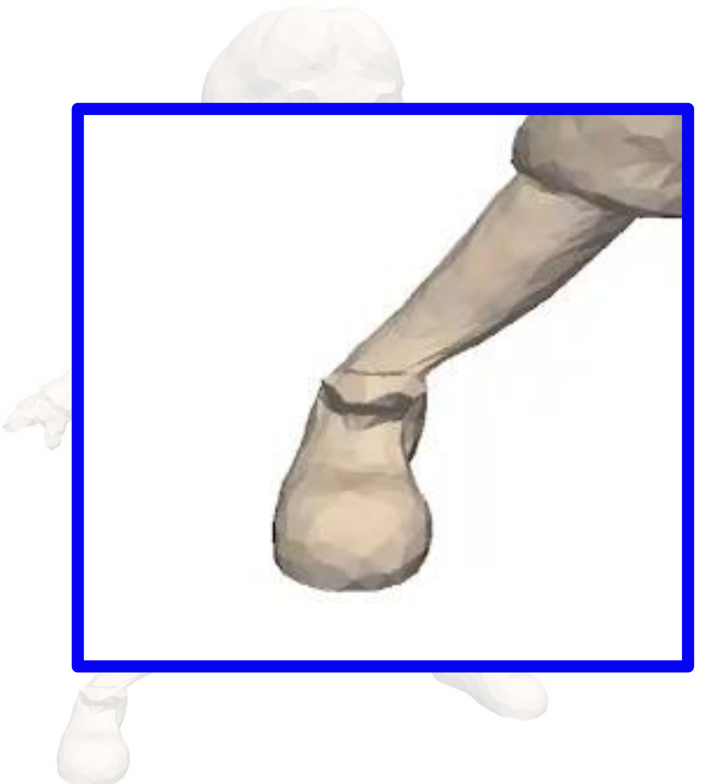
L-BFGS

As-Rigid-As-Possible

Accelerated Quadratic Proxy

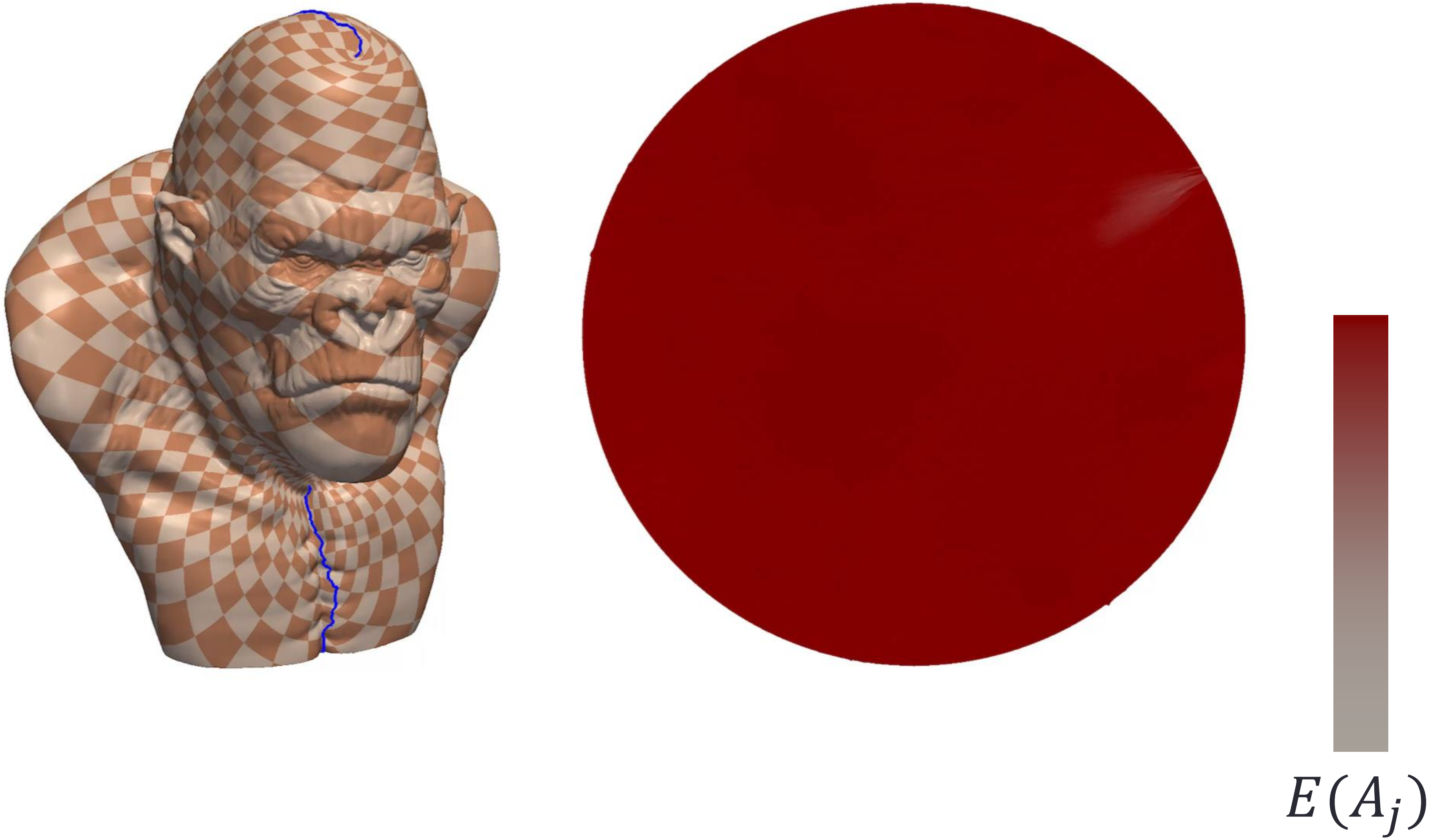
Global-Local

As-Rigid-As-Possible



Accelerated Quadratic Proxy

Global-Local


$E(A_j)$

Observation II

[with Anna Shtengel, Roi Poranne, Olga Sorkine-Hornung and Yaron Lipman]

- Often

$$f = h \circ \boldsymbol{g} = h\left(\begin{bmatrix} g_1 \\ \vdots \\ g_m \end{bmatrix}\right)$$

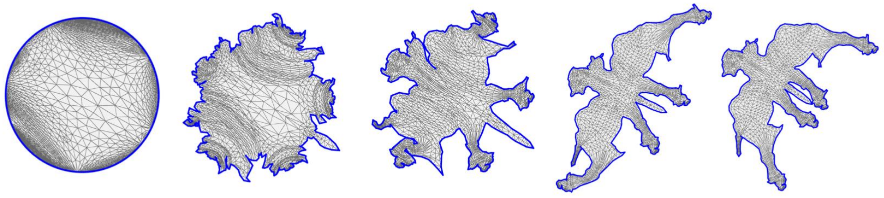
- $h = h^+ + h^-$
- $g_i = g_i^+ + g_i^-$

admit simpler decompositions

Example

Symmetric Dirichlet Energy

[Smith and Schaefer 2015]



$$||A||_F^2 + ||A^{-1}||_F^2$$

For $A \in \mathbb{R}^{2\times 2}$:

$$\Sigma^2 + \sigma^2 + \frac{1}{\Sigma^2} + \frac{1}{\sigma^2}$$

$$= h\left(\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}\right)$$

$$\Sigma^2 = ||A||_2^2$$

$$\sigma^2 = ||A||_F^2 - ||A||_2^2$$

$$= g_1^+$$

$$= g_2^+ + g_2^-$$

Mise en place

$$f = h \circ \boldsymbol{g} = h\left(\begin{bmatrix} g_1 \\ \vdots \\ g_m \end{bmatrix}\right)$$

- Construct

$$\color{green}{\overline{h}}, \color{green}{\overline{g_i}}, \color{red}{\underline{g_i}}$$

Mise en place

$$r = r^{+} + r^{-}$$

$$\overline{r}(x; x_0) = r^{+}(x) + r^{-}(x_0) + \nabla r^{-}(x_0)^T(x - x_0)$$

$$\underline{r}(x; x_0) = -\overline{(-r)}(x; x_0)$$

Mise en place

$$f = h \circ \boldsymbol{g} = h\left(\begin{bmatrix} g_1 \\ \vdots \\ g_m \end{bmatrix}\right)$$

- Construct

$$\overline{h}, \overline{g_i}, \underline{g_i}$$

Mise en place

$$f = h \circ \boldsymbol{g} = h\left(\begin{bmatrix} g_1 \\ \vdots \\ g_m \end{bmatrix}\right)$$

- Construct

$$\overline{h}, \overline{g_i}, \underline{g_i}$$

- Define

$$[g_i] = \begin{cases} \overline{g_i} & \dfrac{\partial \overline{h}}{\partial g_i} \geq 0 \\ \underline{g_i} & \dfrac{\partial \overline{h}}{\partial g_i} < 0 \end{cases}$$

Local Majorization

$$[\boldsymbol{g}] = \begin{bmatrix} \overline{g_1} \text{ or } \underline{g_i} \\ \vdots \\ \overline{g_m} \text{ or } \underline{g_m} \end{bmatrix} \text{ accord. to sign}\left(\frac{\partial \overline{h}}{\partial g_i}\right)$$

- Define

$$\overline{f} = \overline{h} \circ [\boldsymbol{g}]$$

Proposition:

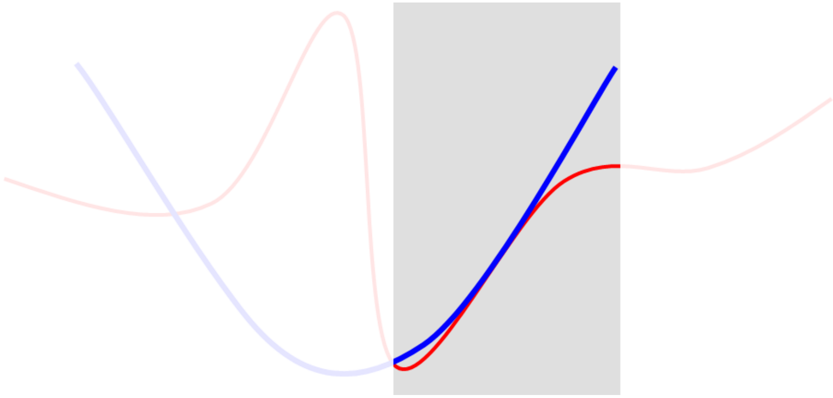
$\overline{f}$ is a ***convex majorizer*** of f over a path-connected set $\Omega(x_0)$ of consistent signature pattern.

Why Majorize?

Q. Why not use Newton's method directly?

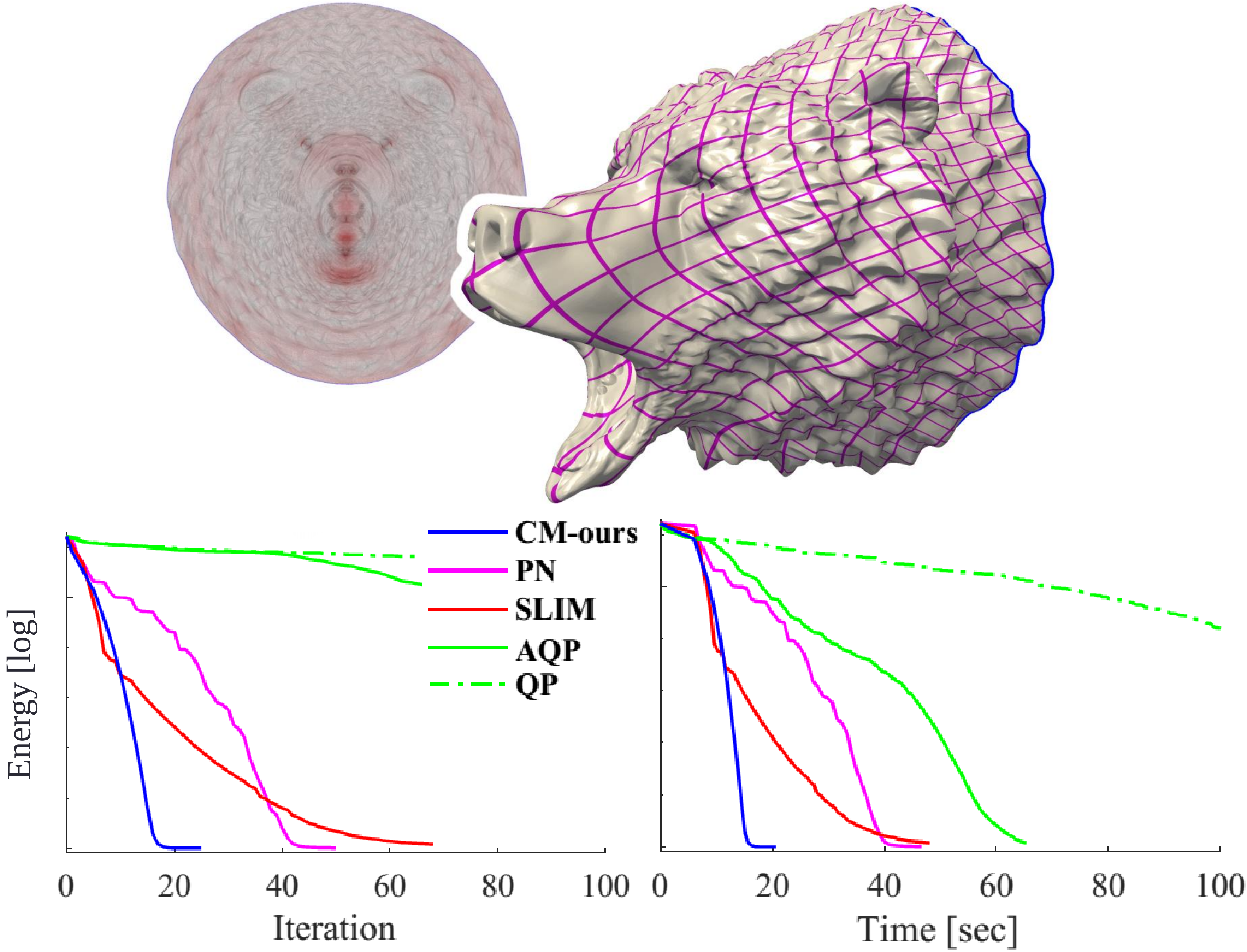
A. $\nabla^2 f$ is generally indefinite

However, $\overline{f}$ is a local ***convex majorizer*** of f



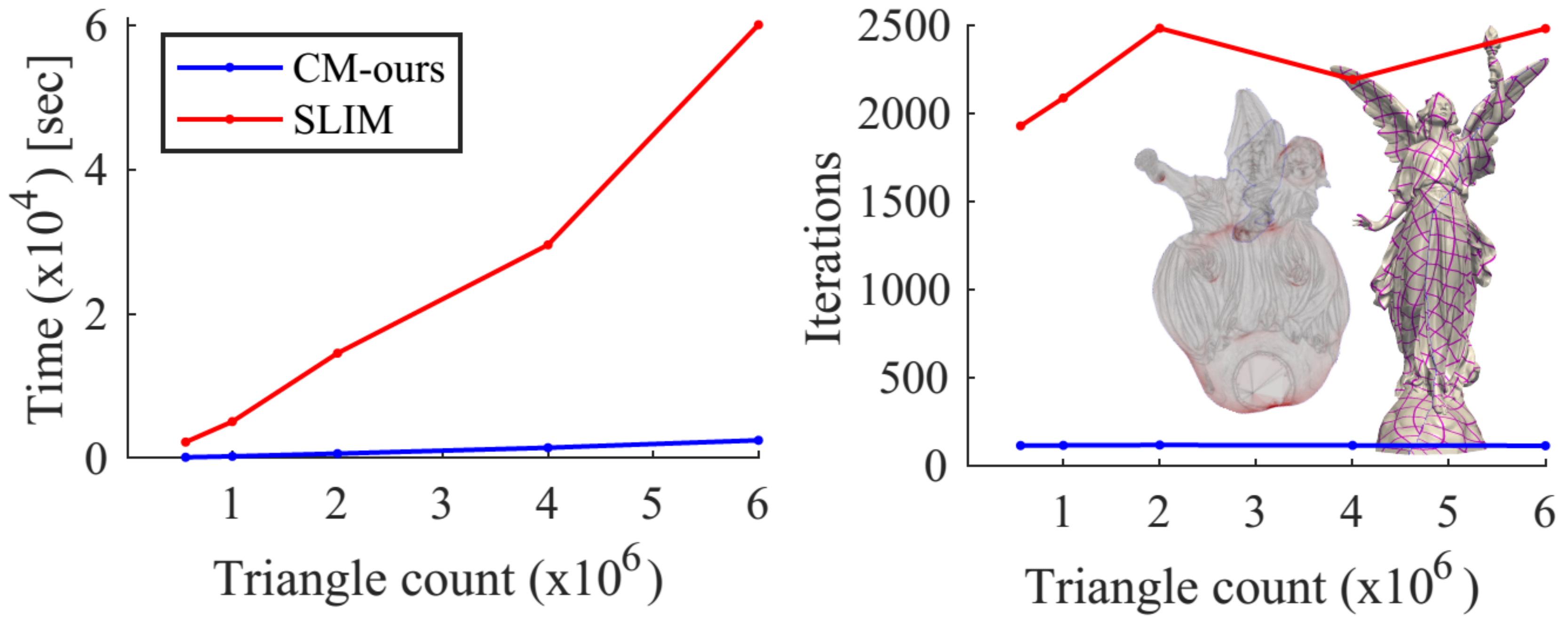
$$p_n = -\left(\nabla^2 \overline{f}\right)^{-1} \nabla \overline{f}$$

is a **decent** direction



CM-ours
PN
SLIM
AQP
QP
Energy [log]
0
20
40
60
80
100
Iteration
0
20
40
60
80
100
Time [sec]

Scalability

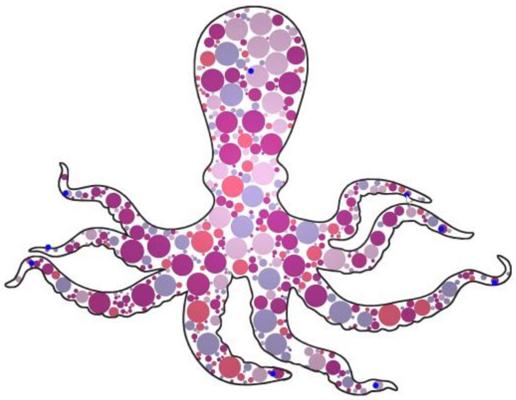


Deformation

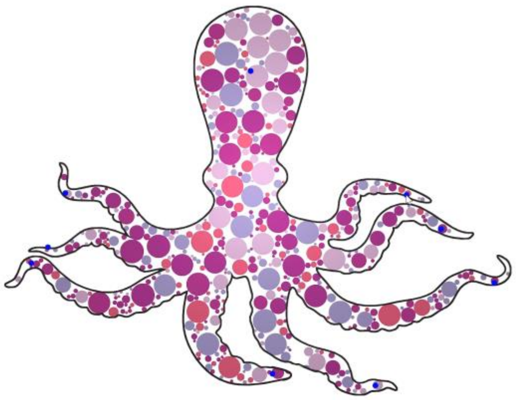
Recorded in realtime
14351 triangles

Projected Newton

[Fu and Liu 16]



Ours



Limitations

Rely on

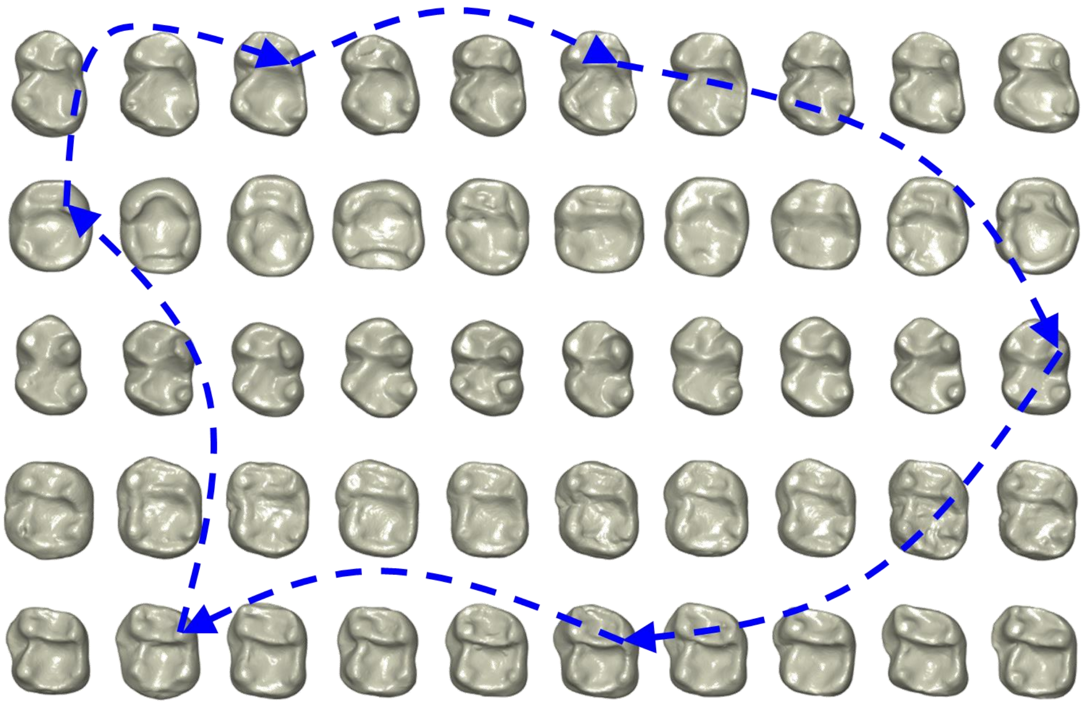
$$f = h \circ \boldsymbol{g}$$
$$h = h^{+} + h^{-}$$
$$\boldsymbol{g} = \boldsymbol{g}^{+} + \boldsymbol{g}^{-}$$

- Not unique
- Not canonical
- Differ in performance

What's next

Learn mappings via geometric optimization:

Learn $\boldsymbol{\theta}$ s.t. $\{\Phi_{kl}^{\boldsymbol{\theta}}\}$ are consistent



Thank you!