MRA-based wavelet frames and digital Gabor filters

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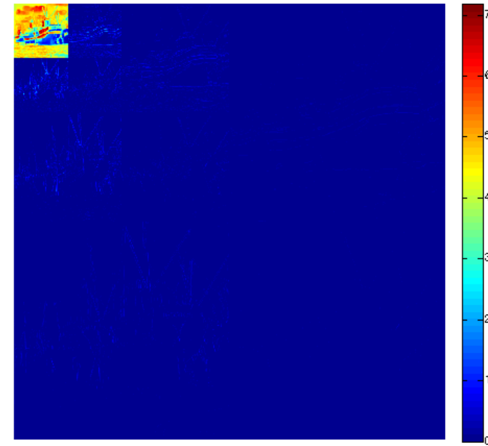
Mathematical representation of images

$$\mathbf{f} = \sum_{n \in I} c_n \mathbf{v}_n$$



$\mathbf{f}$: image $\{\mathbf{v}_n\}_{n \in I}$: system $\{c_n\}_{n \in I}$: coefficients

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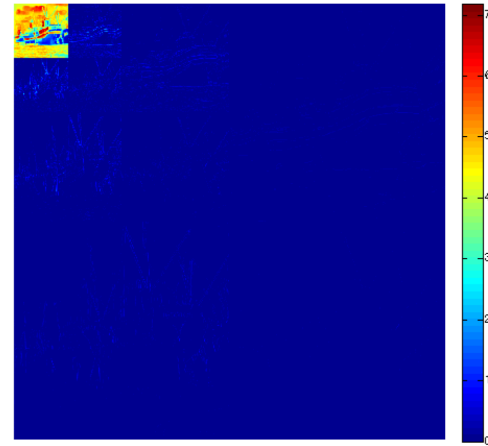


$\mathbf{f}$: image $\quad \{\mathbf{v}_n\}_{n\in I}$: system $\quad \{c_n\}_{n\in I}$: coefficients

A good system for image representation:

- $\{c_n\}$ has physical meanings.
- $\{c_n\}$ is sparse.
- $\{\mathbf{v}_n\}$ is an orthonormal basis or a tight frame.

$$\mathbf{f} = \sum_{n\in I} \langle \mathbf{f}, \mathbf{v}_n \rangle \mathbf{v}_n, \quad \forall \mathbf{f} \in \mathbb{C}^N$$

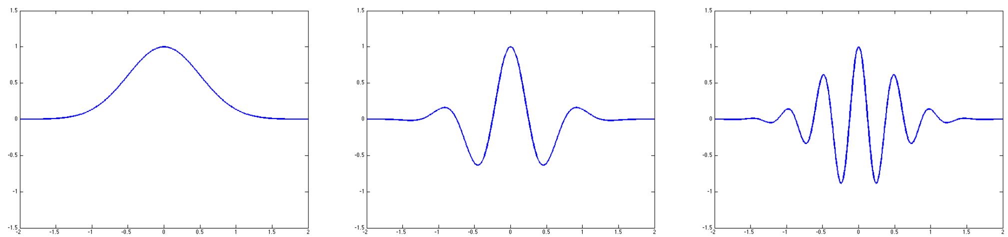
- Atoms $\mathbf{v}_n$'s are localized.

Two kinds of widely used systems

Gabor or Cosine system in $L_2(\mathbb{R})$:

$$\{g(t-ak)e^{2\pi ibt\ell}\}_{k,\ell\in\mathbb{Z}} \quad \text{or} \quad \{g(t-ak)\cos(\pi b\ell t)\}_{k,\ell\in\mathbb{Z}}$$

- Example:



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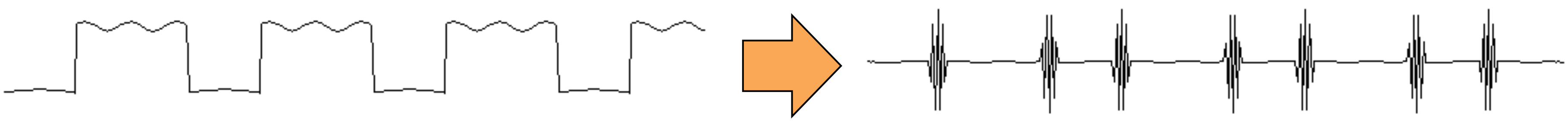
- Provide accurate local time-frequency analysis



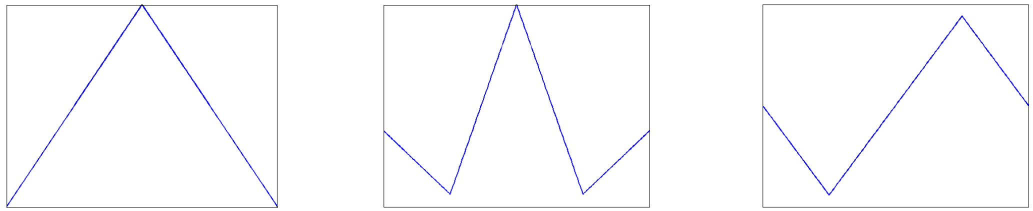
periodic　　　　　　cosine transform

- Discretization of Gabor or Cosine systems: sampling the continuous atoms
- Drawback: lack of multi-scale property

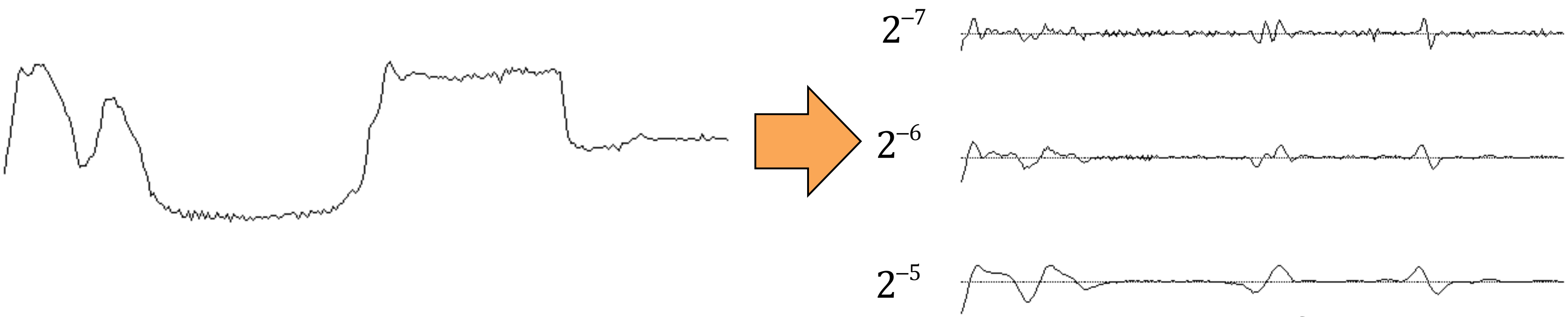
Cont'd

Wavelet system in $L_2(\mathbb{R})$: $\{2^{n/2}\psi_\ell(2^n t - k)\}_{n,k\in\mathbb{Z},1\le\ell\le r}$

- Example: linear spline wavelet



- Providing local discontinuity measurements in multi-scales



piecewise smooth

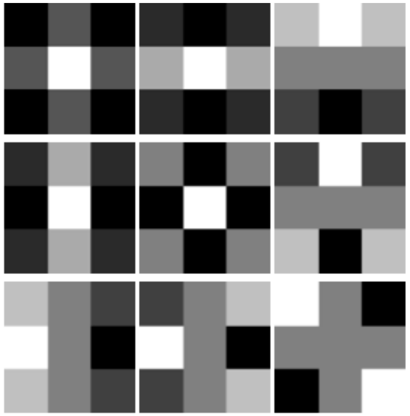
linear spline framelet

- Discrete wavelet systems are generated by the filter bank

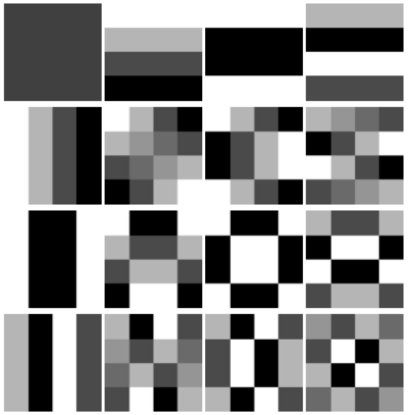
- Drawback: weak local time-frequency analysis

2D system: tensor product of 1D systems

- 2D filters (elementary atoms)



linear spline wavelet



local DCT

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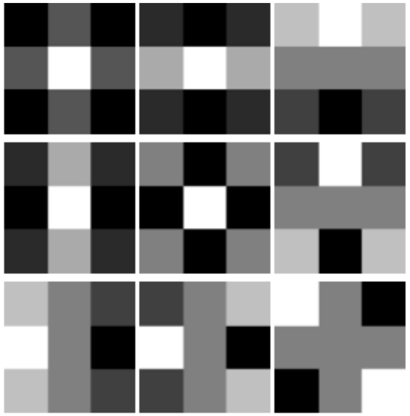

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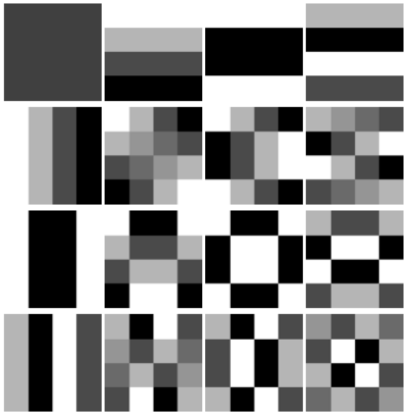

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What goes wrong?

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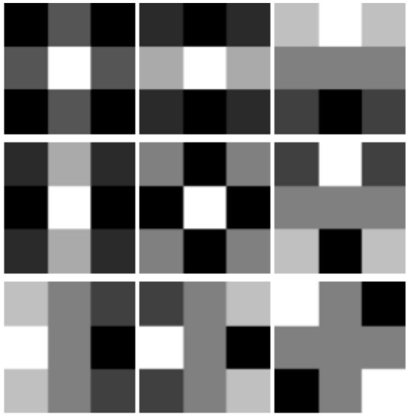

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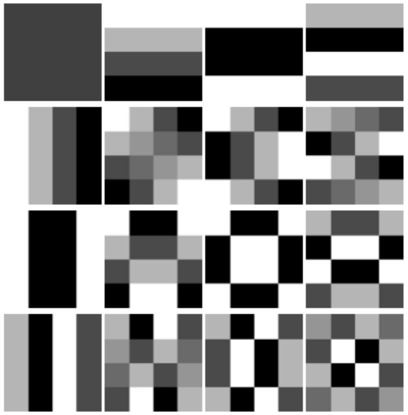

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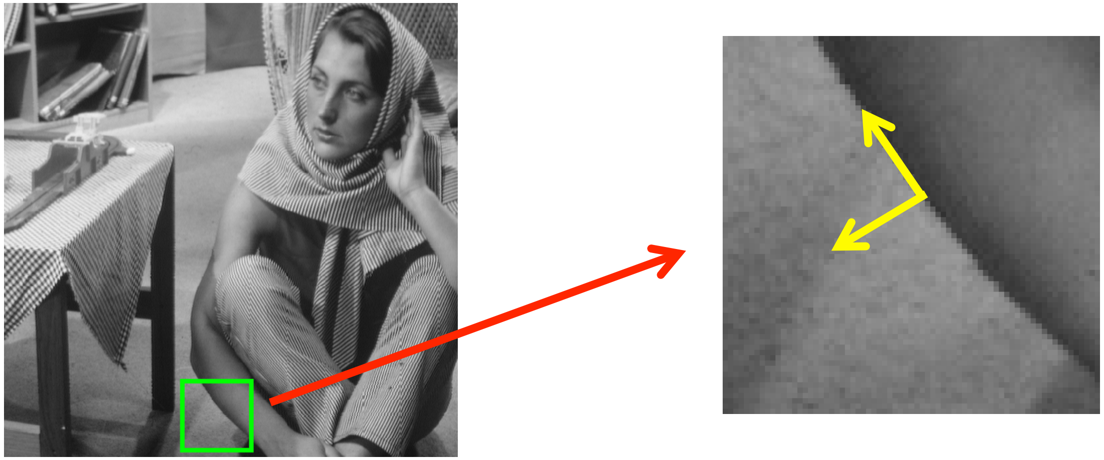
What goes wrong?

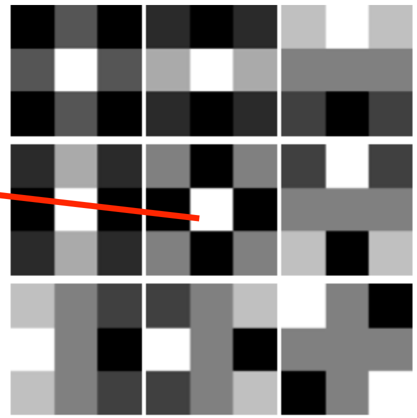
- Geometrical structures in images



2D system: tensor product of 1D systems

- 2D filters (elementary atoms)

$$\frac{1}{16}\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$



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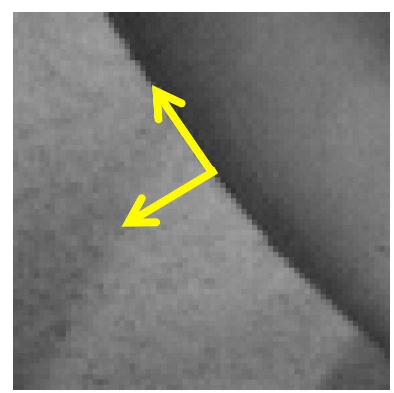
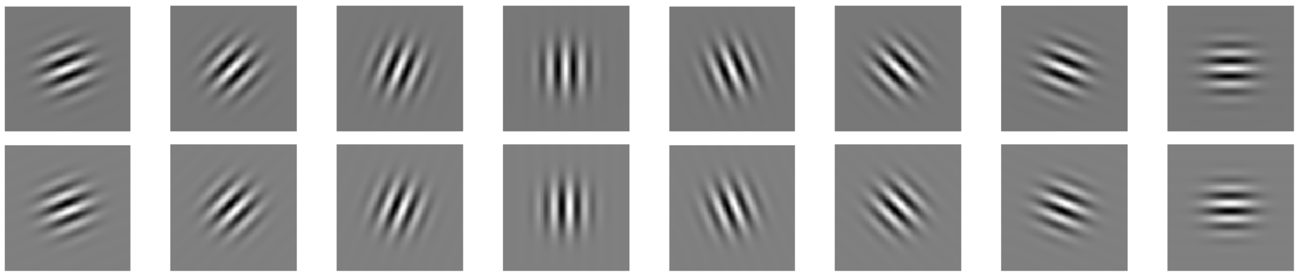
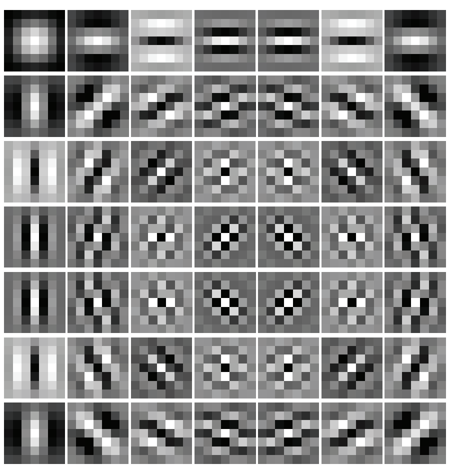


Gabor function for optimal orientation selectivity

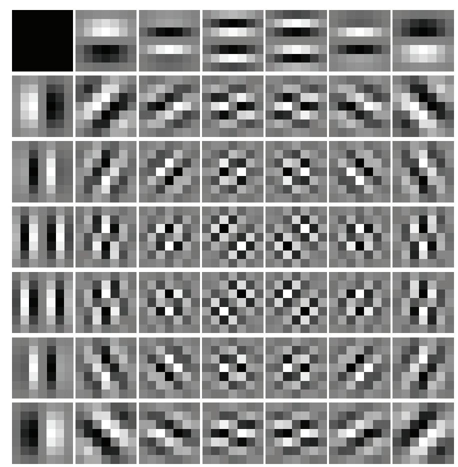
- 2D tensor product Gabor functions $g(t_1)g(t_2)e^{2\pi i(\omega_1 t_1+\omega_2 t_2)}$



- 2D tensor product Gabor filters $\mathbf{g}(m)\mathbf{g}(n)e^{2\pi ib(m\ell_1+n\ell_2)}$



$\mathrm{Real}(\mathbf{g}_{\ell_1,\ell_2})$



$\mathrm{Imag}(\mathbf{g}_{\ell_1,\ell_2})$

Outline of the work

- Motivation: ideal discrete representation for images
  - orientation selectivity
  - local time-frequency analysis
  - multi-scale structure

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  - Relationship between digital Gabor filters and MRA-based wavelet tight frames

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- Applications in image restoration

Definitions and notations

H is a Hilbert space. $\{v_n\}_{n\in I} \subset H$ is

- a *Riesz sequence*: there exists $C_2 \geq C_1 > 0$ s.t.

$$C_1 \sum_{n\in I} |\mathbf{c}(n)|^2 \leq \left\| \sum_{n\in I} \mathbf{c}(n) v_n \right\|_2^2 \leq C_2 \sum_{n\in I} |\mathbf{c}(n)|^2,$$

  for any $\{\mathbf{c}(n)\}_{n\in I} \in \ell^2(I)$.

- an *orthonormal sequence*: a Riesz sequence with $C_1 = C_2 = 1$.
- a *frame* for H : there exist $B \geq A > 0$ s.t.

$$A\|f\|^2 \leq \sum_{n\in I} |\langle f, v_n\rangle|^2 \leq B\|f\|^2, \quad \forall f \in H.$$

- a *tight frame* for H : a frame with $A = B = 1$.
- $\{u_n\}_{n\in I}$ is the *dual frame* of $\{v_n\}_{n\in I}$:

$$f = \sum_{n\in I} \langle f, v_n\rangle\, u_n = \sum_{n\in I} \langle f, u_n\rangle\, v_n, \quad \forall f \in H.$$

Characterization of frame property of discrete Gabor systems

- *Gabor system* $X=(K,L)_{\mathbf{g}}$ in $\mathbb{C}^N$:

  $\{\mathbf{g}_{k,\ell}(m)=\mathbf{g}((m-ak)\bmod N)\,\mathrm{e}^{-2\pi i\ell bm}, 0\le m<N\}_{k\in K,\ell\in L}$,

  with $K:=\{0,1,\ldots,N/a-1\}$, $L:=\{0,1,\ldots,b^{-1}-1\}$.

- Define the *adjoint system* of a Gabor system:

system	$X=(K,L)_{\mathbf{g}}$	$X^*=(L^*,K^*)_{\mathbf{g}}$
shift parameter	(a,b)	$(1/b,1/a)$
lattices	$K=\{0,1,\ldots,N/a-1\}$ $L=\{0,1,\ldots,b^{-1}-1\}$	$L^*=\{0,1,\ldots,Nb-1\}$ $K^*=\{0,1,\ldots,a-1\}$

Cont'd

Theorem 1 (Duality principle)

I. X is a frame for $\mathbb{C}^N$ if and only if $(ab)^{-\frac{1}{2}}X^*$ is a Riesz sequence;

II. X is a tight frame for $\mathbb{C}^N$ if and only if $(ab)^{-\frac{1}{2}}X^*$ is an orthonormal sequence

Cont'd

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Theorem 2 (Construction of Gabor tight frames)

Suppose $\mathbf{g} \in \mathbb{R}^N$ is non-negative with support $\{0,1,\ldots,p-1\}$. $X = (K,L)_{\mathbf{g}}$ is a tight frame for $\mathbb{C}^N$ if and only if

$$\text{(i)} \quad b \le p^{-1}; \quad \text{and} \quad \text{(ii)} \quad \sum_{k=0}^{N/a-1} (\mathbf{g}((\cdot - ak) \bmod N))^2 \equiv b.$$

Removing nonzero DC offset of Gabor tight frames

- Phenomenon of nonzero DC offset

  Example: Cubic B-spline with nodes $[0,1,2,3,4]$ and $a=1,\, b=1/3$.

$$\mathbf{g}_0 = \frac{\sqrt{2}}{6}[1,2,1];\ \mathbf{g}_1 = \frac{\sqrt{2}}{12}[2,-2,-1]+i\frac{\sqrt{6}}{12}[0,-2,1];\ \mathbf{g}_2 = \frac{\sqrt{2}}{12}[2,-2,-1]+i\frac{\sqrt{6}}{12}[0,2,-1].$$

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- Gabor induced frame with zero DC offset and explicit dual frame

$$\bar{\mathbf{g}}_\ell=e^{i\theta_\ell}\mathbf{g}_\ell-\mu_\ell\mathbf{g}_0$$

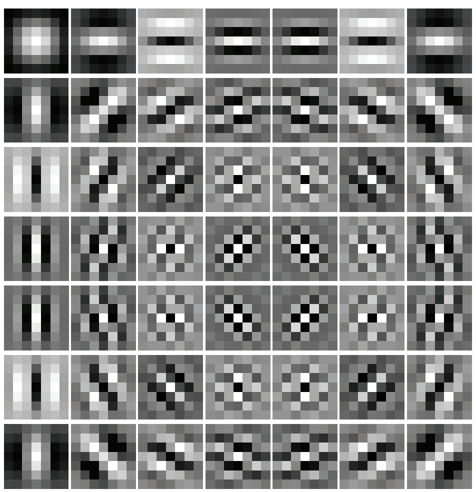
$$\bar{\mathbf{g}}_0=\frac{\sqrt{2}}{6}[1,2,1];\ \bar{\mathbf{g}}_1=\frac{\sqrt{2}}{8}[-1,2,-1]+i\frac{\sqrt{6}}{12}[1,0,-1];\ \bar{\mathbf{g}}_2=\frac{\sqrt{2}}{8}[-1,2,-1]+i\frac{\sqrt{6}}{12}[-1,0,1].$$

Analysis of orientation selectivity

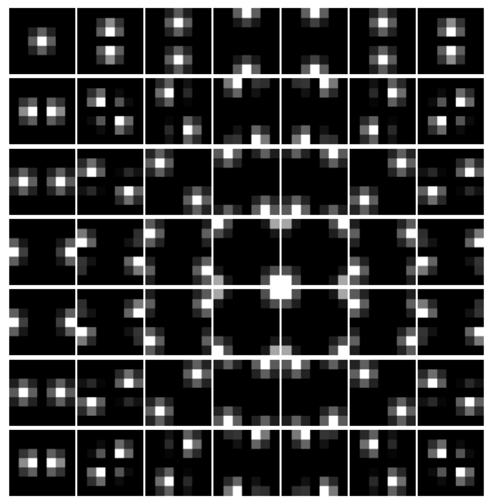
◆ Definition 1 (Orientation selectivity)

$\mathbf{h} \in \mathbb{C}^{p \times p}$ has strong selectivity w.r.t. orientation θ if

1) all maximum points of $|\hat{\mathbf{h}}|$ are on $\omega_x \cos\theta + \omega_y \sin\theta = 0$;
2) the values of $|\hat{\mathbf{h}}|$ away from $\omega_x \cos\theta + \omega_y \sin\theta = 0$ are negligible.



$\mathrm{Real}(\mathbf{g}_{j_1,j_2})$, size 7×7



Absolute value of Fourier of $\mathrm{Real}(\mathbf{g}_{j_1,j_2})$

Gabor induced frames

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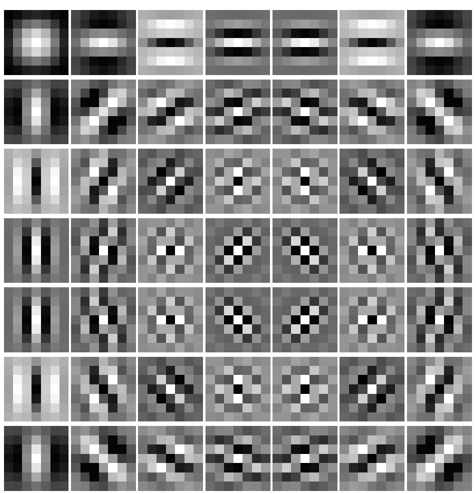

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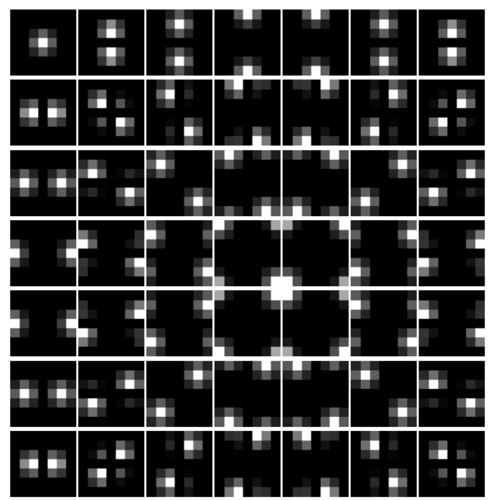
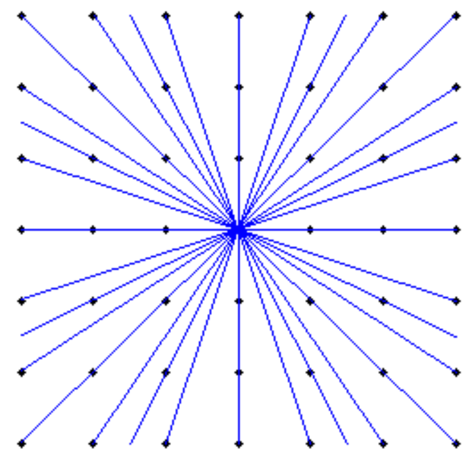


Absolute value of Fourier of Real($\mathbf{g}_{j_1, j_2}$)

Gabor induced frames



7×7

feasible orientations

- What we have done:
  - Derive duality principle for discrete Gabor systems
  - Construct discrete Gabor tight frame and Gabor induced frame providing
    - Local time frequency analysis
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  Can we introduce the multi-scale property into Gabor systems?

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- Next goal:

  Construct MRA-based wavelet tight frame with Gabor structure

Multi-resolution analysis (MRA)

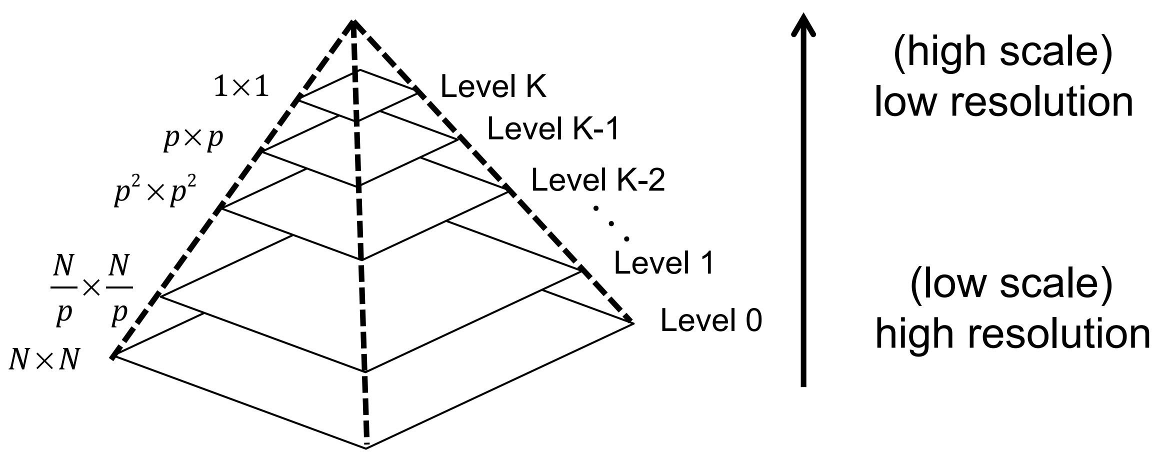
- If $\phi(t) = p \sum_{k\in\mathbb{Z}} \mathbf{a}_0(k)\, \phi(pt-k), \quad t\in\mathbb{R},$
  - $p\in\mathbb{Z}\ (p>1)$: *dilation factor*;
  - $\mathbf{a}_0 \in \ell_2(\mathbb{Z})$: *refinement mask*;
  - $\phi \in L_2(\mathbb{R})$: *p-dilation refinable function*.
- Assume ϕ is *p*-refinable and $\hat{\phi}(0)=1$, define $\{V_n\}_{n\in\mathbf{Z}}$ by

$$V_n = \overline{\mathrm{span}\{\phi(p^n \cdot - k)\}_{k\in\mathbb{Z}}}.$$

- $\{V_n\}_{n\in\mathbb{Z}}$ forms a *multi-resolution analysis (MRA)* for $L_2(\mathbb{R})$ if

$$\text{(i) } V_n \subset V_{n+1}, \quad \text{(ii) } \overline{\cup_n V_n} = L_2(\mathbb{R}), \quad \text{(iii) } \cap_n V_n = \{0\}.$$

Image Pyramid:



MRA based wavelet system and discrete Gabor filters

- By $\psi_\ell(t) = p\sum_{m\in\mathbb{Z}} \mathbf{a}_\ell(m)\phi(pt-m)\ (t\in\mathbb{R})$, define:
  - $\Psi = \{\psi_\ell\}_{\ell=1}^r$: a set of *wavelet functions*;
  - $\mathbf{a}_\ell \subset \ell_2(\mathbb{Z})\ (1\le \ell \le r)$: *wavelet masks*.
- Define *p-dilation wavelet system* $X(\Psi)$

$$X(\Psi) = \{\psi_{\ell,n,k}\}_{\substack{1\le\ell\le r\\ n,k\in\mathbb{Z}}} = \{p^{n/2}\psi_\ell(p^n\cdot - k)\}_{\substack{1\le\ell\le r\\ n,k\in\mathbb{Z}}}.$$

- Discrete multi-scale wavelet tight frames for $\ell_2(\mathbb{Z})$ are generated by the filter bank

$$\{\mathbf{a}_0, \mathbf{a}_1, \ldots \mathbf{a}_r\}.$$

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$$\{\mathbf{a}_0, \mathbf{a}_1, \ldots \mathbf{a}_r\}.$$

Question: can these filter be replaced by Gabor filters with orientation selectivity?

Window: $\mathbf{g}(m) = 0$ if $m \notin [0, q-1]$, masks: $\{\mathbf{g}_\ell(m) = \mathbf{g}(m)\, e^{-2\pi i b\ell m}\}_{\ell=0}^{1/b-1}$

Answer

Theorem 4 (UEP based characterization of Gabor filters induced wavelet tight frames)

ϕ defined from $\mathbf{g}$ generates an MRA for $L_2(\mathbb{R})$ and $X(\Psi)$ form a tight frame for $L_2(\mathbb{R})$, if the followings hold true:

1. $\frac{1}{b} \geq q$;
2. $\sum_{n\in\mathbb{Z}} \mathbf{g}(n) = 1$;
3. $\sum_{n\in\Omega_j} |\mathbf{g}(n)|^2 = \frac{b}{p}$, where $j \in \mathbb{Z}/p\mathbb{Z}, \Omega_j = (p\mathbb{Z} + j) \cap supp(\mathbf{g})$.

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- Unitary Extension Principle (UEP, [1])

Let $\phi \in L_2(\mathbb{R})$ be refinable with $\hat{\phi}(0)=1$, and $\mathbf{a_0}$ is finitely supported
Then $X(\Psi)$ forms a tight frame for $L_2(\mathbb{R})$, if for any $\nu \in p^{-1}\mathbb{Z} \setminus \mathbb{Z}$

$$\sum_{\ell=0}^{r} \hat{\mathbf{a}}_\ell(\omega)\overline{\hat{\mathbf{a}}_\ell(\omega+2\pi\nu)} = \delta(\nu), \quad \text{a.e. } \omega \in [-\pi,\pi]$$

[1] A. Ron and Z. Shen, Affine system in L2(R): the analysis of the analysis operator, J. Funct. Anal. 1997

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Theorem 5 (Solutions to UEP-based characterization)

Finitely supported window $\mathbf{g}$ satisfies Condition 1, 2, 3 for some b, if and only if $\mathbf{g} = b(\ldots,0,0,\underbrace{1,\ldots,1}_{1/b},0,\ldots)$ and $\frac{1}{bp} \in \mathbb{Z}$.

Theorem 6 (Refinable functions in continuum space)

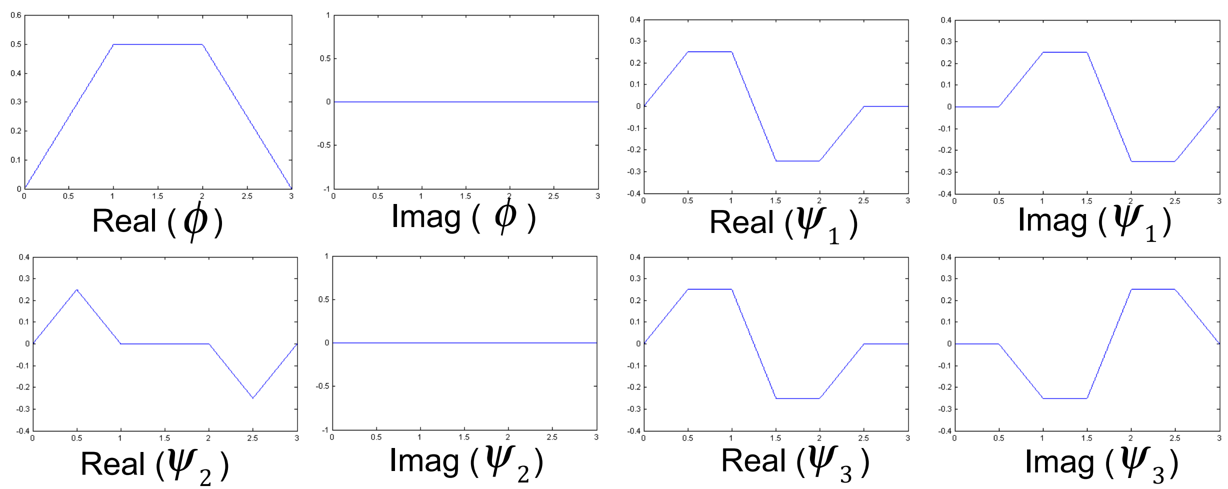
ϕ is refinable with $\hat{\phi}(0)=1$, and $\mathbf{g} = p^{-k}(1,\ldots,1) \in \mathbb{R}^{p^k}$ $(k \geq 2)$

Then ϕ is a $(k-1)$ th order spline.

Example

- Example 2

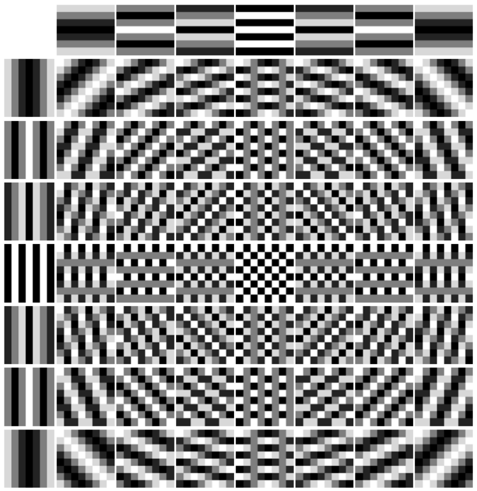
$$p=2,\qquad \text{masks: } \begin{cases} \mathbf{g} = (\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}) \\ \mathbf{g}_1 = (\frac{1}{4},-\frac{1}{4}i,-\frac{1}{4},\frac{1}{4}i) \\ \mathbf{g}_2 = (\frac{1}{4},-\frac{1}{4},\frac{1}{4},-\frac{1}{4}) \\ \mathbf{g}_3 = (\frac{1}{4},\frac{1}{4}i,-\frac{1}{4},-\frac{1}{4}i) \end{cases} \qquad \phi(t)=\begin{cases} \frac{1}{2}t, & t\in[0,1), \\ \frac{1}{2}, & t\in[1,2), \\ \frac{1}{2}(3-t), & t\in[2,3), \\ 0, & \text{otherwise.} \end{cases}$$



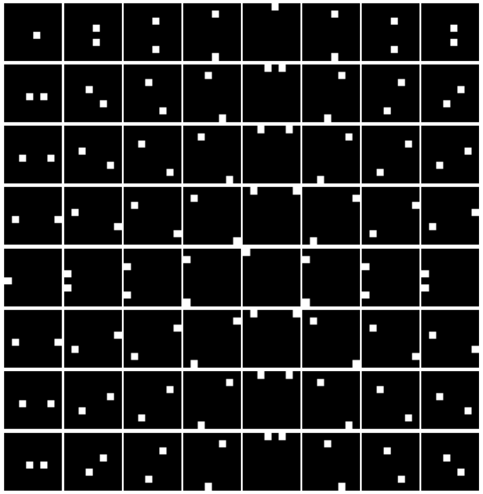
Example

- Example 3

p=8, refinable mask: $\mathbf{g}=(\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8})$



$\mathrm{Real}(\mathbf{g}_{j_1,j_2})$, size 8×8



Absolute value of Fourier of $\mathrm{Real}(\mathbf{g}_{j_1,j_2})$

Application: sparsity based image restoration

Observation:

Blurred

Noisy

?

Truth:

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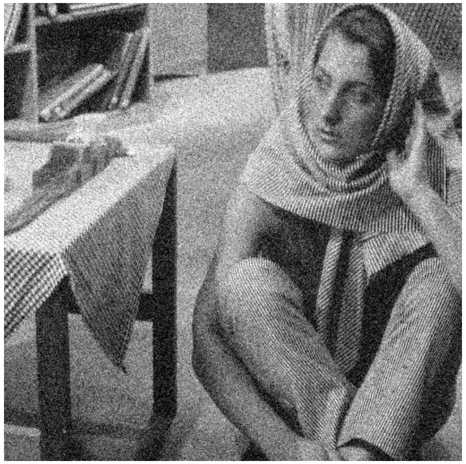

Noisy

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Truth:

- Solve linear system

$$\mathbf{b} = \mathbf{A}\mathbf{f} + \mathbf{n},$$

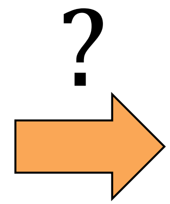
where $\mathbf{b}$ is the observation, $\mathbf{f}$ is the truth and $\mathbf{n}$ is the noise.

Application: sparsity based image restoration

Observation:

Blurred

Noisy



Truth:

- Solve linear system

$$\mathbf{b} = \mathbf{A}\mathbf{f} + \mathbf{n},$$

where $\mathbf{b}$ is the observation, $\mathbf{f}$ is the truth and $\mathbf{n}$ is the noise.

- Suppose $\mathbf{f} = \sum_{n \in I} c_n \mathbf{v}_n = \mathbf{W}_X^* \mathbf{c},$ and $\mathbf{c} = \mathbf{W}_Y \mathbf{f}$ is sparse.

Image recovery using proposed (tight) frame

Recovered image $\hat{\mathbf{u}} = \sum_{k=1}^{m} \mathbf{u}_k$ is solved from *sparsity based multi-layer composite model*,

$$\min_{\{\mathbf{u}_1,\ldots,\mathbf{u}_m\}\subset\mathbb{R}^N} \sum_{k=1}^{m} \lambda_k \|\mathbf{W}_{Y_k}\mathbf{u}_k\|_1, \quad \text{s.t.} \quad \|\mathbf{A}(\sum_{k=1}^{m}\mathbf{u}_k)-\mathbf{b}\|_2 \le \epsilon,$$

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- Proposed and comparison methods:
  - Y_1 : Gabor induced frame by cubic B-spline $a=2, b=1/7, L=7$;

    Y_2 : Gabor induced frame by cubic B-spline $a=4, b=1/15, L=15$.
  - Y_1 : MRA based wavelet tight frame with Gabor structure $p=2, b=1/8, L=8$.

Results

Table 1: PSNR values of denoised images

image	TV [1]	Linear spline framelet [2]	DT-CWT [3]	Multi Gabor induced frame	tight frame w / p=8
Barbara	26.84	29.25	28.90	**30.39**	29.38
Bowl	29.24	30.15	29.43	**30.58**	30.40
Cameraman	28.83	29.00	28.94	29.26	**29.29**
Lena	30.71	31.10	31.49	**31.74**	31.39

[1] Y. Wang, J. Yang, W. Yin, Y. Zhang, A new alternating minimization algorithm for total variation image reconstruction, SIAM J. Imaging Sci. 2008
[2] J. Cai, S. Osher, Z. Shen, Split Bregman methods and frame based image restoration, SIAM J: Multiscale Model. Sim. 2009
[3] I. W. Selesnick, R. G. Baraniuk, N. C. Kingsbury, The dual-tree complex wavelet transform, IEEE Signal Proc. Mag. 2005

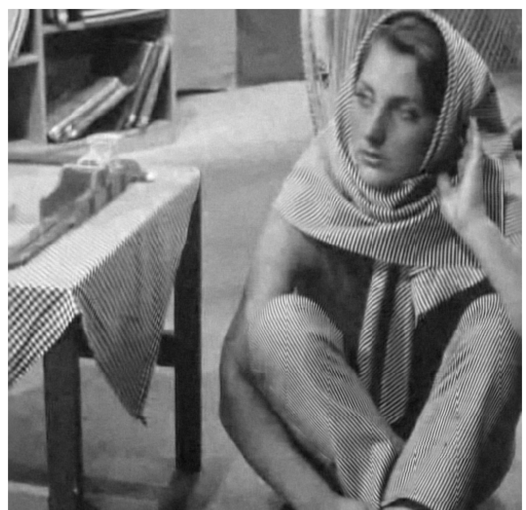
Results

(a) noisy image, $\sigma = 20$

(b) TV

(c) framelet

(d) DT-CWT



(e) Gabor induced frame

(c) tight frame with p=8

Results

(a) noisy image, $\sigma = 20$

(b) TV

(c) framelet

(d) DT-CWT

(e) Gabor induced frame

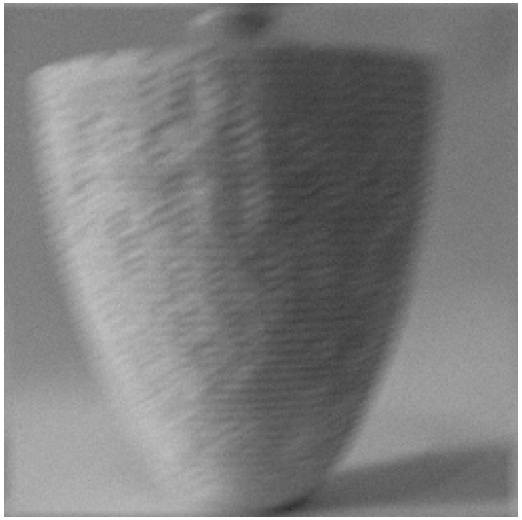
(c) tight frame with p=8

Results

Table 2: PSNR values of deblurred images

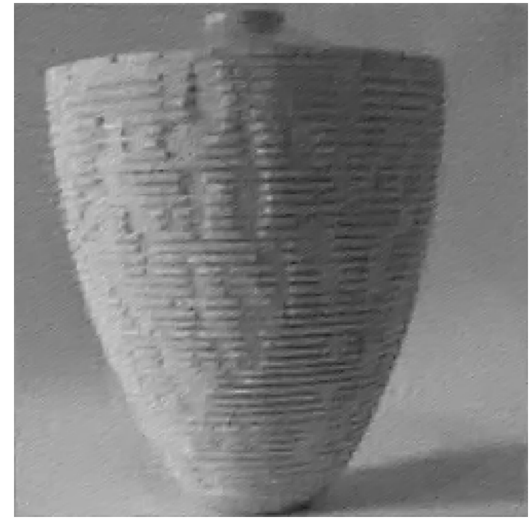
image	kernel	TV	linear spline framelet	DT-CWT	multi Gabor induced frame	tight frame w / p=8
Barbara	disk	24.77	25.17	25.15	**25.65**	25.48
	motion	24.64	24.97	25.00	**25.70**	25.49
	gaussian	24.13	24.14	24.19	**24.21**	24.18
	average	23.99	24.03	24.07	**24.27**	24.10
Bowl	disk	28.73	28.92	28.99	**29.35**	29.13
	motion	28.88	29.08	29.15	**29.67**	29.36
	gaussian	27.96	27.82	28.32	**28.66**	28.46
	average	28.73	28.84	28.94	**29.25**	29.21
Cameraman	disk	26.31	26.83	26.22	**27.01**	26.73
	motion	26.18	**27.14**	26.35	26.93	26.72
	gaussian	24.96	24.84	24.73	**25.04**	24.94
	average	25.08	25.12	25.00	**25.55**	25.30
Lena	disk	32.05	32.17	32.25	**32.53**	32.06
	motion	30.86	30.49	31.21	**31.43**	30.45
	gaussian	31.34	31.26	31.59	**31.74**	31.55
	average	30.10	29.96	30.21	**30.36**	30.06

Results



(a) motion blurred image, $\sigma = 3$

(b) TV



(c) framelet

(d) DT-CWT

(e) Gabor induced frame

(c) tight frame with p=8

Results

(a) average blur kernel, $\sigma = 3$

(b) TV

(c) framelet

(d) DT-CWT

(e) Gabor induced frame

(c) tight frame with $p=8$

Reference

- H. Ji, Z. Shen, Y. Zhao, Directional frames for image recovery: multi-scale discrete Gabor frames, J. Fourier Anal. Appl. 2016
- H. Ji, Z. Shen, Y. Zhao, Digital Gabor filters that generate MRA-based wavelet tight frames, manuscript

Thank you.