Look-down constructions with continuous levels

(Singapore, 01/08/17)

First appeared in a paper by Kurtz (2000)

→ Particle representations for measure-valued population processes with spatially varying birth rates.

[Overcomes the problem that with discrete levels, the branching rates of particles have to be independent of the particle location to keep the exchangeability property of the particles (because once the levels are fixed, the branching rate depends on the levels of the particles and so moving the other levels also changes their branching rates, while inserting new particles in a continuous level space doesn't alter the others).]?

More generally, you can fit as many particles as you want in a continuous compact interval, and so you can deal with measures with infinite mass. Also, levels can overtake each other when they move and so some types of particles can die faster than others without having to impose a preliminary order on the levels.

In the examples we'll see, not only the levels take their values in a continuous space, but they also evolve in time to increase or decrease the reproduction/death rates of the particles according to their current types in some space (genetic, geographic, etc.)

These examples are taken from the Etheridge & Kurtz paper, but the beginning appears also in Kurtz & Rodrigues. | Genealogical constructions of population models.

I Pure death process

We start with a "very" simple example showing the interest in making particles move, and then we'll consider another example showing the interest in having levels take their values in a continuous interval.

Intro/Notation: Levels take their values in $\mathbb{R}_+$. In what we call the "prelimiting model", a particle dies/disappears when its level reaches a given quantity λ. The value of λ won't play a role (at least in the simple example we consider), but then we can make $\lambda \to \infty$ and find conditions under which there is a limiting system model evolving according to a given "high density" model.

λ here is the equivalent of the $\left\{ \begin{array}{l} K \text{ of Anton's part} \\ \text{~~fixed~~ initial level } n_0 \text{ below which} \end{array} \right.$ we look in the discrete level case(*). As in the discrete case, the restriction of the infinite system to the set of particles with a level below some $\lambda \in (0,\infty)$ will give a sample from the full empirical distribution.

(*) though we can put as many particles as we want below a given level.

Model (with and without levels):

State of population with levels: $\eta = \sum_i \delta_{(u_i, x_i)}$ where

u_i = level of particle i (in $\mathbb{R}_+$)

x_i = type of particle i (in some space E) $\to$ geography, genetics, ...

State of pop. without levels $\overline{\eta} = \sum_i \delta_{x_i}$

Suppose we want to make each particle of type x die at rate $d_0(x)$.

Fix $\lambda \in (0,\infty)$ (we don't care about its value at the moment, think of $\lambda = 1$ if you want) and a configuration $\bar{\eta} \in \mathcal{M}_p(E)$ ← point measures

Give iid uniform$([0,\lambda])$ levels to every particle in $\bar{\eta}$.

(Natural thing to do, we'll see later why it is also useful).

We declare that particles die when their levels reach λ.

<u>Recipe</u>: Set $u_i(t) = u_i(0)\, e^{d(x_i)t} \iff \frac{du_i}{dt} = d(x_i)\, u_i$

Then if we call τ_i the death time of particle i, we have

$$\mathbb{P}(\tau_i > t) = \mathbb{P}\left(\frac{u_i(0)}{\lambda} \leq e^{-d(x_i)t}\right) \quad \text{where } \frac{u_i(0)}{\lambda} \sim \mathrm{Unif}[0,1]$$

$$= e^{-d(x_i)t}$$

and so $\tau_i \sim \mathcal{E}xp(d(x_i))$.

⚠ The type of a particle doesn't change through time here.

This is a simple exercise in this case. Another way of seeing it is that for every type x of particles, at time t we want to have thinned the subpopulation of particles of this type by a fraction $1 - e^{-d(x)t}$, i.e. such a fraction should now have levels $\geq \lambda$.

Then, either you use a very easy thinning argument to see that starting from $\bar{\eta} = \sum_i \delta_{x_i}$, the state of the system (without levels) at time t is that each particle has survived with proba $e^{-d(x_i)t}$

Another way of formalizing this is to look at test functions of the form $f(\eta) = \prod_i g(u_i, x_i)$ for some function $g : \mathbb{R}_+ \times E$ such that $g(u,x) = 1$ if $u \geq \lambda$.

The generator of the evolution with levels is

$$A_{pd} f(\eta) = \sum_i \frac{f(\eta)}{g(u_i, x_i)} \partial_{u_i} g(u_i, x_i)\, d(x_i)\, u_i \qquad (*)$$

$$= f(\eta) \int_{\mathbb{R}_+ \times X} \eta(du, dx)\, d(x)\, u\, \frac{\partial_u g}{g}$$

on $\mathbb{R}_+ \times E$

law of a random measure constructed by adding a unif level to every part. in $\bar{\eta}$

Now let's average this function of η with respect to the Uniform$([0,\lambda])$ iid distribution of levels. Write $\alpha(\bar{\eta})$ for the distribution of the levels given to all of the $|\bar{\eta}|$ particles. Then using $(*)$

$$\alpha(\bar{\eta})\left[A_{pd} f \right] = \sum_i \prod_{j \neq i} \bar{g}(x_j)\, d(x_i)\, \frac{1}{\lambda} \int_0^\lambda u\, \partial_u g(u, x_i)\, du$$

$\bar{g}(x) = \frac{1}{\lambda}\int_0^\lambda g(u,x)\,du$

$\bar{f}(\bar{\eta}) = \prod_i \bar{g}(x_i)$

$$= \sum_i \frac{\bar{f}(\bar{\eta})}{\bar{g}(x_i)}\, d(x_i) \left\{ \underbrace{\left[\frac{1}{\lambda} u\, g(u, x_i) \right]_0^\lambda}_{1} - \frac{1}{\lambda}\int_0^\lambda g(u, x_i)\, du \right\}$$

$$= \sum_i \bar{f}(\bar{\eta})\, d(x_i) \left\{ \frac{1}{\bar{g}(x_i)} - 1 \right\}$$

$\Rightarrow$ generator of the pure death model where a particle with type x dies at rate $d_0(x)$.

Not very surprising since we seem to have made everything on purpose, but the calculation does look very magical.

<u>Slight worry</u>: calculations based on the iid uniform structure of the levels, which is true at the very beginning but then needs to be conserved by the evolution of the particles.

So assuming that the iid uniform structure of the levels is conserved and we can just average out the levels to deal with the trajectories, then the law of the process without levels, $(\bar{\eta}_t)_{t\geq 0}$ is that of a pure death process starting at $\bar{\eta}_0 = \bar{\eta}$ and such that particles of type x are killed at rate $d_0(x)$.

$\Rightarrow$ As in the discrete procedure, we need that knowing the trajectories $(x_i(s))_{s\in[0,t]}$ doesn't give you information on the level of each particle.

We'll see later what is the main tool (the Markov mapping theorem) giving a link between the model with and without levels. Before that, let's pass to a "high-density" limit to parallel the construction with discrete levels. We'll adopt a viewpoint which may look slightly different from the Etheridge & Kurtz paper, but that's the way I understand their construction...

<u>High-density limit</u>

Fix $\bar{\eta} \in \mathcal{M}(E)$, not necessarily a point measure and not necessarily finite, and let $\mathcal{P}$ be a Poisson point process on $\mathbb{R}_+ \times E$ with intensity $du \otimes \bar{\eta}(dx)$. Set $\eta^\lambda = \sum_{i:u_i\leq\lambda} \delta_{(u_i,x_i)}$.

Then conditionally on $|\eta^\lambda|$, the levels of these particles are iid Uniform on $[0,\lambda]$. So if we proceed as before, we can construct a pure death process, starting from the particles with levels $\leq\lambda$, and with death rate function d_0. Its initial value is a (random) Poissonian cloud of particles.

Now $\frac{1}{\lambda}\bar{\eta}^{\lambda} = \frac{1}{\lambda}\sum_{i:u_i\leq\lambda}\delta_{x_i} \xrightarrow[\lambda\to\infty]{a.s.} \bar{\eta}$, and $\eta^{\lambda} \xrightarrow[\lambda\to\infty]{a.s.} \mathcal{P}$.

(or $\eta^{\infty} = \sum_{i\in\mathcal{I}}\delta_{(u_i,x_i)}$)

So what happens to the process with and without levels as $\lambda\to\infty$?

Recall that
$$A_{pd}f(\eta^{\lambda}) = f(\eta^{\lambda})\int_{\mathbb{R}_+\times X}\eta^{\lambda}(du,dx)\,d_0(x)\,u\,\frac{\partial_u g(u,x)}{g(u,x)}$$
$$= f(\eta^{\infty})\int_{\mathbb{R}_+\times X}\eta^{\infty}(du,dx)\,d_0(x)\,u\,\frac{\partial_u g(u,x)}{g(u,x)}$$

if $g(u,x)\equiv 1$ when $u\geq\lambda$.

So if we look at test functions of the form $f(\eta) = \prod_i g(u_i,x_i)$ with $g(u,x)\equiv 1$ if $u\geq u_g$, and take $\lambda > u_g$, we see that the generator of the potential limit process $(\eta_t^{\infty})_{t\geq 0}$ is still A_{pd}.

On the other hand, if we integrate this generator with respect to the law $\alpha(\bar{\eta})$ of a PPP on $\mathbb{R}_+\times E$ with intensity $du\otimes\bar{\eta}(dx)$, some standard identities give that if $h(x) := \int_0^{\infty}(1-g(u,x))\,du < \infty$ and $\alpha f(\bar{\eta}) := e^{-\int_E\bar{\eta}(dx)h(x)}$, then
$$\alpha(\bar{\eta})[A_{pd}f] = \alpha f(\bar{\eta})\int_E\bar{\eta}(dx)\,d_0(x)\,h(x).$$

Observe that if we set $\bar{\eta}_t(dx) := e^{-d_0(x)t}\,\bar{\eta}(dx)$, then
$$\frac{d}{dt}(\alpha f)(\bar{\eta}_t) = (\alpha f)(\bar{\eta}_t)\times\frac{d}{dt}\left(-\int_E h(x)e^{-d_0(x)t}\,\bar{\eta}(dx)\right)$$
$$= (\alpha f)(\bar{\eta}_t)\times\int_E d_0(x)\,h(x)\,\bar{\eta}_t(dx)$$
and so we have a candidate for the limit.

But again, we need the Poisson structure to be kept by the evolution of the particles.

So there is a general way of passing from a model without levels to a representation with levels and it goes through the following Markov mapping theorem.

II Markov mapping theorem.

$\boxed{\text{Th. A.2 in E\&K}}$ • Let $\alpha : S_0 \to \mathcal{M}_1(S)$ be a kernel / transition function, with (S_0, d_0) and (S, d) complete, separable metric spaces.

[For us, $S_0 = \mathcal{M}(E)$, $S = \mathcal{M}_p(\mathbb{R}_+ \times E)$ and for $\bar{\eta} \in \mathcal{M}(E)$, $\alpha(\bar{\eta})$ laws of a PPP with intensity $du \otimes \bar{\eta}(dx)$.]

• Let A be an operator with domain $\mathcal{D}(A) \subset \bar{C}(S)$ and $\psi \in C(S)$ such that $\forall f \in \mathcal{D}(A)$, $\exists\, c_f > 0$ s.t.

$$|Af(\eta)| \leq c_f \psi(\eta) \quad \forall \eta \in S,$$

and define $A_0 f(\eta) = \dfrac{Af(\eta)}{\psi(\eta)}$.

Suppose that all these operators are nicely behaved.

• Let $\gamma : S \to S_0$ be a "statistic" which is conserved by α, that is $\alpha(y, \gamma^{-1}(y)) = 1$

[For us : $\gamma(\eta) = \lim\limits_{\lambda \to \infty} \frac{1}{\lambda} \sum\limits_{i : u_i \leq \lambda} \delta_{x_i}$ (the intensity of the projection of η on $\mathcal{M}_p(E)$) .]

+ other technical assumptions.

• Define the operator $C = \{ (\alpha f(\cdot), \alpha(Af)(\cdot)), f \in \mathcal{D}(A) \}$

where $\alpha g(\bar{\eta}) = \int_S g(\eta)\, \alpha(\bar{\eta}, \eta)$.

Operator consisting of averaging with respect to the kernel α.

- Finally, let $\mu_0 \in \mathcal{P}(S_0)$ and $\nu_0 = \int \alpha(\bar\eta, \cdot)\, \mu_0(\bar\eta)$.

Then (we again let aside some technicalities)

(a) If $(\bar\eta_t)_{t\geq 0}$ is a solution to the martingale problem for (C, μ_0), then there exists a solution $(\eta_t)_{t\geq 0}$ of the martingale problem for (A, ν_0) such that $(\bar\eta_t)_{t\geq 0}$ has the same distribution as $(\gamma(\eta_t))_{t\geq 0}$.

(b) For almost every $t \geq 0$, $\mathbb{P}(\eta_t \in \Gamma \mid \hat{\mathcal{F}}_t^{\bar\eta}) = \alpha(\bar\eta_t, \Gamma)$ for $\Gamma \in \mathcal{B}(S)$, where $\hat{\mathcal{F}}_t^{\bar\eta}$ = completion of $\sigma\left(\bar\eta(0), \int_0^r h(\bar\eta_s)\, ds,\ r \leq t,\ h \in B(S_0)\right)$

(c) If in addition, uniqueness holds for the martingale problem for (A, ν_0), then uniqueness holds for the MP for (C, μ_0).

(d) If uniqueness holds for the MP for (A, ν_0), then $\bar\eta$ restricted to the set of times at which $\bar\eta_t$ is $\hat{\mathcal{F}}_t^{\bar\eta}$ measurable is a Markov process.