Learning Session on Genealogies of Interacting Particle Systems

A.Depperschmidt, A.Greven

University Erlangen-Nuremberg

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Tree-valued Markov processes

Contents

Goals

We present here from today's perspective the essentials of a series of papers from 2003-2017 on tree-valued Markov processes with particular focus on some specific stochastic population models and the flow of bridges coding.

The description of processes via flow of bridges has been created in three papers of Bertoin and Le Gall 2003, 2005, 2006:

[1] Bertoin, Jean; Le Gall, Jean-François
Stochastic flows associated to coalescent processes. Probab. Theory Related Fields 126 (2003), no. 2, 261–288.

[2] Bertoin, Jean; Le Gall, Jean-François
Stochastic flows associated to coalescent processes. II. Stochastic differential equations. Ann. Inst. H. Poincaré Probab. Statist. 41 (2005), no. 3, 307–333.

[3] Bertoin, Jean; Le Gall, Jean-Francois
Stochastic flows associated to coalescent processes. III. Limit theorems. Illinois J. Math. 50 (2006), no. 1-4, 147–181.

This approach has been generalized and unified using stochastic equations by Dawson and Li 2012

[4] Dawson, Donald A.; Li, Zenghu
Stochastic equations, flows and measure-valued processes. Ann. Probab. 40 (2012), no. 2, 813–857.

and subsequently a lot of applications where developed:

[5] Labbé, Cyril
Genealogy of flows of continuous-state branching processes via flows of partitions and the Eve property. Ann. Inst. Henri Poincaré Probab. Stat. 50 (2014), no. 3, 732–769.

[6] Foucart, Clément
Generalized Fleming-Viot processes with immigration via stochastic flows of partitions. ALEA Lat. Am. J. Probab. Math. Stat. 9 (2012), no. 2, 451–472.

In particular there is some work which gives a connection with the session on Lookdown constructions.

[7] Gufler, Stephan
Pathwise construction of tree-valued Fleming-Viot processes
arXiv:1404.3682v3

Models

Two populations processes are central:

- Fleming-Viot + Generalized Fleming-Viot with immigration
- Continuous state branching processes with immigration
  and a process appearing as a dual to the first above:
- Kingman and Λ-coalescents.

The first two are related via a rescaling theorem and the first and the third via duality.

Coding

There is a specific coding of their evolving states:

- States: Bridges
- Evolving States: Flows of Bridges

This coding is reminiscent of the ordering in look-down constructions. This allows to obtain representations on one probability space which solve particular stochastic equations which induce weak solutions to some specific martingale problem. Also some genealogical information is nicely represented.

Cannings model

Consider population of fixed size N identified with $\{1, \ldots, N\}$.

$\xi_{i,n} = \#$children of individual $i \in \{1, \ldots, N\}$ in generation $n \in \mathbb{Z}$

Cannings model: For each n, the family $(\xi_{i,n}, i \in \{1, \ldots, N\})$ is exchangeable $\mathbb{N}_0$-valued subject to

$$\sum_{i=1}^{N} \xi_{i,n} = N.$$

Consistent labelling and discrete bridges

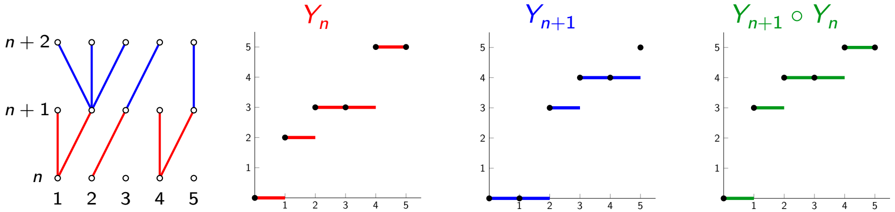
Consistent labelling: For $1 \leq i \leq j \leq N$ the offspring of individual i have smaller labels than that of j.

For each n, $Y_n : \{0, \dots, N\} \to \{0, \dots, N\}$ defined by

$$Y_n(k) := \sum_{i=1}^{k} \xi_{i,n} = \#\text{offspring of first } k \text{ individuals of } n\text{-th generation.}$$

Y_n is non-decreasing, with $Y_n(0) = 0$, $Y_n(N) = N$, i.e. a discrete bridge (Bertoin, Le Gall, 2003).

Cannings model and flows of bridges



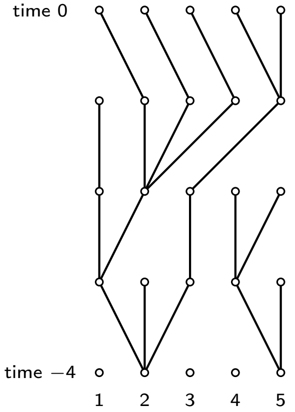
Consistent labelling means that lineages do not “cross-over”.

Interpretation: Increments of Y_n are $\xi_{i,n}$, i.e. offspring numbers of individuals in generation n. In general: Increments of $Y_{n+k-1} \circ \cdots \circ Y_n$ are offspring numbers of individuals in generation n in generation $n+k$.

Note: Normalising leads to bridges from $[0,1]$ to $[0,1]$, i.e. to *distribution functions* with exchangeable jump sizes at deterministic jump times.

Genealogies via partition processes

For $m \in \mathbb{N}_0$: $i, j \in \{1, \ldots, N\}$ are in the same partition element at time $-m$ if the time to most recent common ancestor of individuals i and j from generation 0 is at most m.



$\{\{1\},\{2\},\{3\},\{4\},\{5\}\}$

$\{\{1\},\{2\},\{3\},\{4,5\}\}$

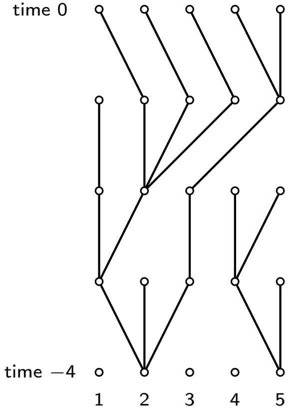
$\{\{1,2,3\},\{4,5\}\}$

$\{\{1,2,3\},\{4,5\}\}$

$\{\{1,2,3,4,5\}\}$

Genealogies via flows of bridges

For $m < n$ set $B_{m,n} = Y_{-m+1} \circ \cdots \circ Y_{-n}$, $\quad B_{m,m} = \mathrm{Id}$.
Increments of $B_{0,n}$ partition the set of individuals of the present (time 0) population according to their ancestral relation n generations ago.



$$B_{0,0} = (0, 1, 2, 3, 4, 5)$$

$$B_{0,1} = (0, 0, 1, 2, 3, 5)$$

$$B_{0,2} = (0, 0, 3, 5, 5, 5)$$

$$B_{0,3} = (0, 3, 3, 5, 5, 5)$$

$$B_{0,4} = (0, 0, 5, 5, 5, 5)$$

Properties of flows of (discrete) bridges

(i) Cocycle property: For every $\ell \leq m \leq n$

$$B_{\ell,m} \circ B_{m,n} = B_{\ell,n}$$

(ii) Law of $B_{m,n}$ depends only on $n - m$ and for all $n_1 \leq n_2 \leq \cdots \leq n_k$

$$B_{n_1,n_2}, \ldots, B_{n_{k-1},n_k}$$

are independent.

Branching models and bridges

Similar objects can be defined for discrete time branching models: bridges have to be replaced by discrete subordinators, i.e. non-decreasing random walks that map $\mathbb{N}_0$ to $\mathbb{N}_0$.

Genealogies of large populations

Under suitable conditions for large population sizes the genealogies of Cannings models can be approximated by Kingman coalescent and more generally by Λ-coalescents and Ξ-coalescents.

Möhle and Sagitov [MS01] give full classification of limits.

Flows of (normalized) bridges are in one-to-one correspondence to such coalescents and to mass coalescents [BLG03].

Mass coalescent in discrete case: Take instead of sizes of partition elements of $\{1, \ldots, N\}$ their relative sizes normalised by N.

Bridges

Definition 1 (Bertoin, Le Gall, 2003).

A bridge is a $[0,1]$-valued random process $B = (B(r), r \in [0,1])$ with

(i) $B(0) = 0$, $B(1) = 1$, paths nondecreasing and right-continuous.

(ii) B has exchangeable increments.

Theorem 2 (General characterization of bridges. Kallenberg, 1973).

B is a bridge iff there is sequence of rv's $\beta = (\beta^i, i \in \mathbb{N})$ with $\beta^1 \geq \beta^2 \geq \cdots \geq 0$ and $\sum_{i=1}^{\infty} \beta^i \leq 1$ and an i.i.d. sequence $U_1, U_2, \ldots$ of uniform on $[0,1]$ rv's (indep. of β) so that

$$B(r) = \Big(1 - \sum_{i=1}^{\infty} \beta^i\Big) r + \sum_{i=1}^{\infty} \beta^i \mathbb{1}_{\{U^i \leq r\}}, \quad r \in [0,1]$$

Iterpretation: U^i and β^i are jump times resp. jump sizes. Between the jumps B increases linearly with drift $(1 - \sum_{i=1}^{\infty} \beta^i)$.

Flows of bridges

Definition 3.

A flow of bridges is a family $(B_{s,t} : -\infty < s \leq t < \infty)$ of bridges so that:

(i) For every $s < t < u$, $B_{s,u} = B_{s,t} \circ B_{t,u}$ a.s.

(ii) The law of $B_{s,t}$ only depends on $t - s$, and for $s_1 < s_2 < \cdots < s_n$, the bridges $B_{s_1,s_2}, B_{s_2,s_3}, \ldots, B_{s_{n-1},s_n}$ are independent.

(iii) $B_{0,0} = \mathrm{Id}$ and $B_{0,t} \to \mathrm{Id}$ in probability as $t \downarrow 0$, in the sense of Skorohod's topology.

Large population limit of Cannings models

Enrich the neutral Cannings model with types in set $E = [0,1]$.

Define $X_t(i) :=$ type of individual i in generation t, and the empirical measure

$$Z_t^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t(i)}.$$

For time scale choose the probability of picking two siblings at random

$$c_N := \frac{E[\xi_{1,N}(\xi_{1,N}-1)]}{N-1} = E\left[\sum_{i=1}^{N} \frac{\xi_{i,N}}{N}\frac{(\xi_{i,N}-1)}{(N-1)}\right].$$

Theorem 4.

Assume $Z_0^N \Rightarrow \mu \in \mathcal{M}_1(E)$ and "further conditions", then $(Z^N_{\lfloor t/c_N \rfloor})$ converges weakly to Fleming-Viot process (Z_t).

"Further conditions": Example

Assume

$$c_N \to 0, \quad \text{and} \quad \frac{E[\xi_{1,N}(\xi_{1,N}-1)(\xi_{1,N}-2)]}{N^2 c_N} \to 0, \quad \text{as } N \to \infty.$$

Then (Z_t) is Fleming-Viot with genealogy given by Kingman's coalescent.

Kingman's *n*-coalescent is a process $(\pi^n_t)_{t\geq 0}$ on partitions of $\{1,\ldots,n\}$ starting with

$$\pi^n_0 = \{\{1\},\ldots,\{n\}\}$$

and when there are k partition elements are present, then each pair coalesces at rate 1, i.e. the total coalescence rate is $\binom{k}{2}$.

Kingman's coalescent is the extension to partitions of $\mathbb{N}$.

Bridges and Kingman mass coalescent

If $\pi = (A_1, A_2, \ldots,)$ is an exchangeable partition of $\mathbb{N}$ define the asymptotic frequency of a block A in the partition by

$$f_A = \lim_{n\to\infty} \frac{|A \cap \{1, \ldots, n\}|}{n} \qquad \text{(exists by de Finetti!).}$$

Consider Kingman's coalescent $(\pi_t)_{t\geq 0}$ starting from partition of $\mathbb{N}$ into singletons. Conditioned on $|\pi_t| = k$ for some $t > 0$, let

$\beta_t^1, \ldots, \beta_t^k$ be the asymptotic frequencies of blocks of π_t,

$U^1, \ldots, U^k$ i.i.d. uniform on $[0, 1]$.

The partition π_t can be represented by the bridge

$$\sum_{i=1}^{k} \beta_t^i \mathbb{1}_{\{U^i \leq r\}}$$

Recall: $\beta_t = (\beta_t^1, \ldots, \beta_t^k)$ is unif. distr. on $\{x \in [0, 1]^k : \sum_{i=1}^k x_i = 1\}$.

Λ-coalescent

$\mathcal{P} :=$ set of partitions of $\mathbb{N}$, Λ a finite measure on $[0,1]$. We assume $\Lambda(\{0\}) = 0$.

Λ-coalescent is a $\mathcal{P}$-valued process $(\Pi_t)_{t\geq 0}$. Its restriction to $\{1,\ldots,n\}$, the n-Λ-coalescent, is a process on partitions of $\{1,\ldots,n\}$.

Jump rates: When there are b blocks ($2 \leq b \leq n$) then for any k ($2 \leq k \leq b$) blocks the rate of merging into one block is given by

$$\lambda_{b,k} = \int_0^1 x^k (1-x)^{b-k} \frac{1}{x^2} \Lambda(dx).$$

Interpretation: Setting $\tilde{\Lambda}(dx) = x^{-2}\Lambda(dx)$ mark at a $\tilde{\Lambda}$-P.P.P. the partitions with probability x and coalesce the marked ones.

This process is dual to the Λ-Fleming-Viot.

Λ-Fleming-Viot particle approximation

Define an N-individuals continuous time Λ-Cannings model:

Take a $\tilde{\Lambda}$-Poisson point process on $[0,1] \times [0,\infty)$ with intensity $d\tilde{\Lambda} \otimes dt$.

Transitions:

- At a point (x,t) mark all N individuals independently
  - with 1 with probability x;
  - with 0 with probability $1-x$.
- Choose among the 1-individuals one individual at random.
- All 1's are killed and replaced by the descendants of the chosen one.

The expected number of marked 1 pairs during time $[0,T]$ is finite:

$$T \int \frac{\Lambda(dr)}{r^2} \frac{N(N-1)}{2} r^2 = T\Lambda([0,1]) \frac{N(N-1)}{2}, \qquad (1)$$

hence only finitely many jumps occur.

Some basic ideas

Next we present some details on the ideas coding the evolution of Λ-Fleming-Viot via flow of bridges:

Generalized Λ-Fleming-Viot processes, which model the evolution of a continuous population with fixed mass 1, have appeared in articles by Donnelly and Kurtz [DK99a], [DK99b], and were studied later in [BLG03], [BLG05],...

Fleming-Viot processes as flows of bridges

It is convenient to view a generalized Fleming-Viot process as a stochastic flow $(F_t, t \geq 0)$ on $[0,1]$, such that for each $t \geq 0, F_t : [0,1] \to [0,1]$ is a (random) right-continuous non-decreasing map with $F_t(0) = 0$ and $F_t(1) = 1$.

We should think of the unit interval as a population, and then of F_t as the distribution function of a (random) probability measure $dF_t(x)$ on $[0,1]$.

The transitions of the flow are Markovian, and more precisely, for every $s, t \geq 0$, we have

$$F_{t+s} = \tilde{F}_s \circ F_t, \tag{2}$$

where $\tilde{F}_s$ is a copy of F_s independent of $(F_r, 0 \leq r \leq t)$.

(Recall the cocycle and indep. increments properties of discrete bridges!)

Partitioning the population via bridges

For every $0 \leq r_1 < r_2 \leq 1$, the interval $(F_t(r_1), F_t(r_2)]$ represents the sub-population at time t which consists of descendants of the sub-population $(r_1, r_2]$ at the initial time.

This means the two-point motions $(F_t(r_1), F_t(r_2))$ play a role for the genealogy.

For example we can ask for the eventual ancestor of the whole population.

Primitive Eve

Define the primitive Eve, e as follows.

$$e := \inf\{y \in [0,1] : \lim_{t\to\infty} F_t(y) = 1\} = \sup\{y \in [0,1] : \lim_{t\to\infty} F_t(y) = 0\} \tag{3}$$

- The process

$$\varrho_t(\{e\}) = F_t(e) - F_t(e-), \quad t \geq 0 \tag{4}$$

is a Markov process on $[0,1]$ with

$$\varrho_t(\{e\}) = F_t(e) - F_t(e-) \xrightarrow{t\to\infty} 1. \tag{5}$$

Interpretation: $\varrho_t(\{y\}) = F_t(y) - F_t(y-)$ corresponds to the mass of the progeny at time t of the individual y from initial population.

Thus primitive Eve is the common ancestor at the initial generation of most of the individuals at time t when t is sufficiently large.

Construction of flow: "simple case"

A simple construction can be given in the case $\int_0^1 x^{-2}\Lambda(dx) < \infty$. Existence and uniqueness of flows in ∞-case via stochastic equations will be explained in the next section.

Let $((T_i, U_i, \xi_i), i \in \mathbb{N})$ denote the sequence of atoms of a Poisson random measure on $[0, \infty) \times [0, 1] \times [0, 1]$ with intensity

$$dt \otimes du \otimes \tilde{\Lambda}(dx), \tag{6}$$

ranked in the increasing order of the first coordinate.

Process $(F_t, t \geq 0)$ starts from $F_0 = \mathrm{Id}$, remains constant on $[T_{i-1}, T_i)$ (with the usual convention $T_0 = 0$), and for every $i \in \mathbb{N}$

$$F_{T_i} = \Delta_i \circ F_{T_{i-1}} \tag{7}$$

where

$$\Delta_i(r) = \xi_i \mathbb{1}_{\{U_i \leq r\}} + r(1 - \xi_i), \quad r \in [0, 1]. \tag{8}$$

At each time T_i, an individual in the population at time T_{i-1} is picked uniformly at random and gives birth to a sub-population of size ξ_i. Simultaneously, the rest of the population shrinks by factor $1 - \xi_i$.

Dual picture

In [BLG03] it is shown that Λ-coalescent started from the partition of $\mathbb{N}$ into singletons is in one-to-one correspondence to the flows of bridges $(B_{s,t}, -\infty < s \leq t < \infty)$.

Consequence of the preceding construction: the mass frequency coalescent at time t has the same distribution as the ranked sequence of jump sizes of F_t.

Define the dual flow $(\hat{B}_{s,t}, -\infty < s \leq t < \infty)$ by

$$\hat{B}_{s,t} = B_{-t,-s}.$$

Then we have the duality relation:

$$\varrho([0, y]) = F_t(y) = \hat{B}_{0,t}(y).$$

A similar picture of flow of bridges can be formulated for branching processes.

These processes appear in asymptotic description of the left corner of the bridge.

Branching and the asymptotics of Λ-Fleming-Viot

Goal:

One wants to study the behaviour of the low ranks in Λ-Fleming-Viot.

Assumptions:

For each integer $k \geq 1$ let $\Lambda_k(dx)$ be a finite measure on $[0,1]$ with $\Lambda_k(\{0\}) = 0$ and let

- $\{X_k(t,v) : t \geq 0, v \in [0,1]\}$ be defined by (12) from a Poisson random measure $\{M_k(ds,dz,du)\}$ on $(0,\infty) \times (0,1]^2$ with intensity $ds\, z^{-2}\Lambda_k(dz)\, du$.
- Suppose that $z^{-2}(z \wedge z^2)\Lambda_k(k^{-1}dz)$ converges weakly as $k \to \infty$ to a finite measure on $(0,\infty)$ denoted by $z^{-2}(z \wedge z^2)\Lambda_\infty(dz)$.

The rescaled p-point motion

$$\{(kX_k(kt, r_1/k), \ldots, kX_k(kt, r_p/k)) : t \geq 0\} \tag{9}$$

converges in distribution to that of the weak solution flow of the stochastic equation (branching)

$$Y_t(v) = v + \int_0^t \int_0^\infty \int_0^\infty x \mathbb{1}_{\{u \leq Y_{s-}(v)\}} \tilde{N}(ds, dx, du), \tag{10}$$

where $\tilde{N}(ds, dx, du)$ is a compensated Poisson random measure on $[0, \infty) \times (0, \infty)^2$ with intensity $ds\, z^{-2} \Lambda_\infty(dz)\, du$.

In order to get such asymptotics, [DL12] apply the machinery of limit theorems for semimartingales [JS03] to the so-called *generalized Fleming-Viot processes*, which were shown in [BLG03] to be duals to the *coalescents with multiple collisions*.

The basic stochastic equations

A key point is to write now out stochastic equations which generate the flow of bridges for the specified models. We execute this now following [DL12] starting with branching, which is technically the simplest case, because of the branching property.

Branching

Suppose that

- $\sigma \geq 0$ and b are constants,
- $v \mapsto \gamma(v)$ is a nonnegative and nondecreasing continuous function on $[0, \infty)$ and
- $(z \wedge z^2)m(dz)$ is a finite measures on $(0, \infty)$.
- Let $\{W(ds, du)\}$ be a white noise on $(0, \infty)^2$ based on the Lebesgue measure $ds\ du$.
- Let $\{N(ds, dz, du)\}$ be a Poisson random measure on $(0, \infty)^3$ with intensity $ds\ m(dz)du$.
- Let $\{\tilde{N}(ds, dz, du)\}$ be the compensated measure of $\{N(ds, dz, du)\}$.

Stochastic equation for Branching

We shall see that for any $v \geq 0$ there is a pathwise unique nonnegative solution of the stochastic equation

$$
\begin{aligned}
Y_t(v) = v + \sigma \int_0^t \int_0^{Y_{s-}(v)} W(ds, du) + \int_0^t [\gamma(v) - bY_{s-}(v)]ds \\
+ \int_0^t \int_0^\infty \int_0^{Y_{s-}(v)} z\tilde{N}(ds, dz, du).
\end{aligned} \tag{11}
$$

It is not hard to show each solution $Y(v) = Y_t(v) : t \geq 0$ is a continuous *state branching process with immigration* (CBI-process).

Theorem 5.

There is a unique nonnegative strong solution to (11).

Then it is natural to call the two-parameter process $\{Y_t(v) : t \geq 0, v \geq 0\}$ a *flow of CBI-processes*. We prove that the flow has a version with the following properties:

(i) for each $v \geq 0, t \mapsto Y_t(v)$ is a càdlàg process on $[0, \infty)$ and solves (11).

(ii) for each $t \geq 0, v \mapsto Y_t(v)$ is a nonnegative and nondecreasing càdlàg process on $[0, \infty)$.

Λ-Fleming-Viot

Suppose that

- $\sigma \geq 0$ and $b \geq 0$ are constants
- $v \mapsto \gamma(v)$ is a nondecreasing continuous on $[0,1]$ with $0 \leq \gamma(v) \leq 1$
- $z^2 \nu(dz)$ is a finite measure on $(0,1]$
- $\{B(ds,du)\}$ is white noise on $(0,\infty) \times (0,1]$ based on $ds\,du$,
- $\{M(ds,dz,du)\}$ is a Poisson random measure on $(0,\infty) \times (0,1]^2$ with intensity $ds\,\nu(dz)du$.

We show that for any $v \in [0,1]$ there is a pathwise unique solution $X(v) = \{X_t(v) : t \geq 0\}$ to the equation

$$
\begin{aligned}
X_t(v) = v &+ \sigma \int_0^t \int_0^1 \left[1_{\{u \leq X_{s-}(v)\}} - X_{s-}(v) \right] B(ds,du) \\
&+ b \int_0^t [\gamma(v) - X_{s-}(v)]\, ds \qquad (12) \\
&+ \int_0^t \int_0^1 \int_0^1 z \left[1_{\{u \leq X_{s-}(v)\}} - X_{s-}(v) \right] M(ds,dz,du).
\end{aligned}
$$

We show that there is a version of the random field $\{X_t(v) : t \geq 0, 0 \leq v \leq 1\}$ with the following properties:

(i) for each $v \in [0, 1], t \mapsto X_t(v)$ is càdlàg on $[0, \infty)$ and solves (12);

(ii) for each $t \geq 0, v \mapsto X_t(v)$ is nondecreasing and càdlàg on $[0, 1]$ with $X_t(0) \geq 0$ and $X_t(1) \leq 1$.

More on generalized Fleming-Viot

Theorem 6.

The path-valued Markov process $\{X(v) : 0 \leq v \leq 1\}$ *has a* ϱ*-càdlàg modification. Consequently, there is a version of the solution flow* $\{X_t(v) : t \geq 0, 0 \leq v \leq 1\}$ *of* (12) *with the following properties:*

(i) *for each* $v \in [0, 1]$, $t \mapsto X_t(v)$ *is càdlàg on* $[0, \infty)$ *and solves* (12)*;*

(ii) *for each* $t \geq 0$, $v \mapsto X_t(v)$ *is nondecreasing and càdlàg on* $[0, 1]$ *with* $X_t(0) \geq 0$ *and* $X_t(1) \leq 1$*.*

We call the solution flow $\{X_t(v) : t \geq 0, v \in [0, 1]\}$ of (12) specified above a *generalized Fleming-Viot flow*. The law of the flow is determined by the parameters (σ, b, γ, ν).

The p-point motion

For any $0 \leq r_1 < \cdots < r_p \leq 1$ the p-point motion

$$\{(B_{-t,0}(r_1), \ldots, B_{-t,0}(r_p)) : t \geq 0\} \tag{13}$$

is equivalent to

$$\{(X_t(r_1), \ldots, X_t(r_p)) : t \geq 0\}. \tag{14}$$

Therefore, the solutions of (12) give a realization of the flow of bridges associated with the Λ-coalescent process.

In Kingman case they showed that the p-point motion $\{(B_{-t,0}(r_1), \ldots, B_{-t,0}(r_p)) : t \geq 0\}$ is a diffusion process in

$$D_p := \{x = (x_1, \ldots, x_p) \in \mathbb{R}^p : 0 \leq x_1 \leq \cdots \leq x_p \leq 1\} \tag{15}$$

with generator A_0 defined by

$$A_0 f(x) = \frac{1}{2} \sum_{i,j=1}^{p} x_{i \wedge j}(1 - x_{i \vee j}) \frac{\partial^2 f}{\partial x_i \partial x_j}(x). \tag{16}$$

The p-point motion of inverses

Given a Λ-coalescent flow $\{B_{s,t} : -\infty < s \le t < \infty\}$, we define the *flow of inverses* by

$$B_{s,t}^{-1}(v) = \inf\{u \in [0,1] : B_{s,t}(u) > v\}, \quad v \in [0,1), \tag{17}$$

and $B_{s,t}^{-1}(1) = B_{s,t}^{-1}(1-)$.

Kingman case: p-point motion $\{(B_{0,t}^{-1}(r_1), \dots, B_{0,t}^{-1}(r_p)) : t \ge 0\}$ is a diffusion process in D_p with generator A_1 given by

$$A_1 f(x) = A_0 f(x) + \sum_{i=1}^{p} \left(\frac{1}{2} - x_i\right) \frac{\partial f}{\partial x_i}(x), \tag{18}$$

More on CBI-processes

Suppose that $\sigma \geq 0$ and b are constants, and $(u \wedge u^2)m(du)$ is a finite measure on $(0, \infty)$. Let φ be a function given by

$$\phi(z) = bz + \frac{1}{2}\sigma^2 z^2 + \int_0^\infty (e^{-zu} - 1 + zu)m(du), \quad z \geq 0. \tag{19}$$

A Markov process with state space $\mathbb{R}_+ := [0, \infty)$ is called a *CB-process* with branching mechanism ϕ if it has transition semigroup $(p_t)_{t \geq 0}$ given by

$$\int_{\mathbb{R}_+} e^{-\lambda y} p_t(x, dy) = e^{-x v_t(\lambda)}, \quad \lambda \geq 0, \tag{20}$$

where $(t, \lambda) \mapsto v_t(\lambda)$ is the unique locally bounded nonnegative solution of

$$\frac{d}{dt} v_t(\lambda) = -\phi(v_t(\lambda)), \quad v_0(\lambda) = \lambda, \quad t \geq 0.$$

Given any $\beta \geq 0$ we can also define a transition semigroup $(q_t)_{t\geq 0}$ on $\mathbb{R}_+$ by

$$\int_{\mathbb{R}_+} e^{-\lambda y} q_t(x, dy) = \exp\Big\{ - x v_t(\lambda) - \int_0^t \beta v_s(\lambda) ds \Big\}. \qquad (21)$$

A nonnegative real-valued Markov process with transition semigroup $(q_t)_{t\geq 0}$ is called a *CBI-process* with branching mechanism ϕ and immigration rate β. It is easy to see that both $(p_t)_{t\geq 0}$ and $(q_t)_{t\geq 0}$ are Feller semigroups.

Theorem 7.

There is a unique nonnegative strong solution of the stochastic equation

$$Y_t = Y_0 + \sigma \int_0^t \int_0^{Y_{s-}} W(ds, du) + \int_0^t (\beta - bY_{s-}) ds + \int_0^t \int_0^\infty \int_0^{Y_{s-}} z\tilde{N}(ds, dz, du).$$

Moreover, the solution $\{Y_t : t \geq 0\}$ is a CBI-process with branching mechanism ϕ and immigration rate β.

Using Ito's formula one can see that $\{Y_t(v) : t \geq 0\}$ solves the martingale problem associated with the generator L defined by

$$Lf(x) = \frac{1}{2}\sigma^2 x f''(x) + (\beta - bx)f'(x) + x\int_0^\infty D_z f(x) m(dz). \qquad (22)$$

Then it is a CBI-process with branching mechanism ϕ and immigration rate β.

Here

$$D_Z =: \Delta_z f(x) - f'(x)z \qquad (23)$$

$$\Delta_z f(x) = f(x+z) - f(x) \qquad (24)$$

Key results of solution theory of Dawson and Li

Setup

Let $\{W(ds, du)\}$ be a white noise on $(0, \infty) \times E$ with intensity $ds\, \pi(dz)$. Let $\{N_0(ds, du)\}$ and $\{N_1(ds, du)\}$ be Poisson random measures on $(0, \infty) \times U_0$ and $(0, \infty) \times U_1$ with intensities $ds\, \mu_0(du)$ and $ds\, \mu_1(du)$, respectively. The s-component represents time.

Suppose that $\{W(ds, du)\}, \{N_0(ds, du)\}$ and $\{N_1(ds, du)\}$ are defined on some complete probability space $(\Omega, \mathcal{F}, P)$ and are independent of each other.

Let $\{\tilde{N}_0(ds, du)\}$ denote the compensated measure of $\{N_0(ds, du)\}$.

Gaussian white noise

Let $(E \times \mathbb{R}_+, \mathcal{E} \otimes \mathcal{B}(\mathbb{R}_+), \nu(dx)dt)$ be a σ-finite measure space.
A *Gaussian white noise* based on $\nu(dx)\, dt$ is a random set function, W, on $\mathcal{E} \otimes \mathcal{B}(\mathbb{R}_+)$ such that

(a) $W(B \times [a, b])$ is a Gaussian random variable with mean zero and variance $\nu(B) \cdot \mid b - a \mid$.

(b) if $(A \times [a_1, a_2]) \cap (B \times [b_1, b_2]) = \varnothing$, then

$$W(A \times [a_1, a_2] \cup (B \times [b_1, b_2])) = W(A \times [a_1, a_2]) + W(B \times [b_1, b_2])$$

and $W(A \times [a_1, a_2])$ and $W(B \times [b_1, b_2])$ are independent.

A nonnegative càdlàg process $\{x(t) : t \geq 0\}$ is called a *solution* of

$$
\begin{aligned}
x(t) = & x(0) + \int_0^t \int_E \sigma(x(s-), u) W(ds, du) \\
& + \int_0^t b(x(s-)) ds + \int_0^t \int_{U_0} g_0(x(s-), u) \tilde{N}_0(ds, du) \\
& + \int_0^t \int_{U_1} g_1(x(s-), u) N_1(ds, du),
\end{aligned} \tag{25}
$$

if it satisfies the stochastic equation almost surely for every $t \geq 0$. We say $\{x(t) : t \geq 0\}$ is a *strong solution* if, in addition, it is adapted to the augmented natural filtration generated by $\{W(ds, du)\}$, $\{N_0(ds, du)\}$ and $\{N_1(ds, du)\}$.

Theorem 8.

Suppose that (σ, b, g_0, g_1) *are admissible parameters satisfying conditions (a)–(d). Then the pathwise uniqueness of solutions holds for* (25).

Theorem 9.

Suppose that (σ, b, g_0, g_1) *are admissible parameters satisfying conditions (a), (c), (e). Then there is a unique strong solution to* (25).

Let E, U_0 and U_1 be Polish spaces and suppose that $\pi(dz)$, $\mu_0(du)$ and $\mu_1(du)$ are σ-finite Borel measures on E, U_0 and U_1, respectively.

We say the parameters (σ, b, g_0, g_1) are *admissible* if:

- $x \mapsto b(x)$ is a continuous function on $\mathbb{R}_+$ satisfying $b(0) \geq 0$;
- $(x, u) \mapsto \sigma(x, u)$ is a Borel function on $\mathbb{R}_+ \times E$ satisfying $\sigma(0, u) = 0$ for $u \in E$;
- $(x, u) \mapsto g_0(x, u)$ is a Borel function on $\mathbb{R}_+ \times U_0$ satisfying $g_0(0, u) = 0$ and $g_0(x, u) + x \geq 0$ for $x > 0$ and $u \in U_0$;
- $(x, u) \mapsto g_1(x, u)$ is a Borel function on $\mathbb{R}_+ \times U_1$ satisfying $g_1(x, u) + x \geq 0$ for $x \geq 0$ and $u \in U_1$.

Let us formulate the following conditions:

(a) There is a constant $K \geq 0$ so that

$$b(x) + \int_{U_1} |g_1(x,u)| \mu_1(du) \leq K(1+x) \tag{26}$$

for every $x \geq 0$.

(b) There is a non decreasing function $x \mapsto L(x)$ on $\mathbb{R}_+$ and a Borel function $(x,u) \mapsto \bar{g}_0(x,u)$ on $\mathbb{R}_+ \times U_0$ so that for every $x \geq 0$:

$$\sup_{0 \leq y \leq x} |g_0(y,u)| \leq \bar{g}_0(x,u)$$

and

$$\int_E \sigma(x,u)^2 \pi(du) + \int_{U_0} [\bar{g}_0(x,u) \wedge \bar{g}_0(x,u)^2] \mu_0(du) \leq L(x). \tag{27}$$

(c) For each $m \geq 1$ there is a non decreasing concave function $z \mapsto r_m(z)$ on $\mathbb{R}_+$ so that $\int_{0+} r_m(z)^{-1} dz = \infty$ and

$$|b_1(x) - b_1(y)| + \int_{U_1} |g_1(x,u) - g_1(y,u)| \mu_1(du) \leq r_m(|x-y|) \quad (28)$$

for every $0 \leq x, y \leq m$.

(d) For each $m \geq 1$ there is a nonnegative non decreasing function $z \mapsto \varrho_m(z)$ on $\mathbb{R}_+$ so that $\int_{0+} \varrho_m(z)^{-2} dz = \infty$,

$$\int_E |\sigma(x,u) - \sigma(y,u)|^2 \pi(du) \leq \varrho_m(|x-y|)^2 \quad (29)$$

and, setting $l_0(x,y,u) := g_0(x,u) - g_0(y,u)$,

$$\int_{U_0} \mu_0(du) \int_0^1 \frac{l_0(x,y,u)^2 (1-t) \mathbb{1}_{\{|l_0(x,y,u)| \leq n\}}}{\varrho_m(|(x-y) + t l_0(x,y,u)|)^2} dt \leq c(m,n) \quad (30)$$

for every $n \geq 1$ and $0 \leq x, y \leq m$. Here $c(m,n) \geq 0$ is a constant.

(e) For each $u \in U_0$ the function $x \to g_0(x,u)$ is non decreasing, and for each $m \geq 1$ there is a nonnegative and non decreasing function $z \mapsto \varrho_m(z)$ on $\mathbb{R}_+$ so that $\int_{0+} \varrho_m(z)^{-2} dz = \infty$ and

$$\int_E |\sigma(x,u) - \sigma(y,u)|^2 \pi(du) + \int_{U_0} |l_0(x,y,u)| \wedge |l_0(x,y,u)|^2 \mu_0(du) \leq \varrho_m(|x-y|)^2 \quad (31)$$

for all $0 \leq x, y \leq m$, where $l_0(x,y,u) = g_0(x,u) - g_0(y,u)$.

The key technical tool in comparison

We say the *comparison property* of solutions holds for a stochastic equation if for any two solutions $\{x_1(t) : t \geq 0\}$ and $\{x_2(t) : t \geq 0\}$ satisfying $x_1(0) \leq x_2(0)$ we have $P\{x_1(t) \leq x_2(t) \text{ for all } t \geq 0\} = 1$.

Theorem 10.

Let (σ, b, g_0, g_1) be admissible parameters satisfying conditions (a)–(d). In addition, assume that for every $u \in U_1$ the function $x \to x + g_1(x, u)$ is non decreasing. Then the comparison property holds for the solutions of (25).

Theorem 11.

Let (σ, b', g_0, g_1') and $(\sigma, b'', g_0, g_1'')$ be two sets of admissible parameters satisfying conditions (a)–(d). In addition, assume that:

(i) *for every $u \in U_1$, $x \to x + g_1'(x,u)$ or $x \to x + g_1''(x,u)$ is non decreasing;*

(ii) *$b'(x) \leq b''(x)$ and $g_1'(x,u) \leq g_1''(x,u)$ for every $x \geq 0$ and $u \in U_1$.*

Suppose that $\{x'(t) : t \geq 0\}$ is a solution of (25) with $(b, g_1) = (b', g_1')$, and $\{x''(t) : t \geq 0\}$ is a solution of the equation with $(b, g_1) = (b'', g_1'')$. If $x'(0) \leq x''(0)$, then $P\{x'(t) \leq x''(t), \text{ for all } t \geq 0\} = 1$.

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