The Algebraic Approach to Duality: Blackboard talk (145 minutes)

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Outline

Giardinà, Redig, Carinci, Giberti, Kurchan, Vafayi: JSP 2009, SPA 2015.

- Dualities as intertwiners between representations of Lie algebras.
  - Markov duality
  - Lie algebras
  - Product spaces
- (Duality based on symmetry).

  Lloyd, Sudbury: AOP 1995, AOP 1997, JTP 2000.
- q-duality
- thinning relations

Markov duality

Ω fin. set, $\mathbb{R}^\Omega := \{f : \Omega \to \mathbb{R}\}$

Markov generator $L : \mathbb{R}^\Omega \to \mathbb{R}^\Omega$ charact. by matrix

$$Lf(x) = \sum_y L(x,y)f(y)$$

$$L(x,y) \geq 0 \quad (x \neq y) \qquad \sum_y L(x,y) = 0.$$

Semigroup $P_t = e^{tL} = \sum_{n=0}^{\infty} \frac{1}{n!} t^n L^n.$

$P_t(x,y)$ transit. probab.

Let $L, \hat{L}$ Markov generators on $\Omega, \hat{\Omega}$
$D : \Omega \times \hat{\Omega} \to \mathbb{R}$ function
<u>Lemma</u> The following are equivalent:

(i) $LD(\,\cdot\,,y)(x) = \hat{L}D(x,\,\cdot\,)(y)$ $\forall x, y$

(ii) $LD = D\hat{L}^\dagger$ $\qquad A^\dagger(x,y) := A(y,x)$ adjoint

(iii) $P_t D = D\hat{P}_t^\dagger$ $\forall t$

(iv) $\mathbb{E}^x\big[D(X_t,y)\big] = \mathbb{E}^y\big[D(x,Y_t)\big]$ $\forall x,y,t$
X_t gener. L, Y_t gener. $\hat{L}$.

<u>Proposition 1</u> $A_i D = D B_i^\dagger$ $(i = 1,2) \Rightarrow$

- $(r_1 A_1 + r_2 A_2)D = D(r_1 B_1 + r_2 B_2)^\dagger$
- $(A_1 A_2)D = D(B_2 B_1)^\dagger$.

<u>Corollary</u> $L, \hat{L}$ Markov generators on $\Omega, \hat{\Omega}$

$$L = r_\emptyset I + r_1 A_1 + r_{23} A_2 A_3 + r_{113} A_1^2 A_3,$$
$$\hat{L} = r_\emptyset I + r_1 B_1 + r_{23} B_3 B_2 + r_{113} B_3 B_1^2,$$

Then

$$A_i D = D B_i^\dagger \quad \forall i \quad \Rightarrow \quad LD = D\hat{L}^\dagger.$$

<u>Remark</u>

	pathwise approach	algebraic approach
building blocks	maps $m, \hat{m}$	operators A_i, B_i
assumption	$D\big(m(x),y\big) = D\big(x,\hat{m}(y)\big)$	$A_i D(\,\cdot\,,y)(x) = B_i D(x,\,\cdot\,)(y)$
result	$D\big(\Phi_{s,t}(x),y\big) = D\big(x,\hat{\Phi}_{-t,-s}(y)\big)$ a.s. dual stochastic flows	$P_t D(\,\cdot\,,y)(x) = \hat{P}_t D(x,\,\cdot\,)(x)$ dual semigroups

Example

Wright-Fisher diffusion with selection $s \geq 0$

$$\begin{aligned} Lf(x) &= x(1-x)\tfrac{\partial^2}{\partial x^2} + sx(1-x)\tfrac{\partial}{\partial x} f(x) \\ &= A^-(1-A^-)A^+(s+A^+)f(x) \end{aligned}$$

with

$$A^- f(x) := (1-x)f(x), \quad A^+ f(x) := \tfrac{\partial}{\partial x} f(x).$$

Observation

$$A^\pm D(\,\cdot\,, y)(x) = B^\pm D(x, \cdot,)(y) \quad \text{with} \quad D(x,n) := (1-x)^n,$$

and

$$B^- f(n) := f(n+1) \quad B^+ f(n) := -nf(n-1).$$

Consequence L dual w.r.t. D to

$$\begin{aligned} \hat{L}f(x) &= (s+B^+)B^+(1-B^-)B^- f(n) \\ &= n(n-1)\big\{f(n-1) - f(n)\big\} + sn\big\{f(n+1) - f(n)\big\}. \end{aligned}$$

!Markov for $s \geq 0$.

Note $A^\pm$ satisfy the commutation relations of the Heisenberg algebra

$$[A^-, A^+] = I, \quad [A^\pm, I] = 0.$$

Idea: take for $A_1, \ldots, A_n$ basis of representation of a Lie algebra.

Lie algebras

Lie algebra g = fin. dim. lin. space with a Lie bracket

- $(a,b) \mapsto [a,b]$ bilinear
- $[a,b] = -[b,a]$ skew symmetry
- $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$ Jacobi identity.

$a_1, \dots, a_n$ basis then Lie bracket determined by <u>commutation relations</u>

$$[a_i, a_j] = \sum_k c_{ijk} a_k.$$

<u>Representation</u> = collection of operators $A_1, \dots, A_n$ on lin. space V s.t.

$$[A_i, A_j] = \sum_k c_{ijk} A_k.$$

<u>faithful</u> iff $A_1, \dots, A_n$ lin. indep.

Given: Two representations $A_1, \dots, A_n$ on V $\quad B_1, \dots, B_n$ on W

Def. <u>Intertwiner</u> = lin. map $\Phi : W \to V$ s.t.

$$A_i \Phi = \Phi B_i \quad \forall i$$

Φ invertible $\Rightarrow$ representations <u>equivalent</u>.

Repres. V *irreducible* iff no nontriv. invariant subspaces:

$$A_i V' \subset V' \quad \forall i \Rightarrow V' = \{0\} \text{ or } V' = V$$

<u>Schur's lemma</u> For equivalent, irreducible representations, the intertwiner is unique up to a multiplicative constant.

<u>Classification of representations</u>

Lie algebra su(2) basis j^-, j^+, j^0 defined by

$$[j^0, j^\pm] = \pm j^\pm \quad [j^-, j^+] = -2j^0.$$

<u>Repres. of su(2)</u> For each $d \geq 2$ $\exists$ irred. repres. of su(2) on $\mathbb{R}^d$ and all irred. repr. with same dim. are equivalent.

Classification theory different for each Lie algebra!

<u>Proposition 2</u> Assume $A_1, \dots, A_n$ and $(B_1)^\dagger, \dots, (B_n)^\dagger$ define equivalent irreducible repres. of same Lie algebra.
$\Rightarrow \exists$ duality function D, unique up to multiplic. cst, s.t.

$$A_i D(\,\cdot\,, y)(x) = B_i D(x, \,\cdot\,)(y) \qquad \forall x, y, i.$$

<u>Remark</u> If

$$[A_i, A_j] = \sum_k c_{ijk} A_k,$$

then

$$[B_i, B_j] = -\sum_k c_{ijk} B_k,$$

commutation relations of conjugate Lie algebra.

Example

Wright-Fisher diffusion with selection $s \geq 0$

$$\begin{aligned} Lf(x) &= x(1-x)\tfrac{\partial^2}{\partial x^2} + sx(1-x)\tfrac{\partial}{\partial x} f(x) \\ &= -A^+(\sqrt{s} - A^+)A^-(\sqrt{s} - A^-)f(x) \end{aligned}$$

with

$$A^- f(x) := \sqrt{s}xf(x), \quad A^+ f(x) := \frac{-1}{\sqrt{s}}\tfrac{\partial}{\partial x} f(x).$$

A^-, A^+, I central representation of Heisenberg algebra

$$[A^-, A^+] = I, \quad [A^\pm, I] = 0.$$

Central = third element represented as I.

Observation $B^+ := A^-$, $B^- := A^+$ satisfy $[B^-, B^+] = -I$
commut. relat. of conjugate Lie algebra.
Hence $(B^-)^\dagger, (B^+)^\dagger, I^\dagger$ def. repres. of Heisenberg algebra.

Stone-von Neumann theorem says more/less:
all central representations of Heisenberg algebra equivalent.

Indeed: $\exists$ intertwiner: $D(x,y) = e^{-sxy}$ satisfies

$$A^\pm D = D(B^\pm)^\dagger$$

Consequence L dual w.r.t. duality function D to

$$\begin{aligned} \hat{L}f(x) &= -B^-(\sqrt{s} - B^-)B^+(\sqrt{s} - B^+)f(x) \\ &= -A^+(\sqrt{s} - A^+)A^-(\sqrt{s} - A^-)f(x) = Lf(x) \end{aligned}$$

self-duality.

Product spaces

Ω_1, Ω_2 finite spaces, $\mathbb{R}^\Omega := \{f : \Omega \to \mathbb{R}\}$

$\mathbb{R}^{\Omega_1} \otimes \mathbb{R}^{\Omega_1} := \mathbb{R}^{\Omega_1 \times \Omega_2}$ tensor product

$$(f \otimes g)(x,y) := f(x)g(y) \qquad (f \in \mathbb{R}^{\Omega_1},\ g \in \mathbb{R}^{\Omega_2}).$$

- $\{f_1,\dots,f_n\}$ basis of $\mathbb{R}^{\Omega_1}$, $\{g_1,\dots,g_m\}$ basis of $\mathbb{R}^{\Omega_2}$ $\Rightarrow$ $\{f_i\otimes g_j\}$ basis of $\mathbb{R}^{\Omega_1}\otimes\mathbb{R}^{\Omega_1}$.

- $\forall\ b:\mathbb{R}^{\Omega_1}\times\mathbb{R}^{\Omega_2}\to V$ bilinear $\exists!$ linear $\overline{b}:\mathbb{R}^{\Omega_1}\times\mathbb{R}^{\Omega_2}\to V$ s.t. $\overline{b}(f\otimes g)=b(f,g)$.

Abstract definition of $V_1\otimes V_2$.

$$(\overline{A}_1\otimes\overline{A}_2)(f\otimes g):=(\overline{A}_1 f)\otimes(\overline{A}_2 g)\qquad(\overline{A}_i\in\mathcal{L}(V_i),\ i=1,2).$$

Let $A_1:=\overline{A}_1\otimes I,\quad A_2:=I\otimes\overline{A}_2$. Then

$$A_1A_2=\overline{A}_1\otimes\overline{A}_2$$

with

$$\left.\begin{aligned}A_1f(x,y)&=\sum_{x'}\overline{A}_1(x,x')f(x',y)\\A_2f(x,y)&=\sum_{y'}\overline{A}_2(y,y')f(x,y')\end{aligned}\right\}\quad\text{A_1,A_2 act only on first, second coordinate.}$$

$\overline{A}_{i,1},\dots,\overline{A}_{i,n_i}$ def. repres. of Lie alg. $\mathfrak{g}_i$ on $V_i=\mathbb{R}^{\Omega_i}$ $(i=1,2)$.

$$A_{1,j}:=\overline{A}_{1,j}\otimes I\quad A_{2,j}:=I\otimes\overline{A}_{2,j}.$$

$$[A_{i,k},A_{j,m}]=0\quad(i\neq j)\quad[A_{i,k},A_{i,m}]=\sum_n c_{kmn}A_{i,n}\quad\text{commut. relat. of } g_i.$$

$\{A_{i,j}: i=1,2,\ j=1,\dots,n_i\}$ def. repres. of $\mathfrak{g}_1\oplus\mathfrak{g}_2$ direct sum of Lie algebras.

Example

S finite set
$\alpha: S\to(0,\infty)$
$q: S\times S\to[0,\infty)$ satisfies $q(i,j)=q(j,i)$ and $q(i,i)=0$

Generator of Brownian energy process (BEP)

$$L:=\tfrac12\sum_{i,j\in S}q(i,j)\big[(\alpha_j z_i-\alpha_i z_j)(\tfrac{\partial}{\partial z_j}-\tfrac{\partial}{\partial z_i})+z_iz_j(\tfrac{\partial}{\partial z_j}-\tfrac{\partial}{\partial z_i})^2\big].$$

- Diffusion $(Z_t)_{t\geq0}$ in $[0,\infty)^S$.

- $\sum_i Z_t(i)$ preserved.

- Drift towards state $z_i = \lambda\alpha_i$ ($\lambda > 0$).

Write $L = \frac{1}{2} \sum_{i,j \in S} q(i,j) \big[A_i^+ A_j^- + A_i^- A_j^+ - 2A_i^0 A_j^0 + \frac{1}{2}\alpha_i\alpha_j \big]$.

$$\begin{aligned} \text{with} \quad A_i^- f(z) &:= z_i \tfrac{\partial^2}{\partial {z_i}^2} f(z) + \alpha_i \tfrac{\partial}{\partial z_i} f(z), \\ A_i^+ f(z) &:= z_i f(z), \\ A_i^0 f(z) &:= z_i \tfrac{\partial}{\partial z_i} f(z) + \tfrac{1}{2}\alpha_i f(z). \end{aligned}$$

Commut. relat.

$$[A_i^0, A_j^\pm] = \pm\delta_{ij} A_i^\pm \quad \text{and} \quad [A_i^-, A_j^+] = 2\delta_{ij} A_i^0.$$

Repres. of direct sum of $|S|$ copies of Lie algebra $\mathrm{su}(1,1)$.

!Representations with <u>different</u> function α_i are <u>not equivalent</u>!

Dual process state space $\mathbb{N}^S$.

$$\begin{aligned} B_i^- f(x) &:= x_i f(x - \delta_i), \\ B_i^+ f(x) &:= (\alpha_i + x_i) f(x + \delta_i), \\ B_i^0 f(x) &:= (\tfrac{1}{2}\alpha_i + x_i) f(x). \end{aligned}$$

satisfy <u>conjugate</u> comm. rel.

$$[B_i^0, B_j^\pm] = \mp\delta_{ij} B_i^\pm \quad \text{and} \quad [B_i^-, B_j^+] = -2\delta_{ij} B_i^0.$$

Intertwiner of <u>product form</u>

$$\Phi = \bigotimes_{i \in S} \overline{\Phi}$$

$$\begin{aligned} A_i^\pm \Phi &= (I \otimes \cdots \otimes I \otimes \overline{A}^\pm \otimes I \otimes \cdots \otimes I)(\overline{\Phi} \otimes \cdots \otimes \overline{\Phi}) \\ &= \overline{\Phi} \otimes \cdots \otimes \overline{\Phi} \otimes \overline{A}^\pm \overline{\Phi} \otimes \overline{\Phi} \otimes \cdots \otimes \overline{\Phi} \\ &= \overline{\Phi} \otimes \cdots \otimes \overline{\Phi} \otimes \overline{\Phi} (\overline{B}^\pm)^\dagger \otimes \overline{\Phi} \otimes \cdots \otimes \overline{\Phi} = \Phi B_i^\pm. \end{aligned}$$

Duality function of <u>product form</u>

$$D(z,x) = \prod_i \overline{D}(z_i, x_i) \quad \text{with} \quad \overline{D}(z,n) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} z^n = z^n \prod_{k=0}^{n-1} (\alpha + k)$$

Dual generator

$$\begin{aligned} \hat{L} := \sum_{i,j \in S} q(i,j) \Big[& \alpha_j x_i \big\{ f\big(x - \delta_i + \delta_j\big) - f\big(x\big) \big\} \\ & + x_i x_j \big\{ f\big(x - \delta_i + \delta_j\big) - f\big(x\big) \big\} \Big]. \end{aligned}$$

Inclusion process

Lloyd Sudbury duals for interacting particle systems

S finite set. $q : S^2 \to [0,\infty)$ symmetric $q(i,j) = q(j,i)$
with $q(i,i) = 0$, $i,j \in S$.

$L = L(a,b,c,d,e)$ is Markov generator (state space $\{0,1\}^S$) with

$$
\begin{array}{llll}
11 \mapsto 00 & \text{at rate} & aq(i,j) & \text{(annihilation)},\\
01 \mapsto 11 & \text{at rate} & bq(i,j) & \text{(branching)},\\
11 \mapsto 01 & \text{at rate} & cq(i,j) & \text{(coalecence)},\\
01 \mapsto 00 & \text{at rate} & dq(i,j) & \text{(death)},\\
01 \mapsto 10 & \text{at rate} & eq(i,j) & \text{(exclusion)}.
\end{array}
$$

(Note that 00 is a trap.)

Examples:

voter	$b = d = 1$	(other par $= 0$),
contact	$b = \lambda,\ c = d = 1$	(other par $= 0$)
symmetric exclusion	$e = 1$	(other par $= 0$)

q-duality:

Duality function D on $\{0,1\}^S \times \{0,1\}^S$ as linear operator acting on

$$\mathbb{R}^{\{0,1\}^S} \cong \bigotimes_{i \in S} \mathbb{R}^{\{0,1\}}$$

Ansatz: duality function of product form w.r.t. margin sites.

$$D(x,y) = \prod_{i \in S} Q(x_i, y_i)$$

(same Q for all sites)

specialize to $Q = Q_q$ for $q \in \mathbb{R} \backslash \{1\}$ with

$$\begin{pmatrix} Q_q(0,0) & Q_q(0,1) \\ Q_q(1,0) & Q_q(1,1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & q \end{pmatrix}$$

$$\Rightarrow \quad D_q(x,y) = q^{\sum_{i \in S} x_i y_i}$$

$$\sum_{i\in S} x_i y_i = |A_x \cap A_y|$$

where $A_x = \{i \in S, x(i) = 1\}$ and $A_x \cap A_y$ is the cardinality of intersection of "occupied sites" in x and y.
(One paper [JTP 2000] also takes a duality function on this intersection set as the starting point to deduce this kind of duality.)

Additive systems duality $\quad q = 0. \quad D_0(x,y) = 1_{\{A_x \cap A_y = \emptyset\}}$

$$= 1_{\{|A_x \cap A_y| = 0\}}$$

Cancellative systems duality $\quad q = -1. \quad D_{-1}(x,y) = (-1)^{|A_x \cap A_y|}$

$$= \begin{cases} 1 & |A_x \cap A_y| \text{ even} \\ -1 & |A_x \cap A_y| \text{ odd} \end{cases}$$

$$= 1 - 2 \cdot 1_{\{|A_x \cap A_y| \text{ odd}\}}$$

$\rightarrow$ These have pathwise interpretations (unlike for other q).

Theorem (q-duality)
$L = L(a,b,c,d,e)$ and $L' = L(a',b',c',d',e')$ are dual with D_q if

$$a' = a + 2q\gamma, \quad b' = b + \gamma, \quad c' = c - (1+q)\gamma, \quad d' = d + \gamma, \quad e' = e - \gamma,$$

where $\gamma = (a + c - d + qb)/(1-q)$.

Duality and intertwining

Suppose L_1, L_2 are dual to $\hat{L}$ with D_1, D_2:

$$L_i D_i = D_i \hat{L}^\dagger, \qquad i = 1,2 \quad (*)$$

If D_i are invertible then

$$D_1^{-1} L_1 = \hat{L}^\dagger = D_2^{-1} L_2 D_2$$

$$\Rightarrow \quad L_1 (\underbrace{D_1 D_2^{-1}}_{K}) = (\underbrace{D_1 D_2^{-1}}_{K}) L_2 \qquad \text{(i)}$$

"intertwining of L_1 with L_2" (instead of $L_2^\dagger$ as in duality)
K "intertwiner"

If L_1, L_2 are generators of Markov processes X^1, X^2
(semigroups P_t^1, P_t^2, distributions μ_t^1, μ_t^2)

and K is a probability kernel then (i) is equivalent to

(ii) $P_t^1 K = K P_t^2$

(iii) $\mu_0^1 K = \mu_0^2 \quad \Rightarrow \quad \mu_t^1 K = \mu_t^2, \quad t \geq 0$

Note that we obtain X^2 from X^1 started from $\mu_0^1 K$ by appying K or in other words we can apply the operator K to a starting distribution and then evolve with the dynamics of X^2 or first evolve with the dynamics of X^1 and then apply the operator K in order to arrive at the same distribution.

In fact, coupling exists:

$$\mathbb{P}\big(X_t^2 \in \cdot \,\big|(X_s^1)_{0 \leq s \leq t}\big) = K(X_t^1, \cdot\,) \text{ a.s., } t \geq 0.$$

Note: We have $K^{-1} L_1 = L_2 K^{-1}$ if K is invertible,
but K^{-1} not a probab. kernel
(reversed roles of L_1 and L_2, intertwining is not symmetric)

q-duality and p-thinning

(Back to interacting particle systems as in q-duality section)

Ansatz: $K(x,y) = \prod_{i \in S} M(x_i, y_i)$ (M probabability kernel "independent coint flips dependent on x_i")

If $M(0,0) = 1$ (natural if 00 is trap) then

$$M_p = \begin{pmatrix} M_p(0,0) & M_p(0,1) \\ M_p(1,0) & M_p(1,1) \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ 1-p & p \end{pmatrix},$$

→ "Thinning kernel" K_p: at each site independently keep particle with probab. p.

Note: $K_p K_{p'} = K_{pp'}$

Easy to see $Q_q Q_{q'}^{-1} = M_p \;\Rightarrow\; D_q D_{q'}^{-1} \overset{(**)}{=} K_p$ if $p = \dfrac{1-q}{1-q'}$

$$\Bigg(\text{from } Q_q^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & q \end{pmatrix}^{-1} = (1-q)^{-1} \begin{pmatrix} -q & 1 \\ 1 & -1 \end{pmatrix} \qquad (q \neq 1),$$

$$Q_q Q_{q'}^{-1} = (1-q')^{-1} \begin{pmatrix} 1 & 1 \\ 1 & q \end{pmatrix} \begin{pmatrix} -q' & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{q-q'}{1-q'} & \frac{1-q}{1-q'} \end{pmatrix} = M_p \Bigg)$$

Proposition: Let L_1, L_2 be generators of Markov processes with state space $\{0,1\}^S$ and $\hat{L}$ an operator s.t.

$$L_i D_{q_i} = D_{q_i} \hat{L}^\dagger \qquad \text{(compare } (*)\text{)}$$

then for $p = \dfrac{1-q_1}{1-q_2} \in [0,1] \quad (q_2 \neq 1)$

$$L_1 K_p = K_p L_2$$

<u>Proof</u> $K = D_{q_1} D_{q_2}^{-1}$ from (i) is the intertwiner $\overset{(**)}{=} K_p$.

<u>Example</u>

Biased voter model $L_{\text{bias}} \quad b = 1+s,\ d = 1$

From the q-duality Theorem (amongst others, restricted to $a' = 0$)

$q = 0 \quad : \quad b' = s,\ c' = 1,\ e' = 1$ branching-coal r.w. (braco)

$q = (1+s)^{-1} \quad : \quad b' = b,\ d' = d$ (biased voter)

$\rightarrow$ self-duality

$\Rightarrow L_{\text{bias}}, L_{\text{braco}}$ are $(1+s)^{-1}$- and 0-dual to L_{bias}

$$\Rightarrow L_{\text{bias}} K_p = K_p L_{\text{braco}} \text{ with } p = \frac{1-(1+s)^{-1}}{1-0} = \frac{s}{1+s}.$$

$$\Rightarrow \mu_0^{\text{bias}} K_p = \mu_0^{\text{braco}} \quad \Rightarrow \quad \mu_t^{\text{bias}} K_p = \mu_t^{\text{braco}}$$