Genealogies of particles on dynamic random networks

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Genealogies of Interacting Particle Systems @ Singapore

- **Interacting particle systems (IPS) on finite networks**
  - David Aldous. "Interacting particle systems as stochastic social dynamics." Bernoulli 19.4 (2013): 1122-1149.
- **Complex network limits: graphons, graphexes, etc.**
  - Christian Borgs et al. "Sparse exchangeable graphs and their limits via graphon processes." arXiv preprint arXiv:1601.07134 (2016).
  - Harry Crane. "Time-varying network models." Bernoulli 21.3 (2015): 1670-1696.
  - Harry Crane. "Dynamic random networks and their graph limits." Ann. Appl. Probab. 26.2 (2016): 691-721.
- **Particles** on (co)**evolving networks**. Some (rare) rigorous works:
  - Luca Avena et al. "Mixing times of random walks on dynamic configuration models." Ann. Appl. Probab. arXiv arXiv:1606.07639 (2016).
  - Emmanuel Jacob, and Peter Mörters. "The contact process on scale-free networks evolving by vertex updating." Royal Society Open Science 4.5 (2017): 170081.
  - Anirban Basak, Rick Durrett, and Yuan Zhang. "The evolving voter model on thick graphs." arXiv:1512.07871 (2015).
- **Open problems.**

Appetizer: A class of finite IPS

- **After:** David Aldous. "Interacting particle systems as stochastic social dynamics." Bernoulli 19.4 (2013): 1122-1149.

Aldous' "Finite Markov Information-Exchange" processes.

- **Agents**: $V := [n]$.
- **Meeting process**: If $\nu_{i,j} > 0$, each undordered pair $\{i,j\} \subset V$ of agents meets at rate $\nu_{i,j}$ independently for different $\{i,j\}$.
- **Meeting geometry**: $G = (V,E)$, $E := \{\{i,j\} \colon \nu_{i,j} > 0\}$ connected graph.
- **States**: $x_i(t) \in S$, $i \in V$, $|S| < \infty$.
- **Update rule**: **Upon meeting at time t, update:**

$$(x_i(t), x_j(t)) := (\mathbf{F}(x_i(t-), x_j(t-)), \mathbf{F}(x_j(t-), x_i(t-))), \quad \{i,j\} \in E,$$

where $\mathbf{F} \colon S^2 \to S$ a (possibly random) mapping.

Example: Voter model

A version

- Assume there are n **possible opinions**: $S := [n]$.
- At time $t = 0$, $x_i(0) = i$, (i.e., the worst possible configuration).
- Upon meeting at time t, flip a fair coin to decide whether:
  - $x_j(t) := x_i(t-)$, i.e., $i \to j$.
  - $x_i(t) := x_j(t-)$, i.e., $j \to i$.

Q: What is the consensus time?

$$T^{\text{voter}} := \min\{t \colon \textbf{all agents have the same opinion}\} = \textbf{?}$$

Flavour:

- This is in the spirit of studies of **mixing/hitting/cover/etc. times of finite Markov chains**.

Goals:

- Study **quantitative dependence of IPS on the "geometry" of the network G**.
- Study
  - $n \to \infty$,
  - $n, t \to \infty$,
  - (rather than just $t \to \infty$ behaviour).

Question

Q:

- Can one describe $n \to \infty$ limit of G**?**
- Is there a limiting object**?**

A class of interesting geometries $G = (V, E)$ is **sparse**, i.e.,

$$|E|/|V|^2 \underset{n\to\infty}{\longrightarrow} 0.$$

Some models:

- Configuration model.
- Preferential attachment.
- ...

Evolving geometries

Many **real-world networks are evolving in time**:

$$G = G(t).$$

This naturally leads to **time-inhomogeneous (and possibly random) meeting (Cox-)Poisson rates**

$$\nu_{i,j} = \nu_{i,j}(t).$$

An inherently multi-scale setup

Scenarios for **speed of the network evolution vs. speed of the agent dynamics**.

- Network is faster than agents.
- Agents are faster than the network.
- Agents and network evolve at the same speed.
  - Adaptive/coevolving agents and network.

A key question:

Q:

- Does the evolution **OF** the network slow down/accelerate the agent dynamics **ON** the network**?**

Outline

1. Introduction

2. Graph limits

3. IPS on evolving networks
   - Mixing times of random walks on dynamic configuration models
   - Contact process on an evolving scale-free network

4. Open problems

Mixing times of random walks on dynamic configuration models

- **After:** Luca Avena et al. "Mixing times of random walks on dynamic configuration models." Ann. Appl. Probab. arXiv arXiv:1606.07639 (2016).

Mixing time

Mixing time of a Markov chain is the **time** it needs to approach its **stationary distribution**

- Popular concept for random walks on static random graphs.
- Provides subtle information about the graph “geometry”.

For evolving graphs, rigorous studies were pioneered by

- Yuval Peres, Alexandre Stauffer, and Jeffrey E. Steif. "Random walks on dynamical percolation: mixing times, mean squared displacement and hitting times." Proba. Theory and Related Fields 162.3-4 (2015): 487-530

Configuration model

Configuration model

The **configuration model** (CM) is a random graph with a given degee sequence.

For **SRW**, on the static CM, the mixing time is of order $\log n$:

- Eyal Lubetzky, and Allan Sly. "Cutoff phenomena for random walks on random regular graphs." Duke Mathematical Journal 153.3 (2010): 475-510.
- Nathanaël Berestycki, Eyal Lubetzky, Yuval Peres, and Allan Sly (2015). Random walks on the random graph. arXiv:1504.01999.

Static configuration model

- Denote by $CM(\underline{d}_n)$ the **set of all graphs on n vertices with given degree sequence**:

$$\underline{d}_n := (d(i))_{i=1}^n.$$

- The **total degree**

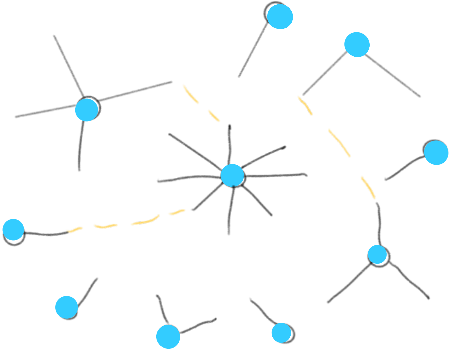
$$|\underline{d}_n| := \sum_{i=1}^n d(i)$$

  is assumed to be even.

- To each degree sequence, we associate a **random graph uniformly drawn from $CM(\underline{d}_n)$**.

Static configuration model: How to generate?

Pair the stubs (a.k.a. halfedges) at random:



$$\underline{d}_n := \{1, 2, 1, 3, 1, 2, 1, 4, 1, 8\},$$
$$n := 10.$$

SRW on dynamic configuration model

For fixed n, draw a starting vertex $u \in V$ and a starting graph configuration $\eta \in CM(d_n)$ and proceed as follows:

1. At each time $t \in \mathbb{Z}_+$, **mark a fraction $\alpha_n \in (0, 1)$ of the edges uniformly at random**.
2. **Refresh/rewire these edges** by using the **configuration model constrained to these edges**, e.g.,

3. Upon rewiring, let the RW **make a step to a random neighbouring vertex**.

Equilibrium in a non-Markovian world?

- Discrete time evolving configuration model: at each unit of time a **fraction $\alpha_n \in (0,1)$ of the edges is refreshed (rewired)**.
- The **rewiring preserves the prescribed degree**.
- Therefore, the **stationary distribution of the SRW does not change in time**.
- Therefore, the notion of **mixing time is well defined**.

Regularity assumptions

Regularity assumptions

Let D_n be the degree of a randomly chosen vertex. There exists a random variable D such that

- $\lim_{n\to\infty} D_n \stackrel{\text{distr}}{=} D$.
- $\lim_{n\to\infty} \mathbb{E}[D_n^2] = \mathbb{E}[D^2] < \infty$.
- $\mathbb{P}\{D_n \geq 3\} = 1$ for all $n \in \mathbb{N}$.

NB! These conditions ensure that

- the probability for a random graph to be **simple** is positive,
- the probability for a random graph to be **connected** tends to one.

Mixing time

- Denote by $\mathbb{P}_{u,\eta}$ the **joint law** of the RW and the dynamic CM.
- Denote by X_t the **location of the RW** at time $t \in \mathbb{Z}_+$.

Definition

The ε**-mixing time** is defined as

$$t^n_{\mathrm{mix}}(\varepsilon; u, \eta) := \inf\{t \in \mathbb{Z}_+ : \|\mathbb{P}_{u,\eta}\{X_t = \cdot\} - \pi_n(\cdot)\|_{\mathrm{TV}} < \varepsilon\},$$

where $\pi_n(i) := {}^{d(i)}/_{|d_n|}$ is the **stationary distribution**.

NB! It is not the usual worst (w.r.t. the initial configuration) case mixing time.

Mixing time

Theorem 1 [Rough asymptotics of mixing time]

If $\lim_{n\to\infty} \alpha_n (\log n)^2 = \infty$, then, for every $\varepsilon > 0$, **with high probability w.r.t. the uniform distribution on u and η**, as $n \to \infty$,

$$(1+o(1))\frac{\sqrt{2}}{\sqrt{\alpha_n}}\sqrt{\log(1/\varepsilon)} \leq t^n_{\mathrm{mix}}(\varepsilon; u, \eta) \leq (1+o(1))\frac{2\sqrt{3}}{\sqrt{\alpha_n}}\sqrt{\log(1/\varepsilon)}.$$

In words: the statement is for **typical** u and η (as opposed to the worst case ones).

Mixing time

Theorem 2 [Sharp asymptotics for slow graph dynamics]

If $\lim_{n\to\infty} \alpha_n (\log n)^2 = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$, then, for every $\varepsilon > 0$, with high probability w.r.t. the uniform distribution on u and η, as $n \to \infty$,

$$t^n_{\mathrm{mix}}(\varepsilon; u, \eta) = (1 + o(1)) \frac{\sqrt{2/a}}{\sqrt{\alpha_n}} \sqrt{\log(1/\varepsilon)},$$

where $a \in (0,1)$ is the **escape probability from the root** for SRW on the **GW-tree** with offspring distribution f given by

$$f(k) := \frac{(k+1)\mathbb{P}\{D = k+1\}}{\mathbb{E}[D]}, \quad k \in \mathbb{Z}_+,$$

i.e., the **size-biased version of D**.

Discussion

1. The mixing time is of order

$$1/\sqrt{\alpha_n},$$

which shows that the **graph dynamics can speed up mixing** (if “severe” enough, i.e., $\alpha_n \gg 1/(\log n)^2$, cf. Theorem 1).

2. Sharp asymptotics for the slow graph dynamics (Theorem 2). The constant involves a $a \in (0,1)$, which shows that the **mixing time is an outcome of the interplay between the particle and random graph dynamics**.

3. Proofs are based on a **stopping time argument**: the first time the RW moves along an edge that has been **relocated** is a **strong uniform time**.

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Contact process (a.k.a. susceptible-infectious-susceptible (SIS) model)

- **After:** Emmanuel Jacob, and Peter Mörters. "The contact process on scale-free networks evolving by vertex updating." Royal Society Open Science 4.5 (2017): 170081.

Contact process on a finite graph of n agents:

- Each agent can be either **infected** or **healthy**.
- Start in a configuration with all (=worst case) infected agents.
- Upon meeting, an infected agent **infects** its vis-à-vis at rate $\lambda > 0$.
- An infected vertex **recovers** at **rate one**.
- (No immunity: Once recovered, a vertex is again susceptible.)

Fact

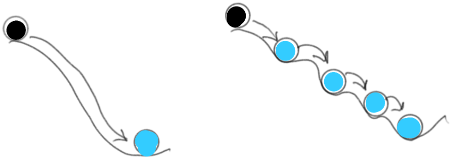
After a **random extinction time** $T^{\text{ext}} < \infty$, **all vertices become healthy** (i.e., absorbing state).

The contact process

Q: How big is T^{ext} as $n \to \infty$?

Two scenarios:

- **Quick extinction:** $\mathbb{E}[T^{\text{ext}}]$ is at most **polynomial in n**.
- **Slow extinction:** W.h.p. T^{ext} is at least **exponential in n**.



Sketch of the **"infection landscape"**:
quick vs. slow extinction due to **metastability**.

Static scale-free network

Scale-free (a.k.a. power law) degrees:

$$\text{proportion of nodes with degree } k \approx k^{-\tau},$$

where $\tau > 0$ is some **power law exponent**.

A class of models:

- $V := [n]$.
- **Idea:** Smaller index implies bigger influence.
- Each pair $\{i,j\}$ of vertices connects independently with probability

$$p_{i,j} := n^{-1} k(i/n, j/n) \wedge 1.$$

Consider:

- **Factor kernel (Chung-Lu model):** $k(x,y) := \beta x^{-\gamma} y^{-\gamma}$
  $\Rightarrow p_{i,j} := \frac{\beta n^{2\gamma-1}}{i^\gamma j^\gamma} \wedge 1.$
  where $\beta > 0$ and $\gamma \in (0,1)$ are the parameters of the model.
- $\Rightarrow \mathbb{E}[\deg(i)] \approx C(n/i)^\gamma \Rightarrow \tau = 1 + 1/\gamma.$

Results for static scale-free networks

Mean-field prediction of Pastor-Sattoras and Vespignani (2001):

- $\tau < 3$, the infection survives for an exponential time for all $\lambda > 0$ (**slow extinction**).
- $\tau > 3$, the expected extinction time is polynomial for small $\lambda > 0$ (existence of **quick extinction**).

Proved to be **WRONG** by Chatterjee and Durrett (2009), Berger et al. (2005): always **slow extinction.** Refinement by Mountford, Valesin and Yao (2013).

Question

Assumption: Network evolution is on the **same time scale** as the spread of the disease.

Q: What happens if we allow for **evolving interaction networks?**

An evolving scale-free network

Consider a continuous-time evolving network $(G(t))_{t\in\mathbb{R}_+}$:

- $V_t := [n]$, $t \geq 0$.
- E_0 consists of independently chosen edges $\{i,j\}$ each with probability

$$p_{i,j} := \frac{1}{n} k(i/n, j/n).$$

- **Vertex driven updating:**
  - Every vertex initiates **independent updates** at **rate** $\kappa > 0$.
  - Upon update initiated by $i \in V$, all adjacent edges are removed and new edges $\{i,j\}$ are formed with probability $p_{i,j}$, $j \in [n] \setminus \{i\}$.

NB! $\Rightarrow G_t \sim G_0$, $t > 0$.

Contact process on evolving scale-free network

Theorem

Consider the contact process, where at $t = 0$ everybody is infected. Then

- **[Slow extinction]** If $\tau < 4$ ($\Leftrightarrow \gamma > 1/3$), then, for all parameters,

$$\mathbb{P}\{T^{\text{ext}} \leq \mathrm{e}^{cn}\} \leq \mathrm{e}^{-cn}.$$

- **[Quick extinction]** If $\tau > 4$ ($\Leftrightarrow \gamma < 1/3$), then there exists a parameter $\lambda_c > 0$ such that, for all $\lambda < \lambda_c$, there exists $C > 0$ such that uniformly in $n > 0$:

$$\mathbb{E}[T^{\text{ext}}] \leq C n^{\gamma} \log n.$$

NB! Here, quick extinction is possible but with a bigger power law exponent (=4) than the (wrongly) predicted one (=3) in the static case.

Heuristics

Scale freeness $\rightsquigarrow$ $\exists$ agents of high degree (= **"stars"**).

Static network:

- Stars can keep infection alive for a long time:
  - If a star gets infected $\rightsquigarrow$ it **infects a fraction of its neighbours**.
  - But once it recovers, it will quickly be **reinfected by its infected neighbours**.
- Therefore, **metastable states arise**, when a fraction of stars become infected.

Evolving network:

- An infected star can **get rewired and subsequently recover** before infecting its neighours $\rightsquigarrow$ quick reinfection is unlikely.
- Therefore, stars can hold infection for a shorter time, and if they are not sufficiently connected (τ big enough), this can destroy metastability.
- **NB!** $\rightsquigarrow$
  - Rewiring can help the SIS to get out of metastable states.
  - Rewiring **speeds up** extinction.

Heuristics

However, it can go the other way around:

- Probability that a star **rewires and then recovers** before infecting its neighbours is $\Theta(1/\deg)$ (= **“successful recovery”**).
- Therefore, the # of updates of an infected star before a successful recovery is $\Theta(\deg)$.
- At each update a star gets $\Theta(\deg)$ neighbours.
- Therefore, an infected star infects $\Theta(\deg^2)$ agents before successful recovery.
- **Mean-field calculation** $\rightsquigarrow$ **phase transition at $\tau = 4$** (instead of the (wrong) mean-field prediction $\tau = 3$ in the static case).
- **NB!** $\rightsquigarrow$
  - Rewiring can help the SIS to infect more vertices.
  - Rewiring **slows down** extinction.
- **The main idea of the proof**: coupling with a mean-field process.

Open (meta-)problems

- Study your fav. finite IPS on your fav. evolving network
  - E.g., finite voter model on evolving network: consensus time? Duality with coalescing RW on evolving network?
- Scaling limits/universality.
- Are exchangeable graph/particle models provable scaling limits of any finite IPS on evolving networks?
- Characterization of the Markovian complex network dynamics for sparse edge exchangeable random networks?
- Adaptive (coevolving) models: allowing for interactions between agent states and graph evolution.
- Infer the network geometry from the behaviour of an interacting particle system on it.
- ...

Summary

- Finite IPS on networks.
- Network limits.
- Evolving networks.
- Finite IPS on evolving networks: Examples.
- Open problems.